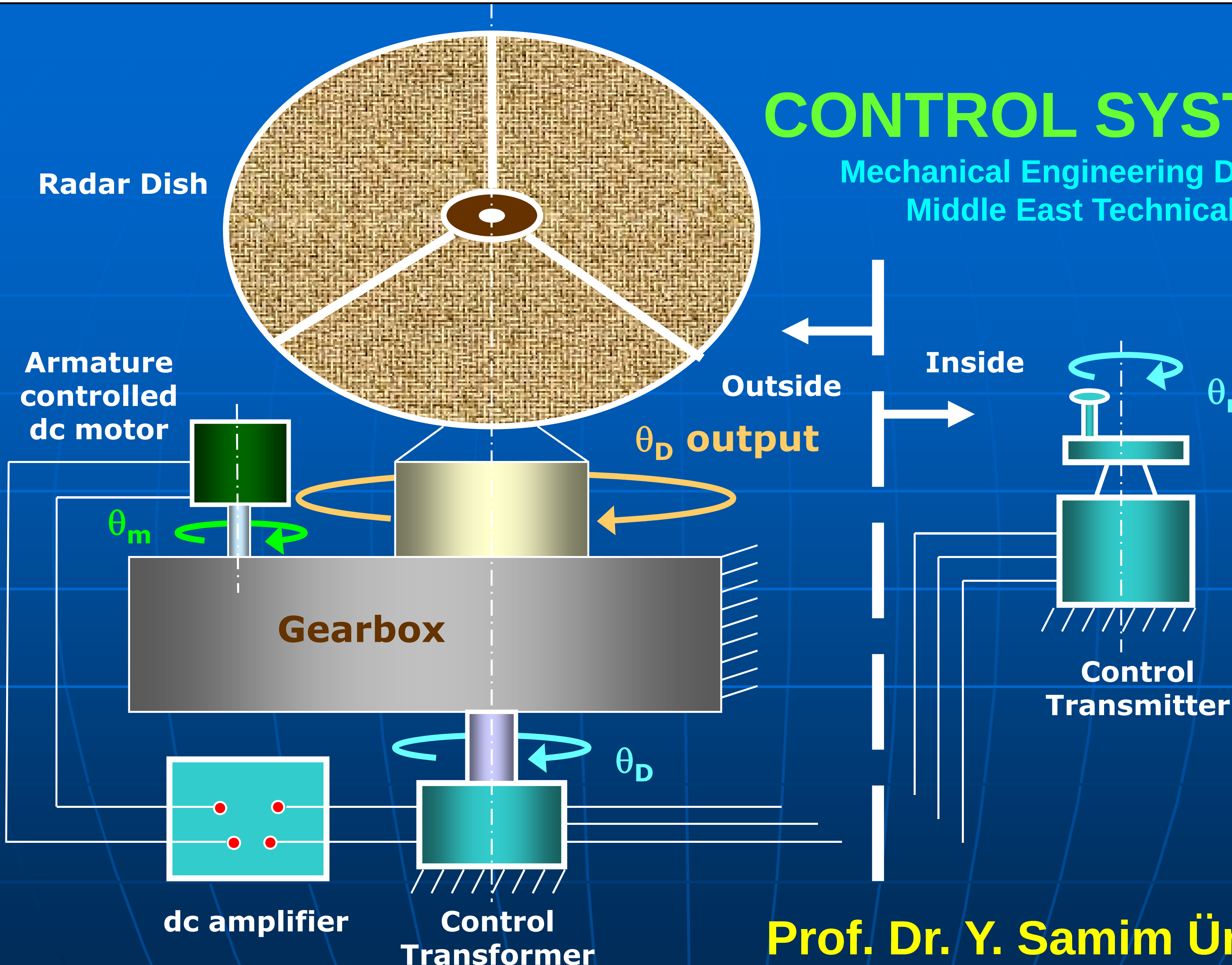
ME 304

CONTROL SYSTEMS

Mechanical Engineering Department,
Middle East Technical University



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CH VI

COURSE OUTLINE

I. INTRODUCTION & BASIC CONCEPTS
II. MODELING DYNAMIC SYSTEMS
III. CONTROL SYSTEM COMPONENTS
IV. STABILITY
V. TRANSIENT RESPONSE
VI. STEADY STATE RESPONSE
VII. DISTURBANCE REJECTION
VIII. BASIC CONTROL ACTIONS & CONTROLLERS
IX. FREQUENCY RESPONSE ANALYSIS
X. SENSITIVITY ANALYSIS
XI. ROOT LOCUS ANALYSIS

STEADY STATE RESPONSE - OBJECTIVES

In this chapter :

- **Open loop transfer functions of feedback control systems will be classified.**
- **Steady state error of feedback control systems due to step, ramp, and parabolic inputs will be investigated.**
- **Selection of controller parameters for a specified steady state error and the measures to be taken to reduce steady state error will be examined.**

STEADY STATE RESPONSE

- **Steady State Response is the response of a system as time goes to infinity.**
- **It is particularly important since it provides an indication of the accuracy of a control system when its output is compared with the desired input.**
- **If they do not agree exactly, then a steady state error exists.**

STEADY STATE RESPONSE

- The Steady State Response of a system is judged by the steady state error due to step, ramp, and parabolic (acceleration) inputs.
- These inputs may be associated with the ability of a control system to :
  - Position itself relative to a stationary target,
  - Follow a target moving at constant speed, and
  - Track an object that is accelerating.

STEADY STATE RESPONSE

- **Note that for the steady state response to exist, the system must be stable.**
- **Therefore before going into steady state analysis it would be good practise to check the stability of the system.**

STEADY STATE RESPONSE

- **The steady state response and error can be obtained by using the final value theorem.**
- **The final value theorem :**
- **The final value of a time signal can be found from the Laplace transform of the signal in the s-domain.**

$$\lim_{t\to\infty} y(t) = \lim_{s\to o} sY(s)$$

STEADY STATE RESPONSE

- **The final value theorem :**

In the case of the output of a system

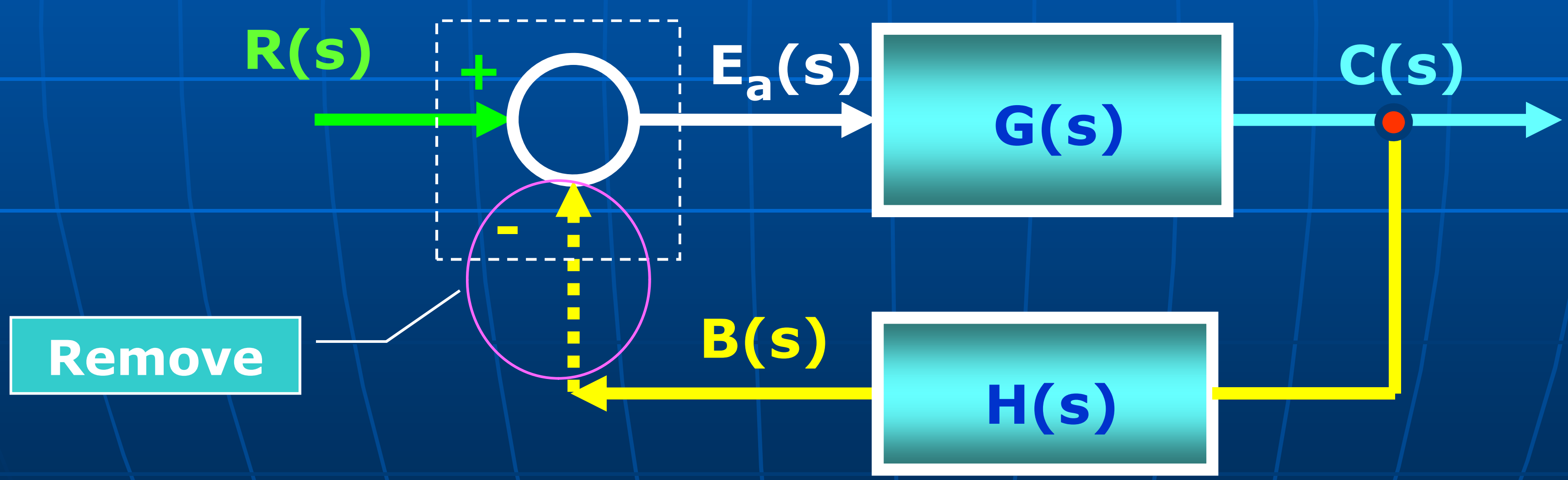
$$C(s) = G(s)R(s)$$

$$\lim_{t \to \infty} c(t) = \lim_{s \to 0} sG(s)R(s)$$

OPEN LOOP TRANSFER FUNCTION

- **The open loop transfer function of a general feedback control system is given by :**

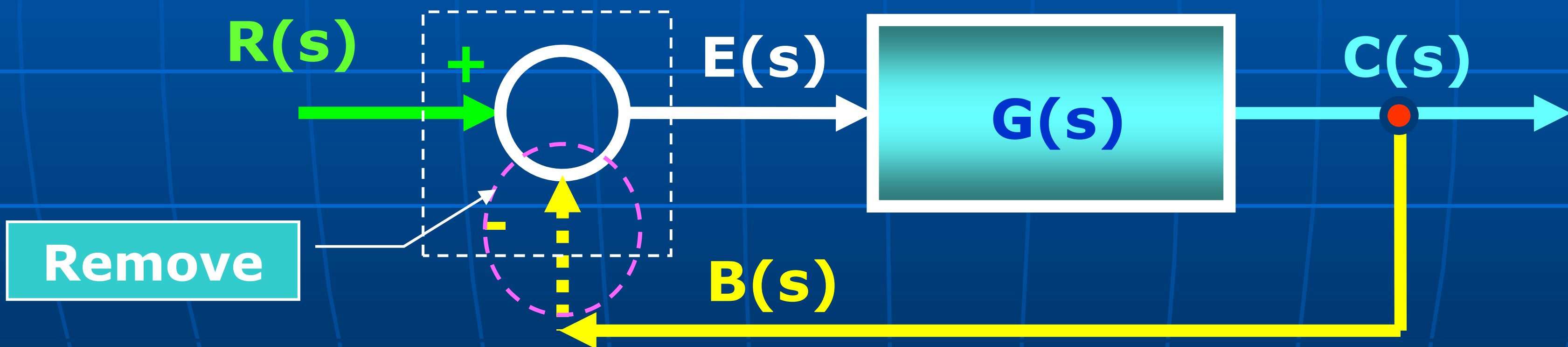
$$\frac{B(s)}{R(s)} = G(s)H(s)$$



OPEN LOOP TRANSFER FUNCTION

- The open loop transfer function of a unity feedback control system is given by :

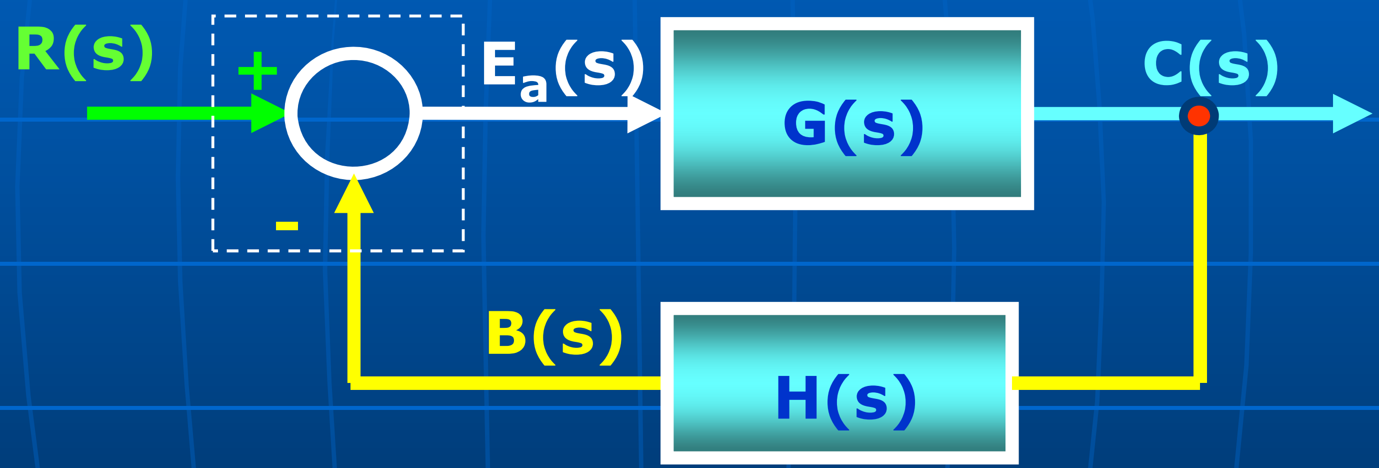
$$\frac{B(s)}{R(s)} = G(s)$$



STEADY STATE ERROR

Nise Ch. 7.1-7.5, Dorf & Bishop Section 4.5, 5.7

The Laplace transform of actuating error, $E_a(s)$, for a general closed loop system :



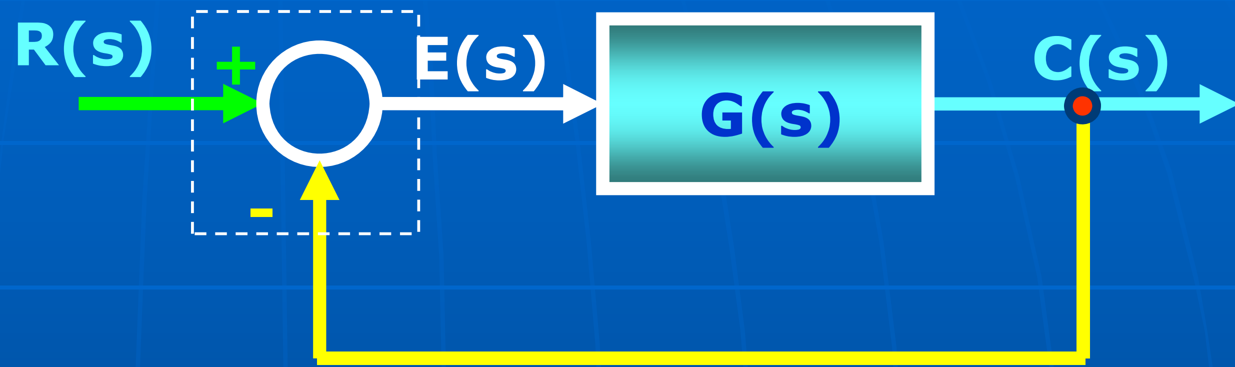
$$E_a(s) = R(s) - H(s)C(s)$$
$$= R(s) - H(s)G(s)E_a(s)$$

$$E_a(s) = \frac{1}{1 + G(s)H(s)} R(s)$$

$$E_a(s) = \frac{R(s)}{1+G(s)H(s)}$$

STEADY STATE ERROR

$$H(s) = 1$$



$$E(s) = \frac{R(s)}{1+G(s)}$$

Note that the actuating and actual errors will be identical only for a unity feedback system, i.e. with an ideal sensor.

For a non-unity feedback system, the actual error may not be zero when the actuating error is zero.

STEADY STATE RESPONSE

- **Here, with a systematic approach, it will be shown that the steady state error for a unity feedback system due to a certain type of input depends on the type of its open loop transfer function.**
- **For a non-unity feedback system, the analysis is somewhat more involved* and will not be covered here.**

*** See Nise, Section 7.6, and Kuo, pp.248-249.**

SYSTEM TYPE

Nise 7.3, Dorf & Bishop 5.7

- The open loop transfer function of a unity feedback control system can be written in the general form :

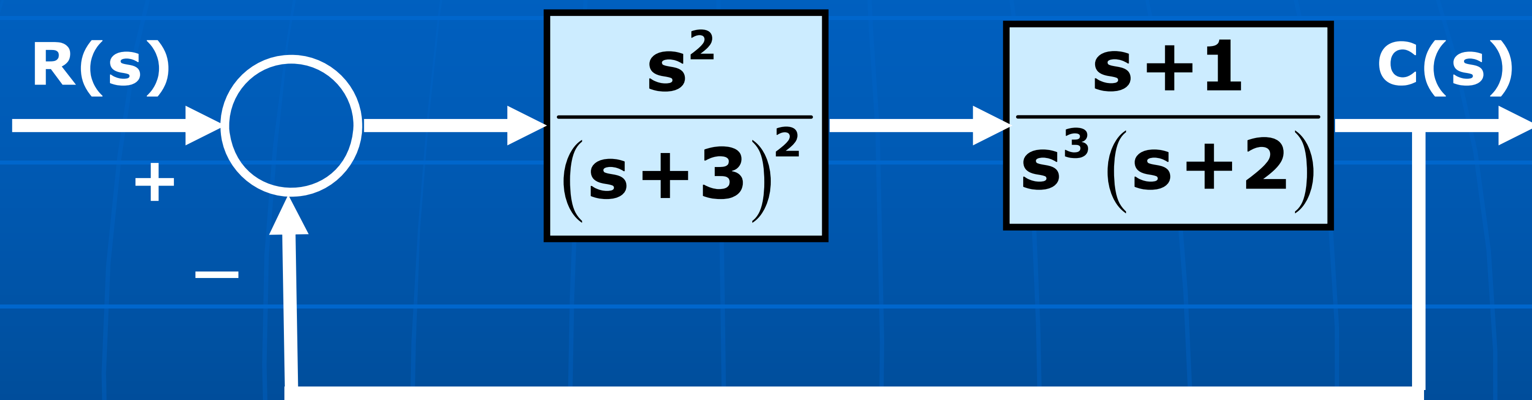
$$T(s) = \frac{K \prod_{p=1}^{P} \left(1 + T_p s\right)}{s^N \prod_{m=1}^{M} \left(1 + \tau_m s\right) \prod_{q=1}^{Q} \left(1 + 2\xi_q \frac{s}{\omega_{n_q}} + \frac{s^2}{\omega_{n_q}^2}\right)}$$

- A unity feedback control system with this open loop transfer function is called a type N system.

s^N : N poles at the origin (free integrators),

K : open loop gain.

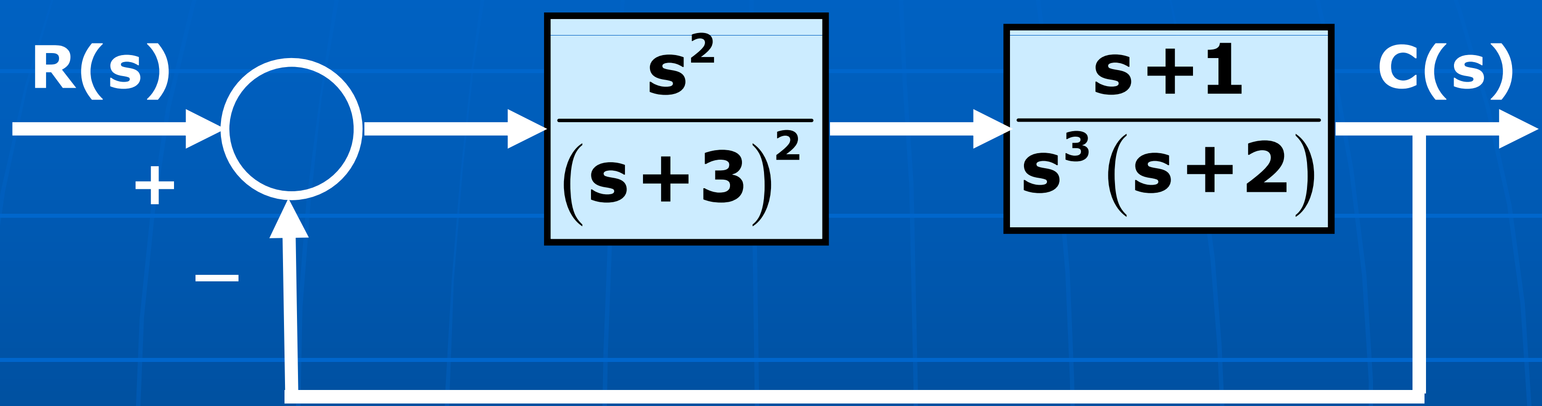
SYSTEM TYPE – EXAMPLE 1a



$$G(s) = \frac{s^2}{(s+3)^2}\frac{s+1}{s^3(s+2)} = \frac{s+1}{s(s+2)(s+3)^2}$$

N=1 : type 1 system

SYSTEM TYPE – EXAMPLE 1b



$$G(s)=\frac{s+1}{s(2)\left(\frac{1}{2}s+1\right)(3)^2\left(\frac{1}{3}s+1\right)^2}=\frac{\frac{1}{18}(s+1)}{s\left(\frac{1}{2}s+1\right)\left(\frac{1}{3}s+1\right)^2}$$

Open loop gain : 1/18

$$E(s) = \frac{R(s)}{1+G(s)}$$

STEADY STATE ERROR

Using the final value theorem :

$$e_{ss} = \lim_{t\to\infty} e(t)$$

$$= \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Note that the steady state error depends on the input and the open loop transfer function of the system.

STEADY STATE ERROR
Step Input

Step Input :

$$r(t) = R$$

$$R(s) = \frac{R}{s}$$

$$e_{ss} = \lim_{s \to 0} \frac{s\left(\frac{R}{s}\right)}{1 + G(s)} = \frac{R}{1 + \lim_{s \to 0} G(s)}$$

$$e_{ss} = \frac{R}{1 + \lim_{s \to 0} G(s)}$$

STEADY STATE ERROR

Define the position error constant

$$K_s = \lim_{s \to 0} G(s)$$

Then

$$e_{ss} = \frac{R}{1 + K_s}$$

STEADY STATE ERROR

$$K_s = G(0)$$

$$e_{ss} = \frac{R}{1+K_s}$$

$$G(s) = \frac{K(T_1 s+1)(T_2 s+1)\ldots(T_m s+1)}{s^N(\tau_1 s+1)(\tau_2 s+1)\ldots(\tau_n s+1)}$$

$$K_s = \begin{cases} K & \text{for type 0 systems} \\ \infty & \text{for type 1 or higher systems} \end{cases}$$

$$e_{ss} = \begin{cases} \dfrac{R}{1+K} & \text{for type 0 systems} \\ 0 & \text{for type 1 or higher systems} \end{cases}$$

STEADY STATE ERROR
Ramp Input

Ramp Input :

$$r(t) = Rt$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$R(s) = \frac{R}{s^2}$$

$$e_{ss} = \lim_{s\to 0} \frac{s\left(\frac{R}{s^2}\right)}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{R}{s+sG(s)}$$

$$= \frac{R}{\lim_{s\to 0} sG(s)}$$

$$e_{ss} = \frac{R}{\lim_{s \to 0} sG(s)}$$

STEADY STATE ERROR

Define the velocity error constant

$$K_v = \lim_{s \to 0} sG(s)$$

Then

$$e_{ss} = \frac{R}{K_v}$$

$$K_v = \lim_{s\to 0} sG(s)$$

$$e_{ss} = \frac{R}{K_v}$$

STEADY STATE ERROR

$$G(s) = \frac{K(T_1 s+1)(T_2 s+1)\ldots(T_m s+1)}{s^N(\tau_1 s+1)(\tau_2 s+1)\ldots(\tau_n s+1)}$$

$$K_v = \begin{cases} 0 & \text{for type 0 systems} \\ K & \text{for type 1 systems} \\ \infty & \text{for type 2 or higher systems} \end{cases}$$

$$e_{ss} = \begin{cases} \infty & \text{for type 0 systems} \\ \dfrac{R}{K} & \text{for type 1 systems} \\ 0 & \text{for type 2 or higher systems} \end{cases}$$

STEADY STATE ERROR
Acceleration Input

Parabolic (acceleration) Input :

$$r(t)=\frac{R}{2}t^2$$

$$L\left\{t^n\right\}=\frac{n!}{s^{n+1}}$$

$$R(s)=\frac{R}{s^3}$$

$$e_{ss}=\lim_{s\to 0}\frac{s\left(\frac{R}{s^3}\right)}{1+G(s)}$$

$$=\lim_{s\to 0}\frac{R}{s^2+s^2G(s)}$$

$$=\frac{R}{\lim_{s\to 0}s^2G(s)}$$

$$e_{ss} = \frac{R}{\lim_{s\to 0} s^2 G(s)}$$

STEADY STATE ERROR

Define the acceleration error constant

$$K_a = \lim_{s\to 0} s^2 G(s)$$

Then

$$e_{ss} = \frac{R}{K_a}$$

$$K_a = \lim_{s\to 0} s^2 G(s)$$

$$e_{ss} = \frac{R}{K_a}$$

STEADY STATE ERROR

$$G(s) = \frac{K(T_1 s+1)(T_2 s+1)\ldots(T_m s+1)}{s^N(\tau_1 s+1)(\tau_2 s+1)\ldots(\tau_n s+1)}$$

$$K_a = \begin{cases} 0 & \text{for type 0 and 1 systems} \\ K & \text{for type 2 systems} \\ \infty & \text{for type 3 or higher systems} \end{cases}$$

$$e_{ss} = \begin{cases} \infty & \text{for type 0 and 1 systems} \\ \frac{R}{K} & \text{for type 2 systems} \\ 0 & \text{for type 3 or higher systems} \end{cases}$$

STEADY STATE ERROR – Summary

System Type	Step Input	Ramp Input	Parabolic Input
0	$\frac{R}{1+K}$	∞	∞
1	0	$\frac{R}{K}$	∞
2	0	0	$\frac{R}{K}$
3	0	0	0

STEADY STATE ERROR
Multiple Inputs

For linear systems, the steady state error for the simultaneous application of two or more inputs will be the superposition of the steady state errors due to each input applied separately.

STEADY STATE ERROR

For example, for an input of

$$r(t) = 1 + 2t + \frac{3t^2}{2}$$

the steady state error is given as the superposition of the steady responses to each of the inputs.

$$e_{ss} = \frac{1}{1+K_s} + \frac{2}{K_v} + \frac{3}{K_a}$$

STEADY STATE ERROR

Therefore the steady state error of the system subjected to the composite input will be :

$$e_{ss} = \frac{1}{1+K_s} + \frac{2}{K_v} + \frac{3}{K_a}$$

$$= \frac{1}{1+K} + \frac{2}{0} + \frac{3}{0} = \infty$$ **for type 0 systems**

$$= \frac{1}{1+\infty} + \frac{2}{K} + \frac{3}{0} = \infty$$ **for type 1 systems**

$$= \frac{1}{1+\infty} + \frac{1}{\infty} + \frac{3}{K} = \frac{3}{K}$$ **for type 2 systems**

$$= \frac{1}{1+\infty} + \frac{1}{\infty} + \frac{3}{\infty} = 0$$ **for type 3 or higher systems**

STEADY STATE ERROR - Observations

It is observed that :

i) When e_{ss} is finite, increasing the open loop gain decreases the steady state error.

Step : $\frac{R}{1+K}$

Ramp : $\frac{R}{K}$
Acceleration : $\frac{R}{K}$

K : open loop gain

STEADY STATE ERROR - Observations

ii) **As the type of the system is increased, e_{ss} decreases.**

Therefore one may attempt to improve the steady state response by including an integrator in the controller.

This, however, may cause stability problems which become critical for type 3 or higher systems.

STEADY STATE ERROR - Observations

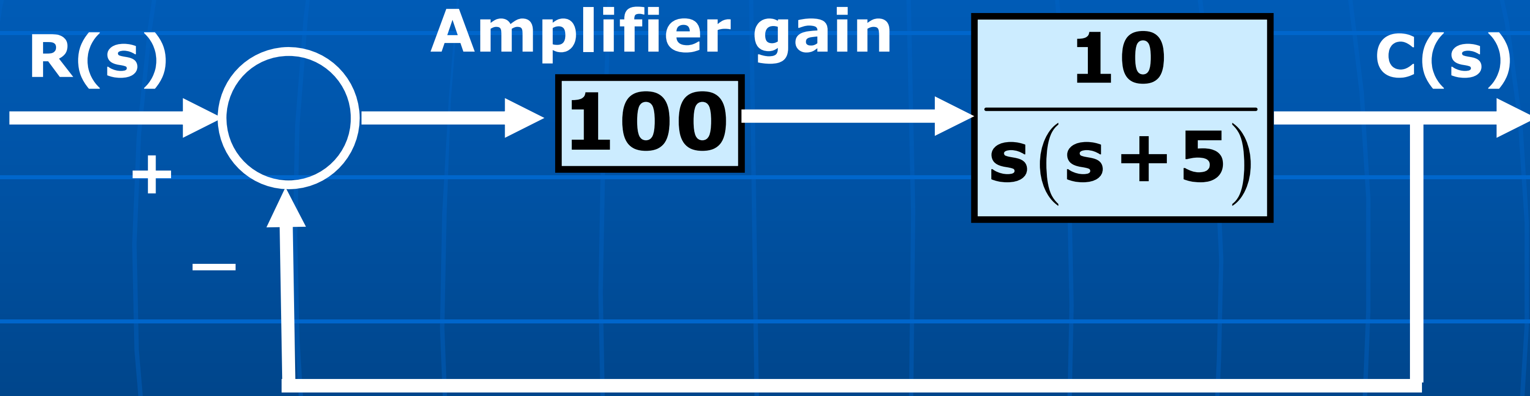
iii) **The approach to the minimization of the steady state error in control systems may thus be :**

Determine the maximum possible open loop gain and check the maximum allowable open loop gain that will not result in instability.

Choose the smaller of the two open loop gain values.

STEADY STATE ERROR – Example 1a

Determine the steady state error for $r(t)=2+3t$



$$G(s)=\frac{1000}{s(s+5)}=\frac{1000}{s(5)(0.2s+1)}=\frac{200}{s(0.2s+1)}$$

N=1 : type 1 system, K=200

STEADY STATE ERROR – Example 1b

$$G(s) = \frac{200}{s(0.2s+1)}$$

N=1 : type 1 system, K=200

$$e_{ss} = 0 \text{ for } r(t) = 2 \text{ (step input)}$$

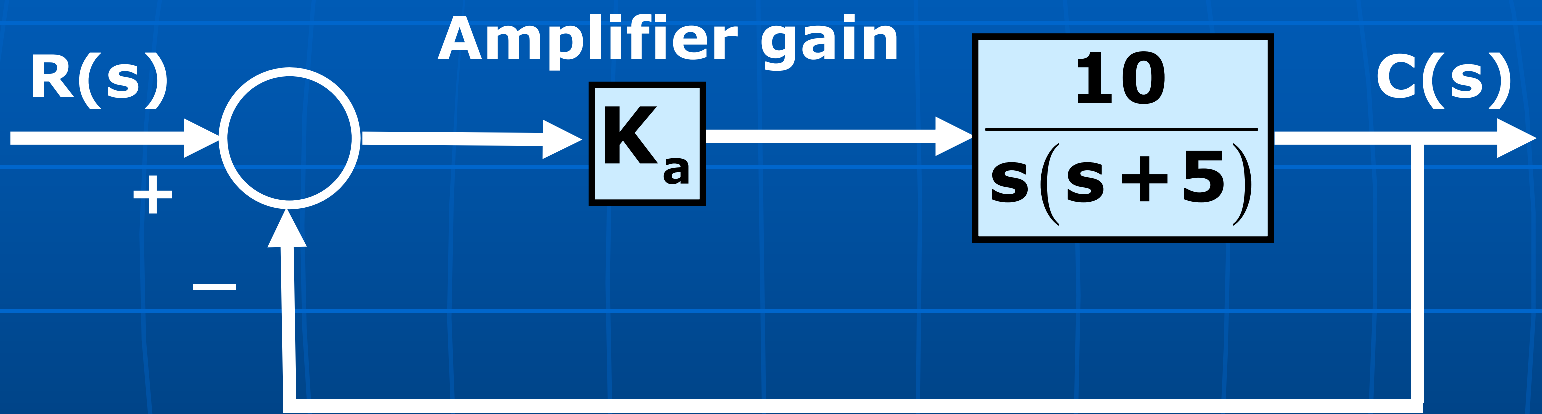
$$e_{ss} = \frac{R}{K} = \frac{3}{200} = 0.015 \text{ for } r(t) = 3t \text{ (ramp input)}$$

$$\text{Hence } e_{ss} = 0.015 = 1.5\,\%$$

It is obvious that if you increase the value of the open loop gain K, say by increasing amplifier gain, steady state error will decrease.

STEADY STATE ERROR – Example 1c

Let us check the maximum value of the amplifier gain without causing instability.



$$\frac{C(s)}{R(s)}=\frac{\frac{10K_a}{s(s+5)}}{1+\frac{10K_a}{s(s+5)}}=\frac{10K_a}{s^2+5s+10K_a}$$

STEADY STATE ERROR – Example 1d

Thus the characteristic polynomial is given by :

$$D(s) = s^2 + 5s + 10K_a$$

It is obvious that it passes Hurwitz test. Application of Routh's stability criterion will result in only one condition on stability :

s^2	1	$10K_a$
s^1	5	0
s^0	$10K_a$	

$$K_a > 0$$

STEADY STATE ERROR – Example 1e

As an alternative solution, let us calculate the steady state error using the final value theorem.

$$\frac{C(s)}{R(s)} = \frac{\dfrac{1000}{s(s+5)}}{1+\dfrac{1000}{s(s+5)}} = \frac{1000}{s(s+5)+1000}$$

$$r(t) = 2 + 3t$$

$$R(s) = \frac{2}{s} + \frac{3}{s^2}$$

$$E(s) = R(s) - C(s) = R(s) - \frac{1000}{s(s+5)+1000}R(s)$$

$$E(s) = \frac{s(s+5)}{s^2+5s+1000}R(s) = \frac{s(s+5)}{s^2+5s+1000}\left(\frac{2}{s}+\frac{3}{s^2}\right)$$

STEADY STATE ERROR – Example 1f

Thus with the error expression in Laplace domain :

$$E(s) = \frac{s+5}{s^2+5s+1000}\left(\frac{2s+3}{s}\right)$$

$$e_{ss} = \lim_{s\to 0} sE(s)$$

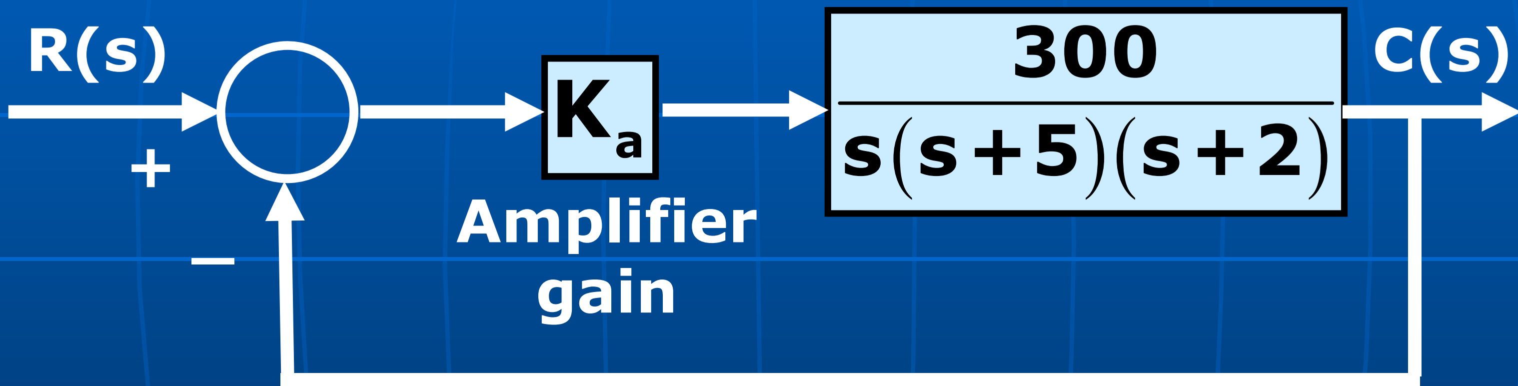
$$= \lim_{s\to 0} s\left(\frac{2s+3}{s}\right)\frac{s+5}{s^2+5s+1000}$$

$$= \frac{15}{1000} = 0.015 = 1.5\ \%$$

It is obvious that this method requires considerably higher effort.

STEADY STATE ERROR – Example 2a

Determine the value of amplifier gain K_a such that the steady state error for a ramp input $r(t)=3t$ is going to be at most 2 %.



$$G(s)=\frac{300K_a}{s(s+2)(s+5)}=\frac{300K_a}{s(2)(5)(0.5s+1)(0.2s+1)}=\frac{30K_a}{s(0.5s+1)(0.2s+1)}$$

N=1 : type 1 system, $K=30K_a$

STEADY STATE ERROR – Example 2b

$$G(s) = \frac{30K_a}{s(0.5s+1)(0.2s+1)}$$

N=1 : type 1 system, $K=3K_a$; R=3

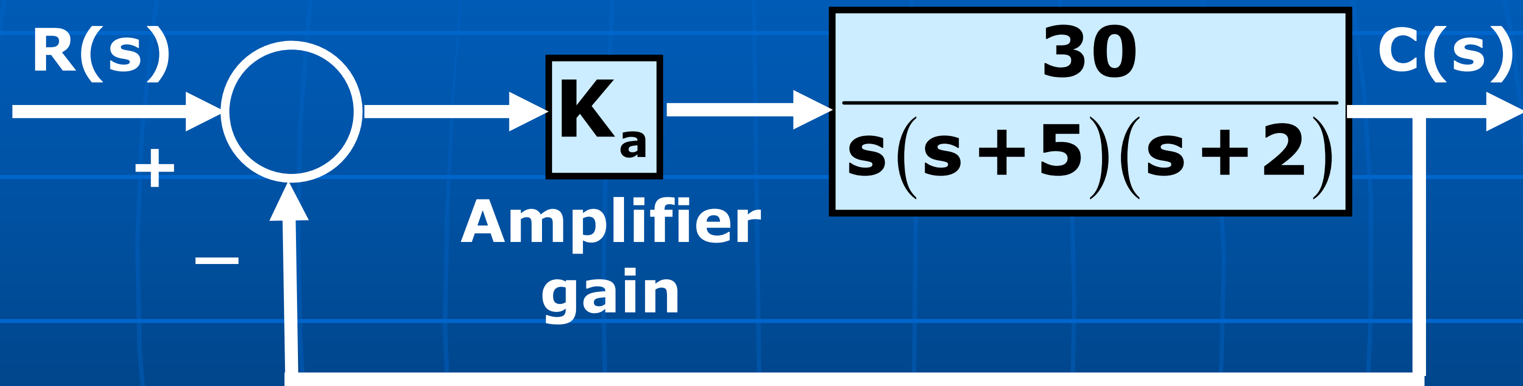
$$e_{ss} = \frac{R}{K} = \frac{3}{30K_a} \leq 0.02 \text{ for } r(t) = 3t \text{ (ramp input)}$$

$$\text{Hence } K_a \geq \frac{3}{30(0.02)} = 5$$

Now , the limit on K_a with respect to stability should be checked.

STEADY STATE ERROR – Example 2c

Let us check the maximum value of the amplifier gain without causing instability.



$$\frac{C(s)}{R(s)} = \frac{\dfrac{30K_a}{s(s+2)(s+5)}}{1+\dfrac{30K_a}{s(s+2)(s+5)}} = \frac{30K_a}{s^3+7s^2+10s+30K_a}$$

STEADY STATE ERROR – Example 2d

Thus the characteristic polynomial is given by :

$$D(s) = s^3 + 7s^2 + 10s + 30K_a$$

It is obvious that it passes Hurwitz test. Application of Routh's stability criterion will result in :

s^3	1	10
s^2	7	$30K_a$
s^1	$\dfrac{70-30K_a}{7}$	0
s^0	$30K_a$	

$$K_a > 0$$

$$10 - \frac{30}{7}K_a > 0 \qquad K_a < \frac{70}{30} = 2.33$$

STEADY STATE ERROR – Example 2e

It is obvious that the value of K_a required for the specified steady state error will make the system unstable.

Therefore, the minimum possible steady state error will be higher than the desired value.

Can you determine the minimum possible steady state error for this system ?