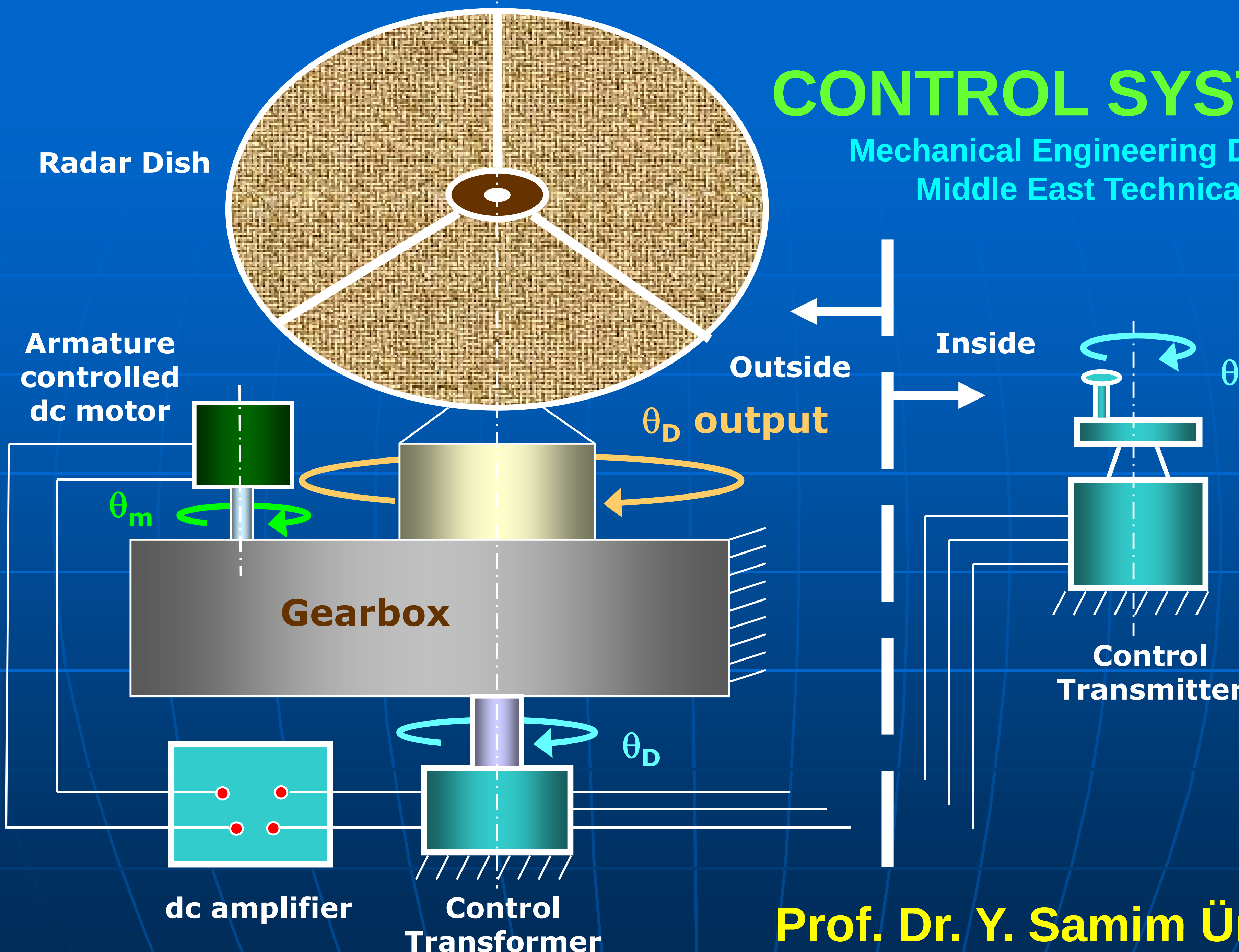
ME 304
CONTROL SYSTEMS

Mechanical Engineering Department,
Middle East Technical University



Prof. Dr. Y. Samim Ünlüsoy

CH V

COURSE OUTLINE

I. INTRODUCTION & BASIC CONCEPTS
II. MODELING DYNAMIC SYSTEMS
III. CONTROL SYSTEM COMPONENTS
IV. STABILITY
V. TRANSIENT RESPONSE
VI. STEADY STATE RESPONSE
VII. DISTURBANCE REJECTION
VIII. BASIC CONTROL ACTIONS & CONTROLLERS
IX. FREQUENCY RESPONSE ANALYSIS
X. SENSITIVITY ANALYSIS
XI. ROOT LOCUS ANALYSIS

TRANSIENT RESPONSE OBJECTIVES

Completed !

In this chapter :

- Time response of general first and second order systems to standard test inputs will be obtained.
- Specification of transient response as performance characteristics for control systems will be examined.
- The selection of controller parameters to meet transient response specifications will be explored.

We are here !

TRANSIENT RESPONSE of SECOND ORDER SYSTEMS

Nise Section 4.4, 4.5, 4.6

- **There exists a large number of second order systems which are represented by the same general second order differential equation and the corresponding transfer function :**

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_1 \frac{dy}{dt} + b_0 y$$

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

$$G(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

TRANSIENT RESPONSE of SECOND ORDER SYSTEMS

- There exists two more general representations of the transfer functions of second order systems. The first is :

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(\eta\omega_n s + \omega_n^2\right)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$K = G(0) = \frac{b_0}{a_0}$$

$$\eta = \frac{b_1}{b_0}\sqrt{\frac{a_0}{a_2}}$$

$$\omega_n = \sqrt{\frac{a_0}{a_2}} \qquad \xi = \frac{a_1}{2a_2}\sqrt{\frac{a_2}{a_0}}$$

K : steady state or dc gain.
η : characteristic time ratio.
ξ : damping ratio.
ω_n : undamped natural frequency.

$$G(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

TRANSIENT RESPONSE of SECOND ORDER SYSTEMS

- The second general form of the transfer function for the second order systems is in the form :

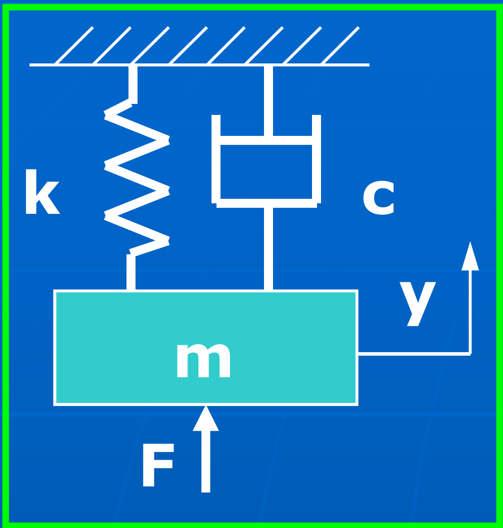
$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(T_0 s + 1\right)}{T^2 s^2 + 2\xi T s + 1}$$

$$K = G(0) = \frac{b_0}{a_0}$$

$$T = \sqrt{\frac{a_2}{a_0}} \qquad T_0 = \frac{b_1}{b_0}$$

$$\xi = \frac{a_1 T}{2 a_2}$$

K : steady state (or dc) gain.
T : system characteristic time.
T_0 : numerator characteristic time.
ξ : damping ratio.



SECOND ORDER SYSTEMS

$$m\ddot{y} + c\dot{y} + ky = F$$

$$G(s) = \frac{1}{ms^2 + cs + k} = \frac{\frac{1}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}} = \frac{\frac{1}{k}\left(\frac{k}{m}\right)}{s^2 + \left(\frac{c}{m}\right)s + \left(\frac{k}{m}\right)}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(\eta\omega_n s + \omega_n^2\right)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

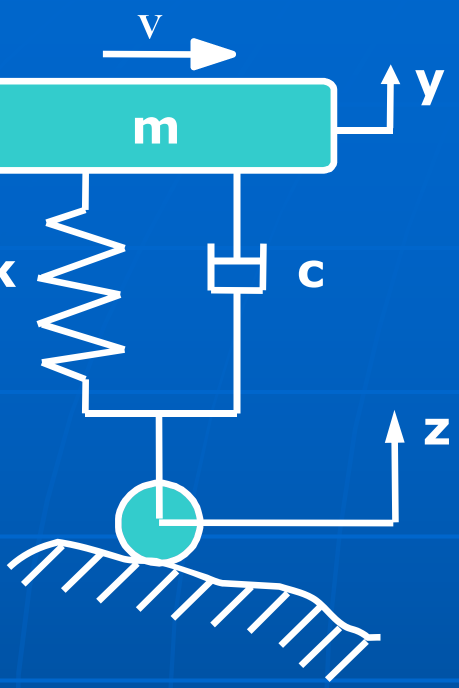
$$K = \frac{1}{k} \qquad \eta = 0 \qquad \omega_n^2 = \frac{k}{m} \qquad \xi = \frac{c}{2m\omega_n}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(T_0 s + 1\right)}{T^2 s^2 + 2\xi T s + 1}$$

$$K = \frac{1}{k} \qquad T = \frac{1}{\omega_n} = \sqrt{\frac{m}{k}} \qquad \xi = \frac{c}{2m\omega_n}$$

$$T_0 = 0$$

SECOND ORDER SYSTEMS



$$m\ddot{y} + c\dot{y} + ky = c\dot{z} + kz$$

$$G(s) = \frac{cs + k}{ms^2 + cs + k} = \frac{\left(\frac{c}{m}\right)s + \left(\frac{k}{m}\right)}{s^2 + \left(\frac{c}{m}\right)s + \left(\frac{k}{m}\right)}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(\eta\omega_n s + \omega_n^2\right)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$K = 1$ $\quad \eta = \frac{c\omega_n}{k}$ $\quad \omega_n^2 = \frac{k}{m}$ $\quad \xi = \frac{c}{2m\omega_n}$

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(T_0 s + 1\right)}{T^2 s^2 + 2\xi Ts + 1}$$

$K = 1$ $\quad T_0 = \frac{c}{k}$ $\quad T = \sqrt{\frac{m}{k}}$ $\quad \xi = \frac{c}{2m\omega_n}$

SECOND ORDER SYSTEMS

Nise Section 4.5, Dorf&Bishop Section 5.3, Ogata pp.226-229

- **In this course, we will limit the transient response studies of the second order systems to the step response of the systems represented by the general transfer function of the form :**

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- **Thus numerator dynamics will not be considered.**

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

SECOND ORDER SYSTEMS

- **Such a system is classified as :**

  - undamped if $\xi = 0$,

  - underdamped if $0 < \xi < 1$,

  - critically damped if $\xi = 1$, **and**

  - overdamped if $\xi > 1$.

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

SECOND ORDER SYSTEMS

- The characteristic equation for the second order system is obtained by setting the denominator of the transfer function equal to zero.

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

- The roots of the characteristic equation can be written, in general, as :

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

SECOND ORDER SYSTEMS

- The roots of the characteristic equation are called the **poles** of the system.

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

- Similarly, the roots of the numerator polynomial of the transfer function are called the **zeroes** of the system.

$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

SECOND ORDER SYSTEMS

Thus, for

- **undamped** (ξ=0) systems there are
  - two purely imaginary roots (poles),

$$s_{1,2} = \pm j\omega_n$$

- **critically damped** (ξ=1)systems, there are
  - two identical (repeated) real roots,

$$s_{1,2} = -\omega_n$$

- **overdamped** (ξ>1) systems, there are
  - two distinct negative real roots, and

$$s_1 = -\omega_{n1}$$
$$s_2 = -\omega_{n2}$$

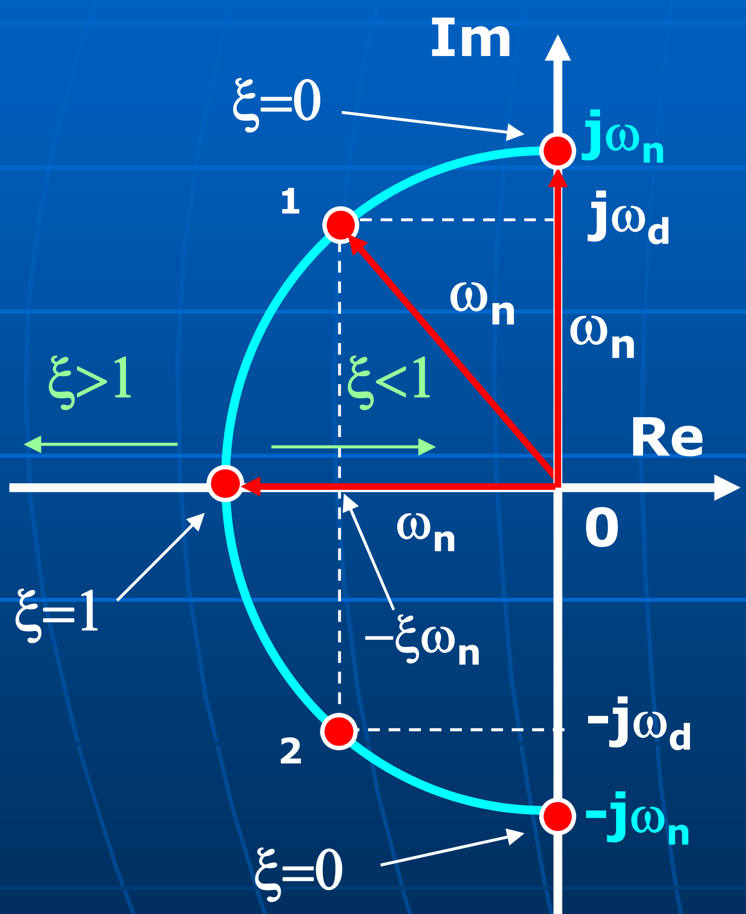
- **underdamped** (ξ<1) systems, there are
  - two complex conjugate roots.

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}$$

$$s_{1,2} = -\xi\omega_n \pm j\omega_d$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

SECOND ORDER SYSTEMS



- The roots of the characteristic equation on the complex plane.

$$\xi = 0 \quad \Rightarrow \quad s_{1,2} = \pm j\omega_n$$

$$\xi = 1 \quad \Rightarrow \quad s_{1,2} = -\omega_n$$

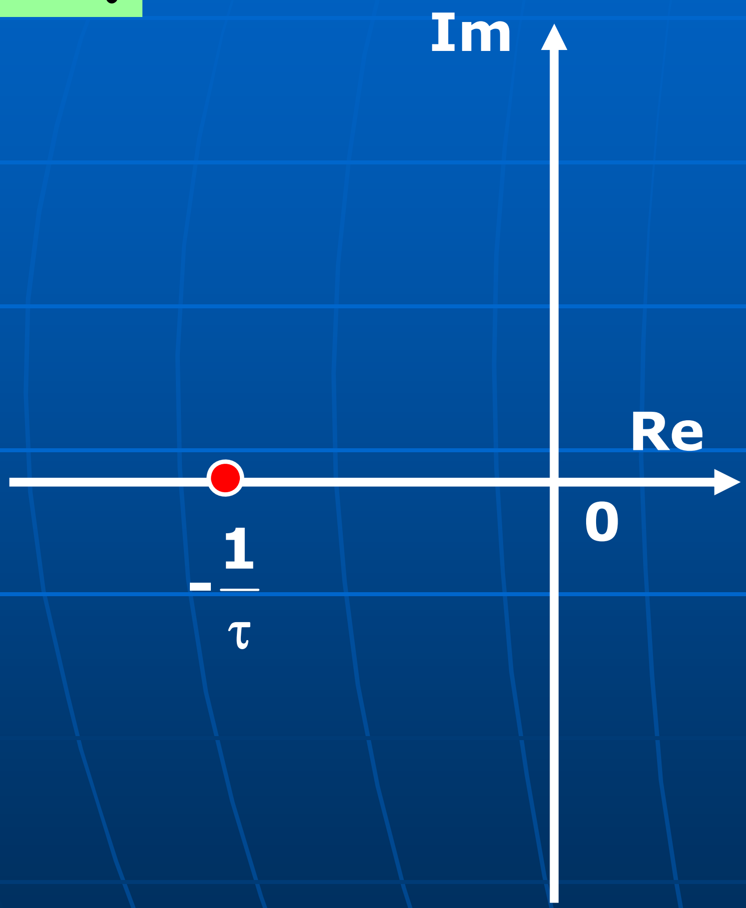
$$\xi > 1 \quad \Rightarrow \quad s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$\xi < 1 \quad \Rightarrow \quad s_{1,2} = -\xi\omega_n \pm j\omega_d$$

$$G(s) = \frac{K}{\tau s + 1}$$

$$s = -\frac{1}{\tau}$$

FIRST ORDER SYSTEMS



- **The root of the characteristic equation is always negative and real.**
- **Thus the response will contain an exponential term.**

$$G(s)=\frac{C(s)}{R(s)}=\frac{K\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

UNDAMPED SECOND ORDER SYSTEMS

- **Undamped System (ξ = 0) – Step input :**

$$G(s)=\frac{K\omega_n^2}{s^2+\omega_n^2}$$

$$R(s)=\frac{R}{s}$$

$$C(s)=\frac{K\omega_n^2}{s^2+\omega_n^2}\frac{R}{s}=KR\frac{\omega_n^2}{s\left(s^2+\omega_n^2\right)}$$

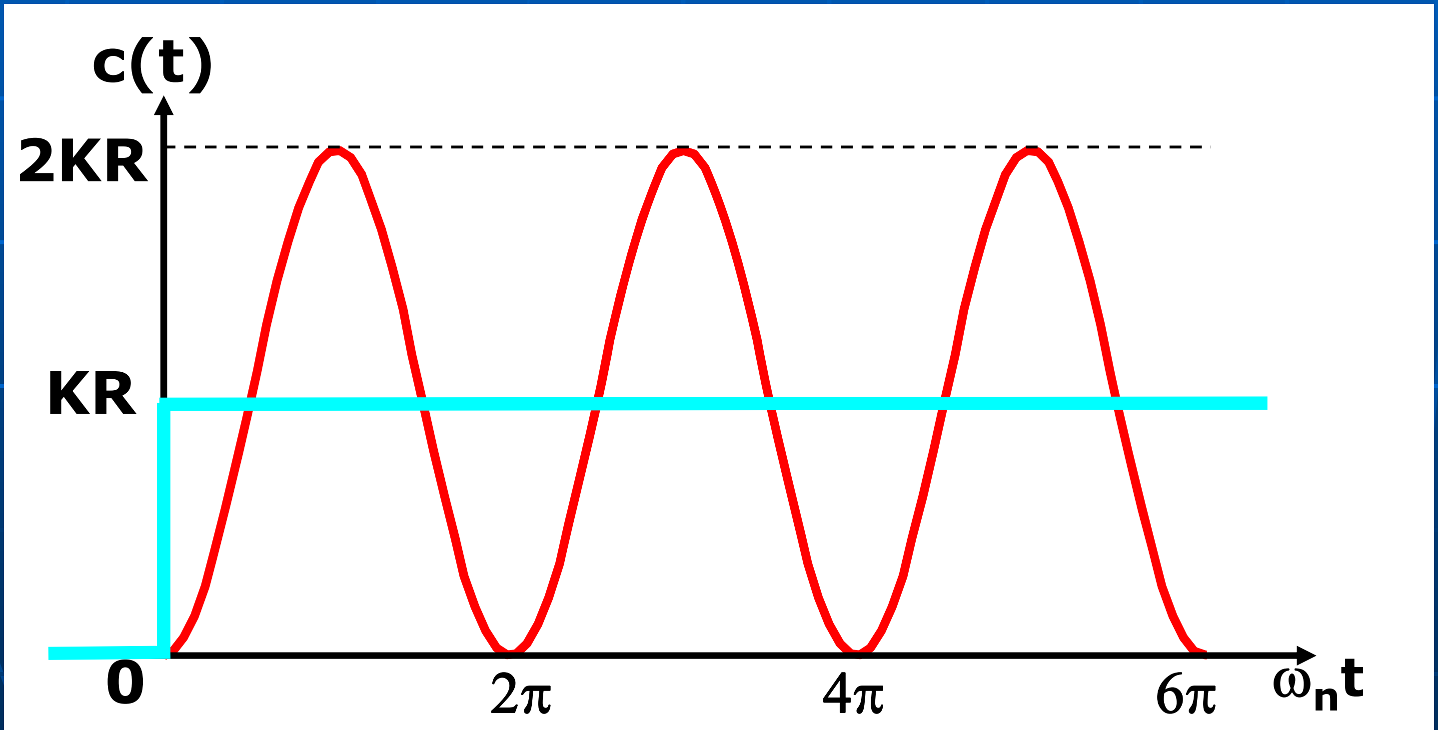
$$L^{-1}\left[\frac{\omega^2}{s\left(s^2+\omega^2\right)}\right]=1-\cos\omega t$$

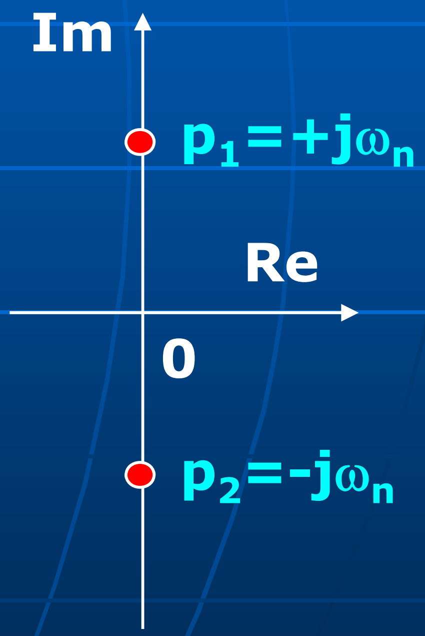
$$c(t)=KR\left(1-\cos\omega_n t\right)$$

$$c(t) = KR\left(1 - \cos\omega_n t\right)$$

UNDAMPED SECOND ORDER SYSTEMS

- It is clear that the step response of a general **undamped second order system** is a **harmonic function with frequency ω_n**, superimposed on a **step function**.





UNDERDAMPED SECOND ORDER SYSTEMS

- **Underdamped System ($0<\xi<1$) – Step input :**

$$G(s)=\frac{C(s)}{R(s)}=\frac{K\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$R(s)=\frac{R}{s}$$

$$C(s)=\frac{K\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\frac{R}{s}$$

$$c(s)=KR\left[\frac{1}{s}-\frac{s+\xi\omega_n}{\left(s+\xi\omega_n\right)^2+\omega_d^2}-\frac{\xi\omega_n}{\left(s+\xi\omega_n\right)^2+\omega_d^2}\right]$$

UNDERDAMPED
SECOND ORDER SYSTEMS

$$c(s) = KR\left[\frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right]$$

- **Taking the inverse Laplace transform :**

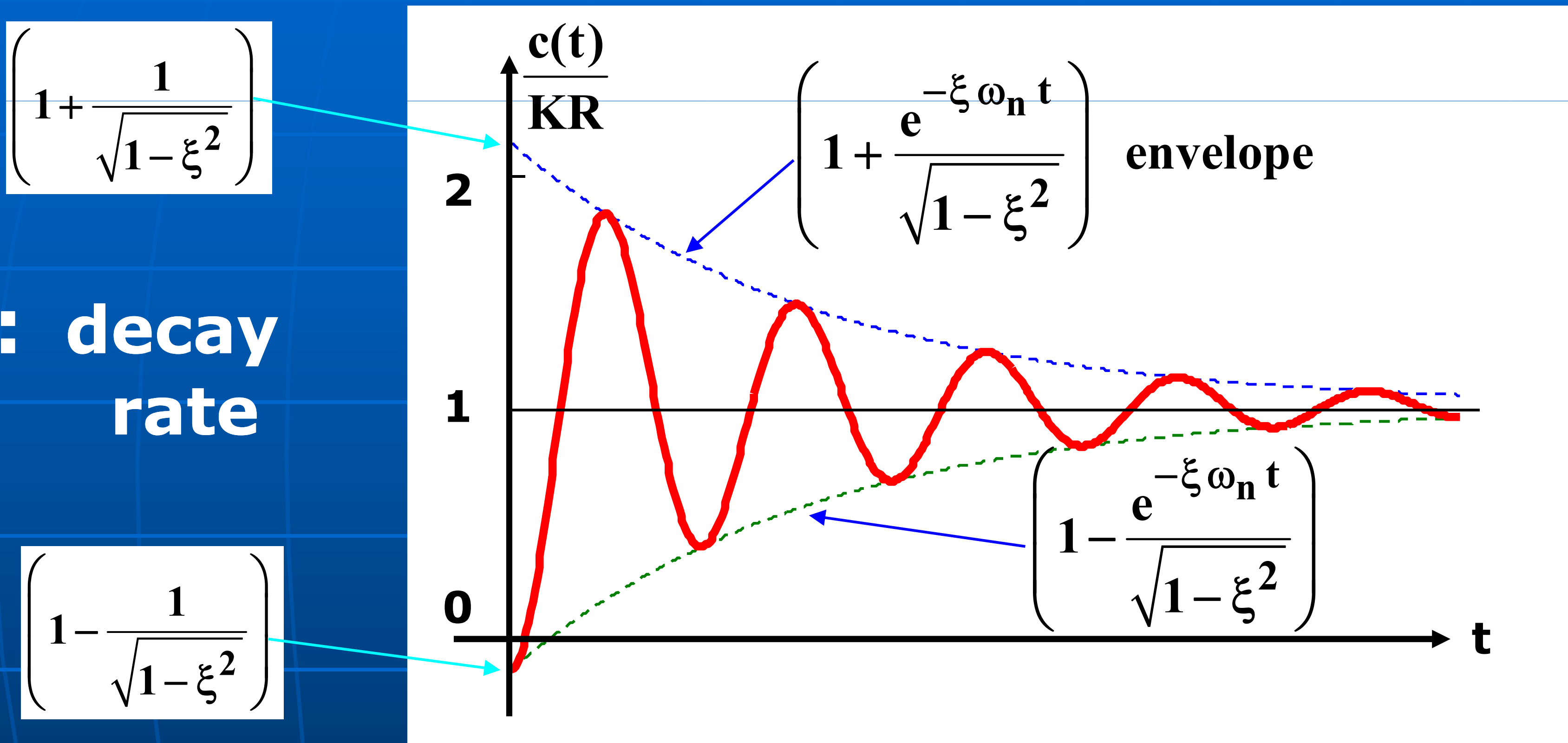
$$c(t) = KR\left[1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)\right] \qquad t \geq 0$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$
$$(\omega_d < \omega_n)$$

the damped natural frequency.

$\xi\omega_n$ **decay rate.**

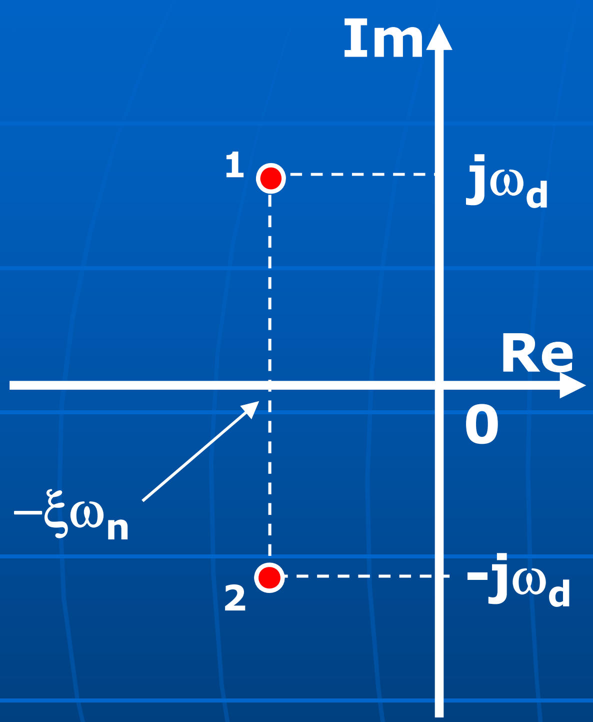
UNDERDAMPED
SECOND ORDER SYSTEMS

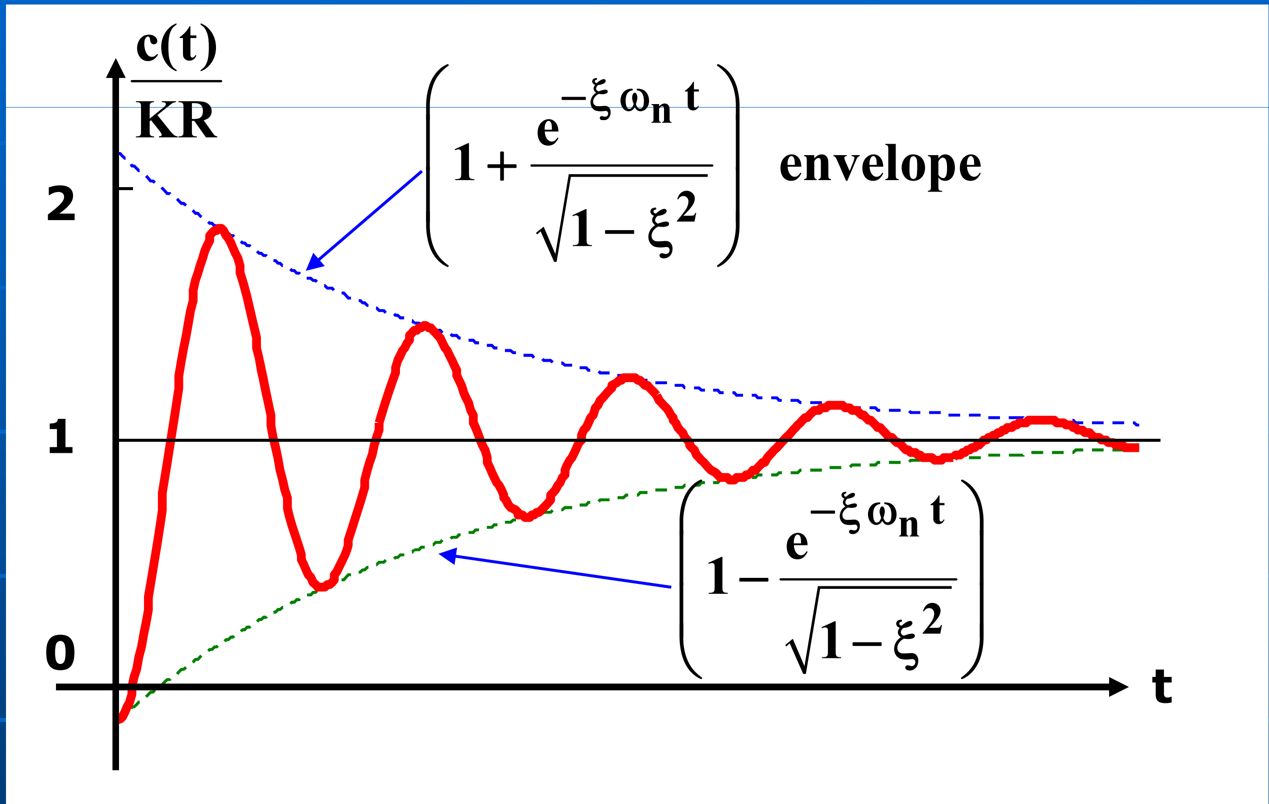


$\xi\omega_n$: decay rate

$$\frac{c(t)}{KR} = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right) \qquad t \geq 0$$

UNDERDAMPED
SECOND ORDER SYSTEMS





$$\frac{c(t)}{KR}=1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left(\omega_d t+\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right)\qquad t\geq 0$$

CRITICALLY DAMPED
SECOND ORDER SYSTEMS

- **Critically Damped System (ξ=1) – Step input :**

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$R(s) = \frac{R}{s}$$

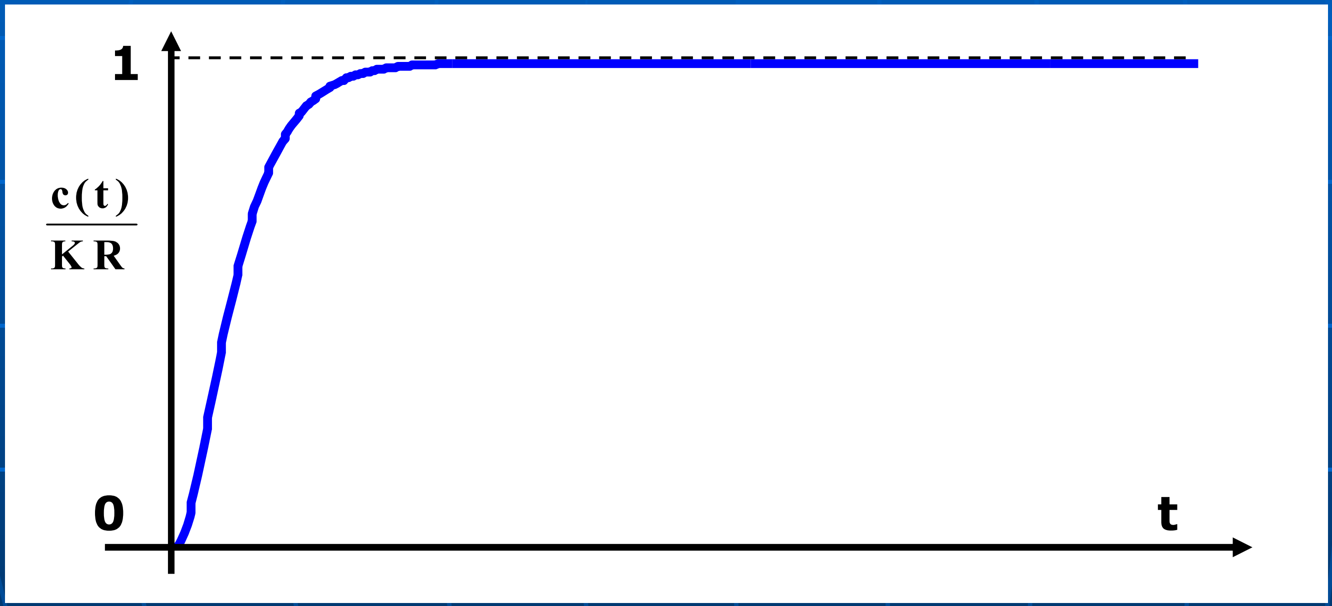
$$C(s) = \frac{K\omega_n^2}{\left(s + \omega_n\right)^2}\frac{R}{s}$$

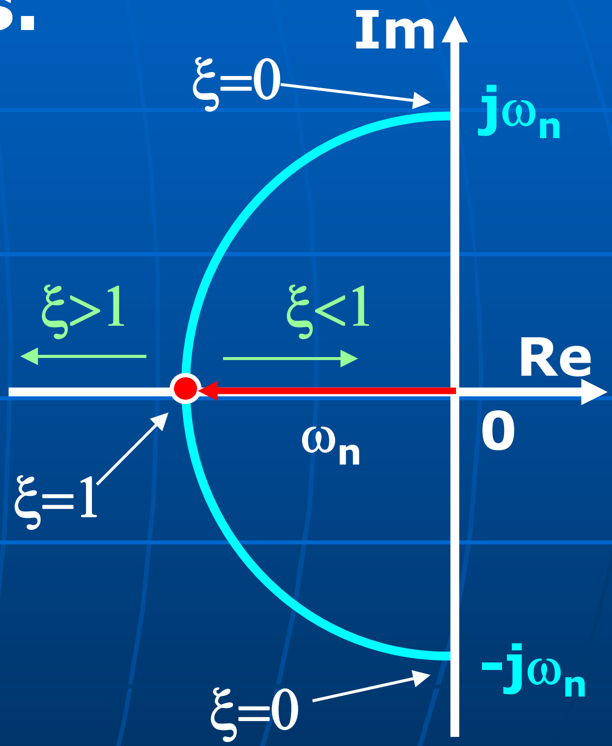
$$c(t) = KR\left[1 - e^{-\omega_n t}\left(1 + \omega_n t\right)\right] \qquad t \geq 0$$

CRITICALLY DAMPED
SECOND ORDER SYSTEMS

Critically damped second order system - $\xi=1$

Fast response with no oscillations.





$$\frac{c(t)}{KR} = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right) \qquad t \geq 0$$

OVERDAMPED
SECOND ORDER SYSTEMS

- **Overdamped System ($\xi>1$) – Step input :**

$$G(s)=\frac{C(s)}{R(s)}=\frac{K\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

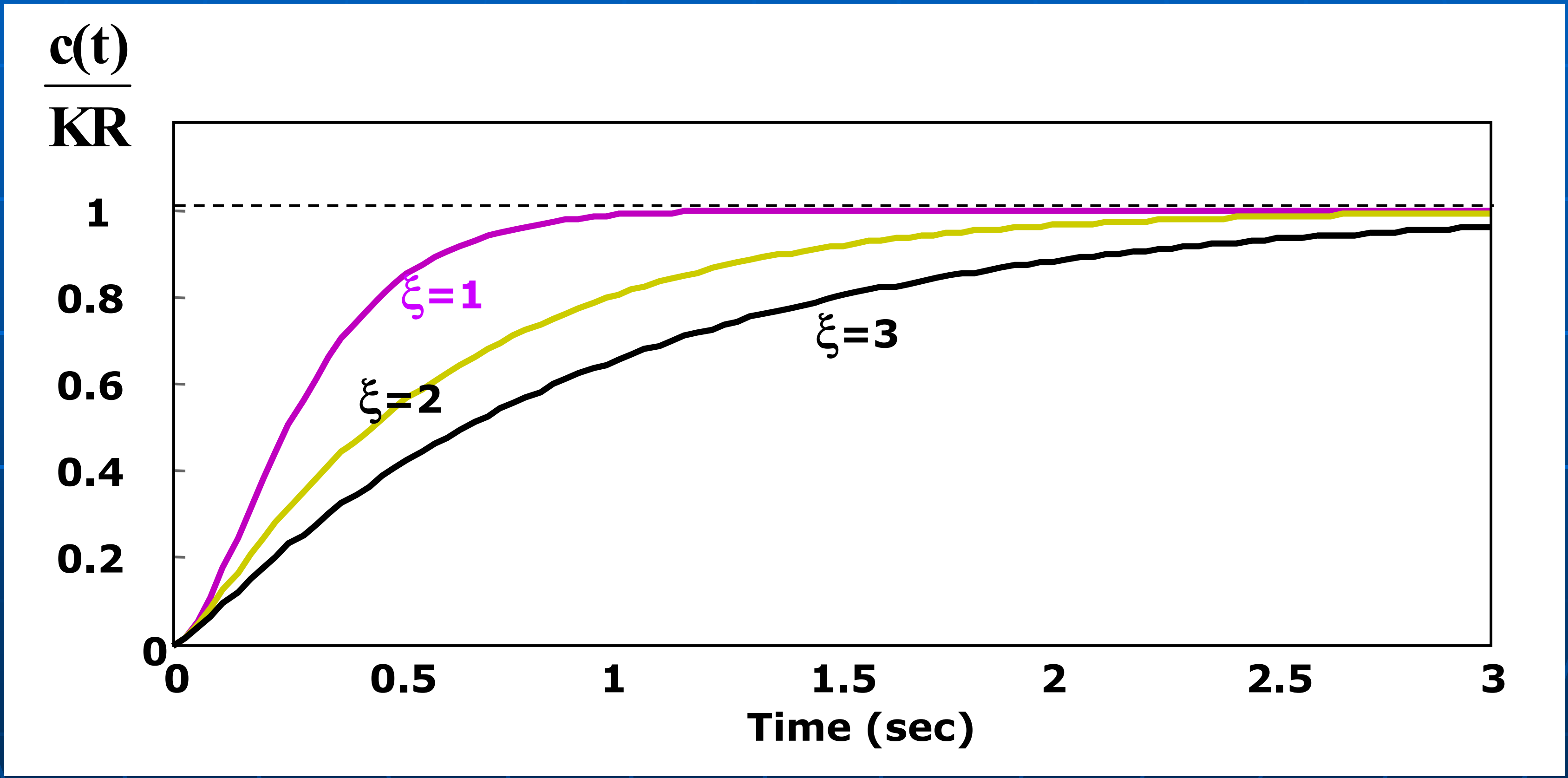
$$R(s)=\frac{R}{s}$$

$$G(s)=\frac{K\omega_n^2}{\left(s+\xi\omega_n+\omega_d\right)\left(s+\xi\omega_n-\omega_d\right)}$$

$$c(t)=KR\left[1+\frac{\omega_n}{2\sqrt{\xi^2-1}}\left(\frac{e^{-s_1 t}}{s_1}-\frac{e^{-s_2 t}}{s_2}\right)\right] \qquad t\geq 0$$

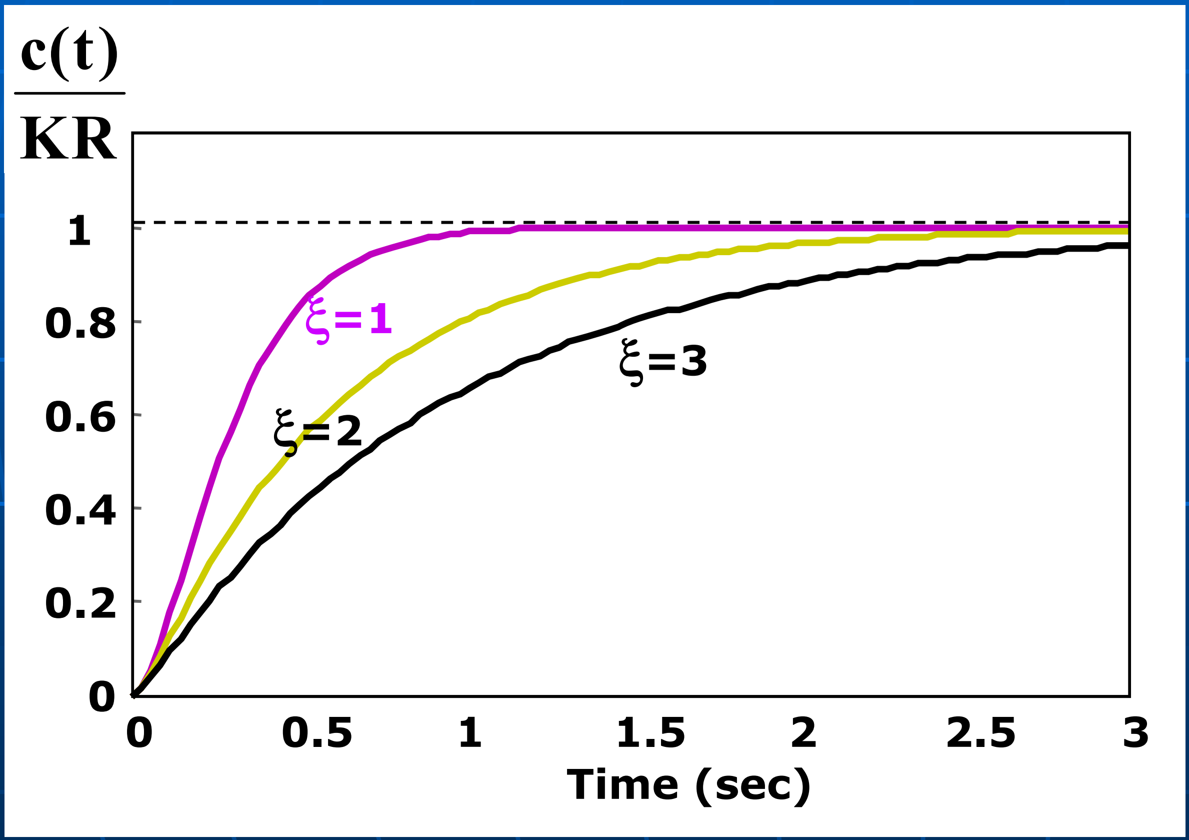
OVERDAMPED
SECOND ORDER SYSTEMS

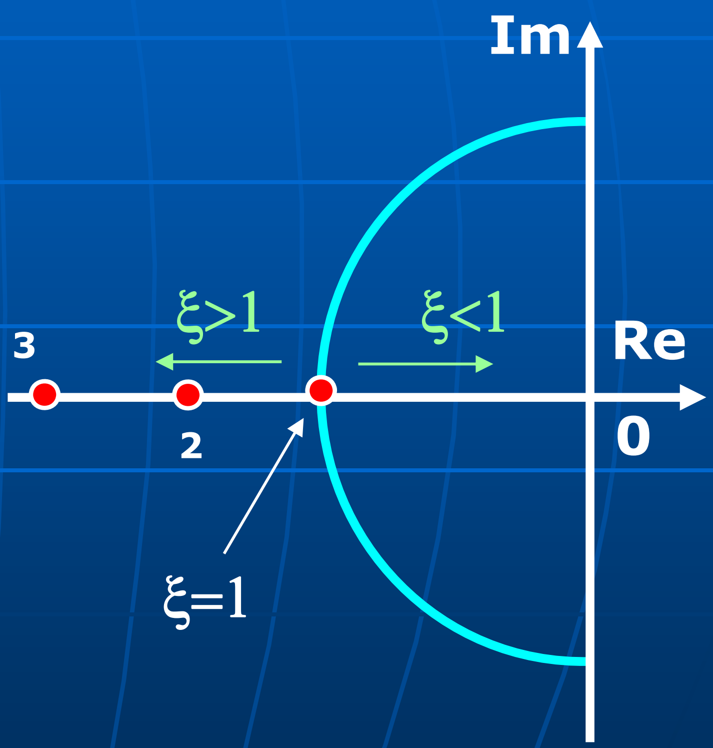
- **Overdamped System (ξ>1) – Step input :**



OVERDAMPED
SECOND ORDER SYSTEMS

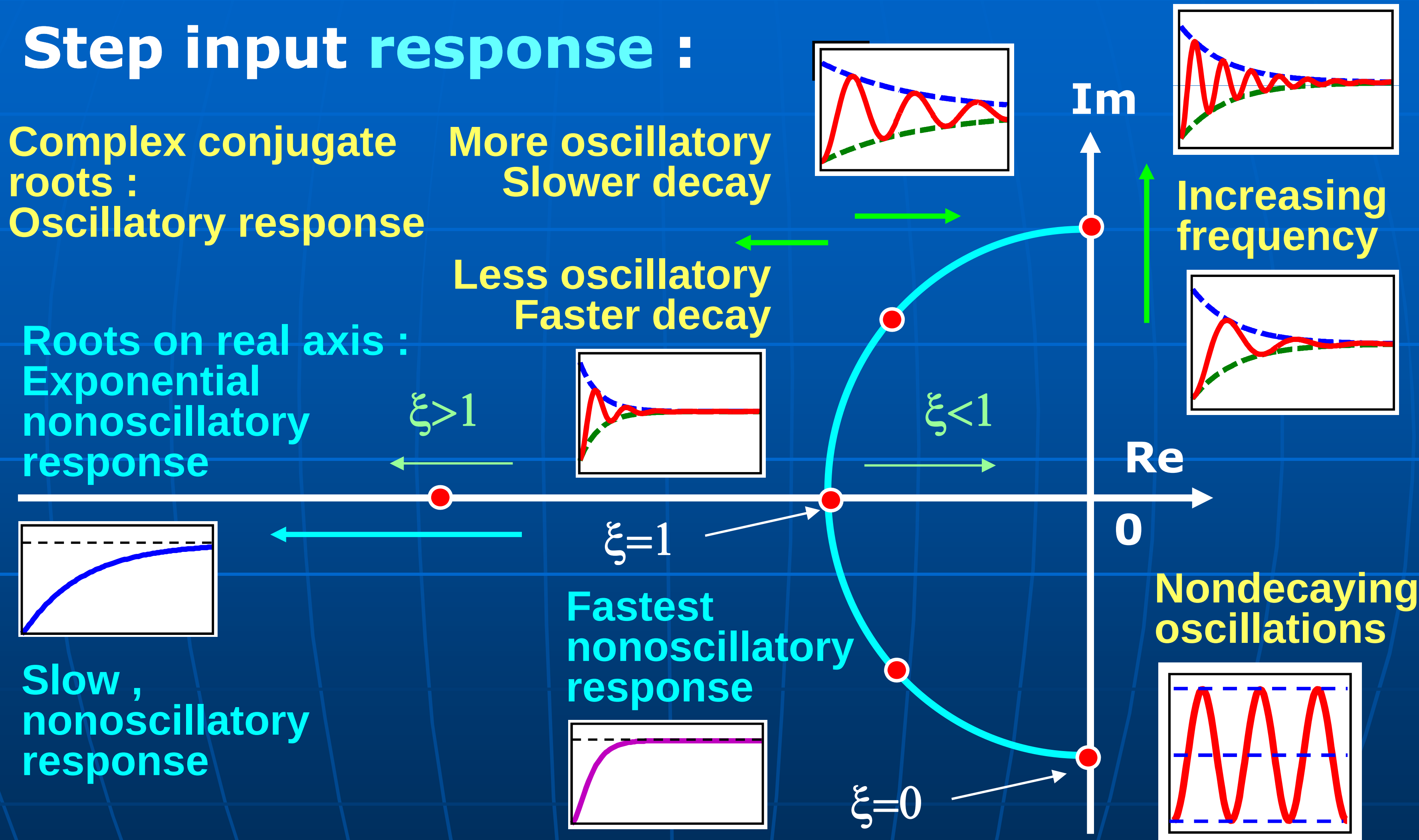
- **Overdamped System (ξ>1) – Step input :**





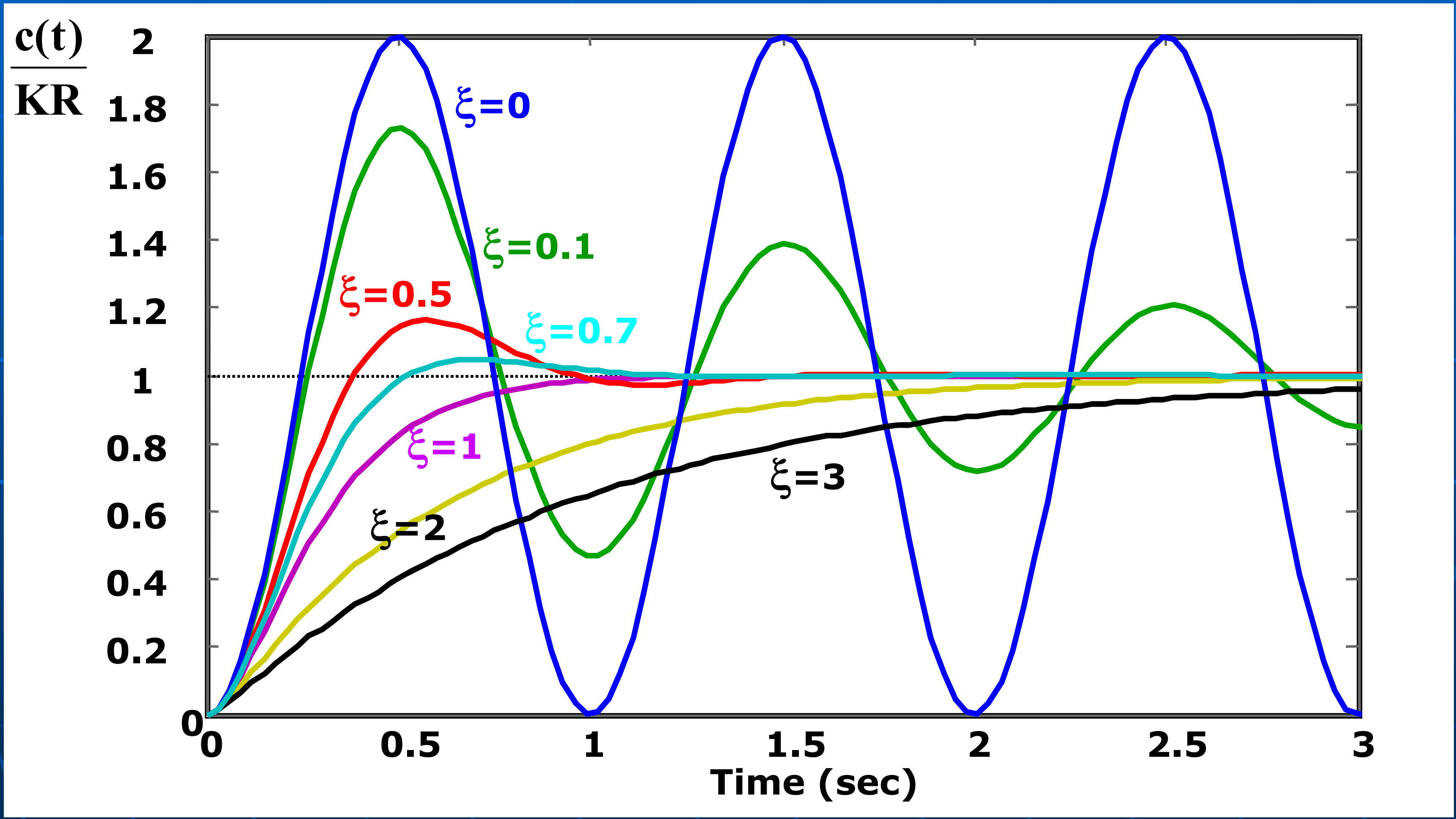
SECOND ORDER SYSTEMS

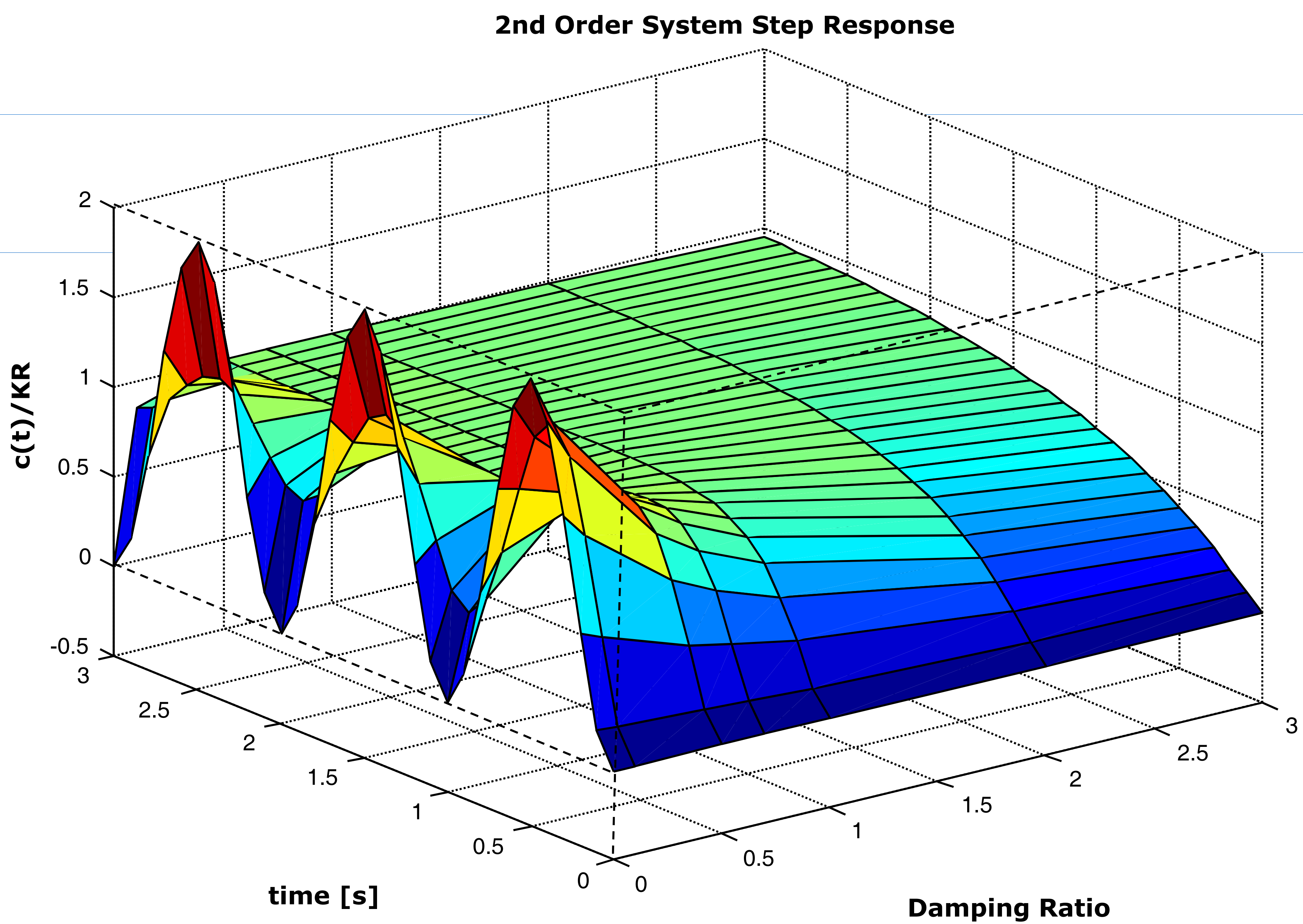
- **Step input response :**



SECOND ORDER SYSTEMS

- **Step input response :**




2nd Order System Step Response
c(t)/KR
2
1.5
1
0.5
0
-0.5
3
2.5
2
1.5
1
0.5
0
time [s]
0
0.5
1
1.5
2
2.5
3
Damping Ratio

TRANSIENT RESPONSE OBJECTIVES

In this chapter :

- Time response of general first and second order systems to standard test inputs will be obtained.
- Specification of transient response as performance characteristics for control systems will be examined.
- The selection of controller parameters to meet transient response specifications will be explored.

Now we are here !

TRANSIENT RESPONSE SPECIFICATIONS

Nise Sect. 4.5, 4.6, Dorf&Bishop Section 5.3, Ogata pp.229-235

- **The performance of a control system is usually specified in terms of its transient response to a unit step input.**
- **To be able to compare different systems under the same conditions, the system is assumed to be stable and initially at rest with zero output.**

TRANSIENT RESPONSE SPECIFICATIONS

- **In specifying the transient response of a control system in relation to its speed of response and relative stability, the most commonly used parameters are :**
  - **Delay Time (t_d),**
  - **Rise Time (t_r),**
  - **Peak Time (t_p),**
  - **Settling Time (t_s),**
  - **Maximum Overshoot (M_p).**

Speed of Response (Delay Time, Rise Time, Peak Time, Settling Time)

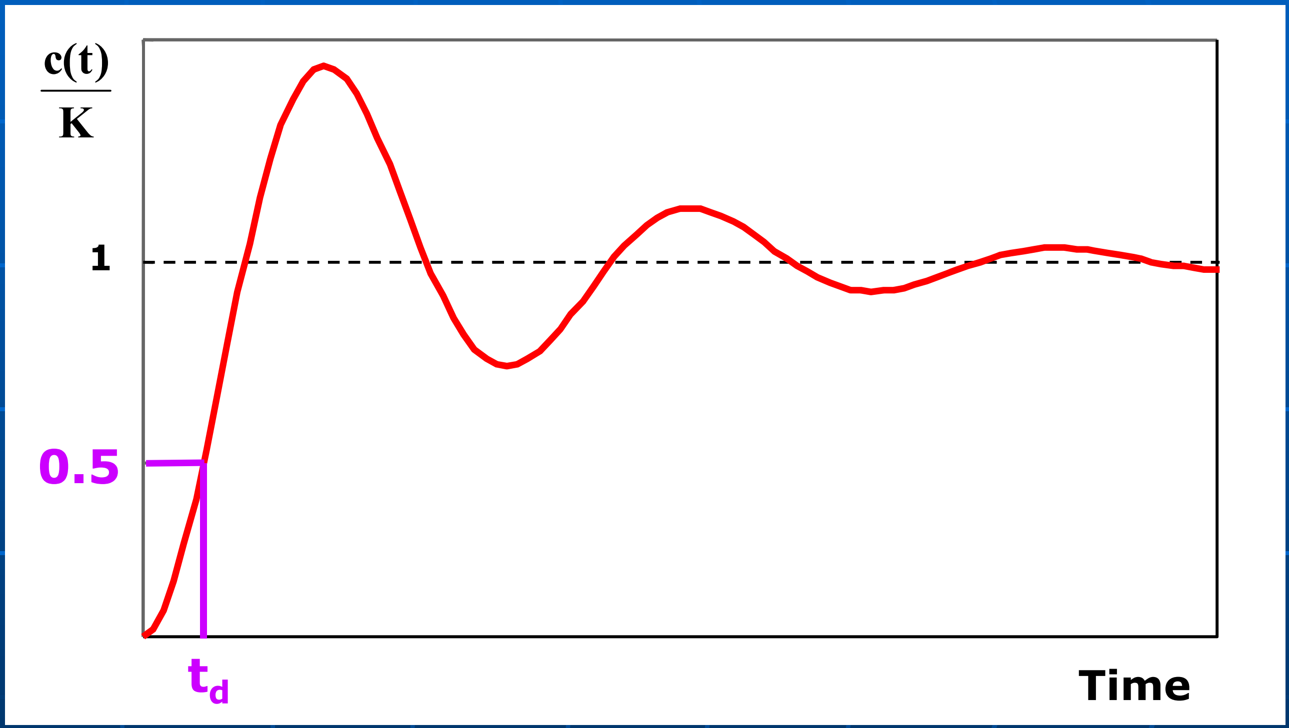
Relative Stability (Maximum Overshoot)

TRANSIENT RESPONSE SPECIFICATIONS

- The **speed of response** is judged by how fast the response reaches the final or steady state value.
- **Relative Stability** is related to how oscillatory the system will be before reaching the steady state.

TRANSIENT RESPONSE SPECIFICATIONS

- **Delay Time (t_d)**
- **Time required for the <u>unit step response</u> to reach half of the final value for the <u>first time</u>.**

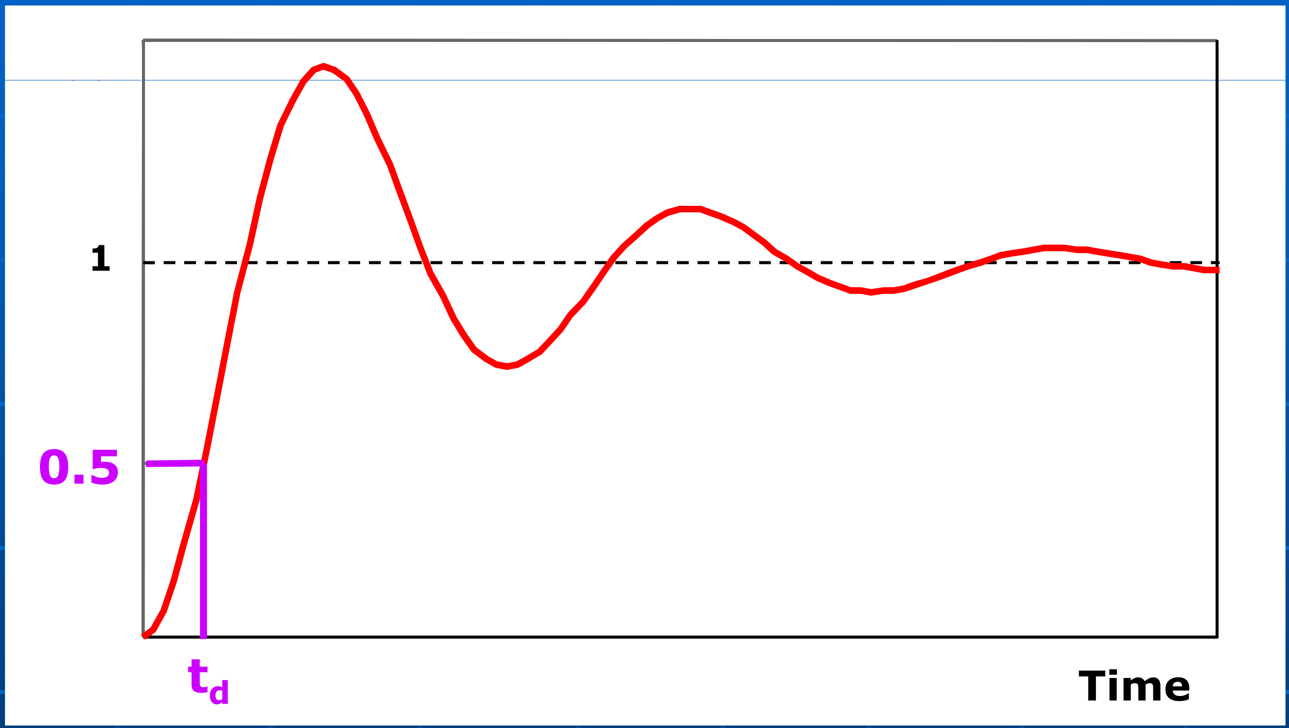


Why do we need to say that ?

TRANSIENT RESPONSE SPECIFICATIONS

- **Delay Time (t_d)**

$$t_d \cong \frac{1+0.7\xi}{\omega_n}$$



- **A better approximation :**

$$t_d \cong \frac{1+0.125\xi+0.469\xi^2}{\omega_n}$$

TRANSIENT RESPONSE SPECIFICATIONS

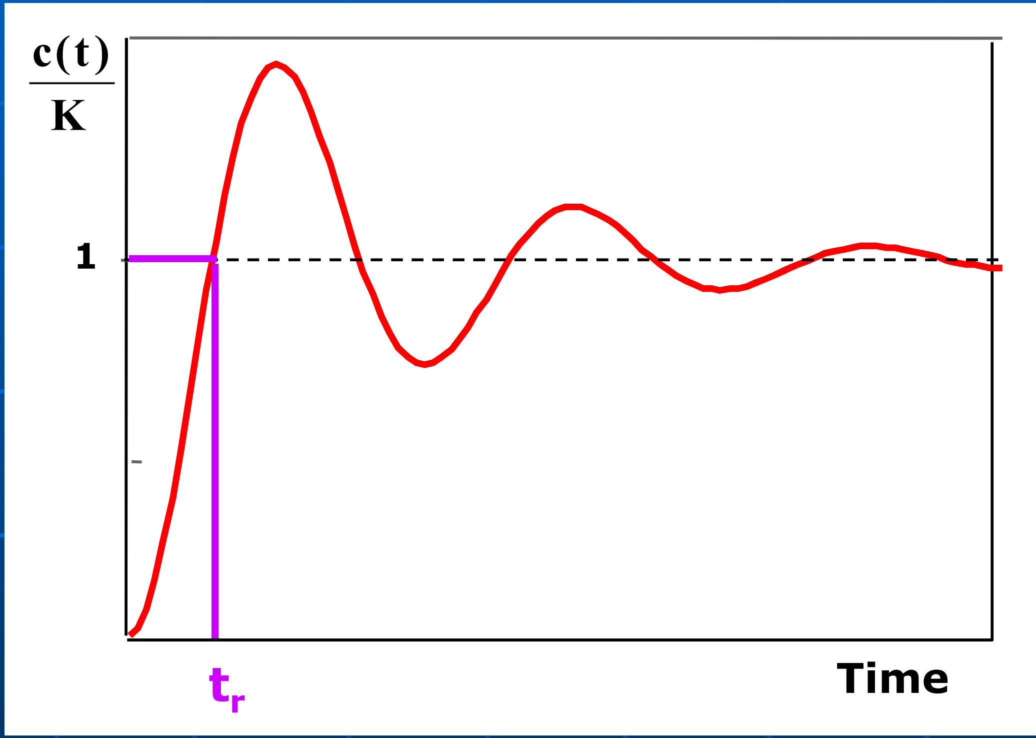
- **Rise Time (t_r)**

  For underdamped systems :

- **Time required for the unit step response to reach :**

  from 0 to 100 %

  of the final value.



TRANSIENT RESPONSE SPECIFICATIONS

- **Rise Time (t_r)**

To find the value of t_r, set :

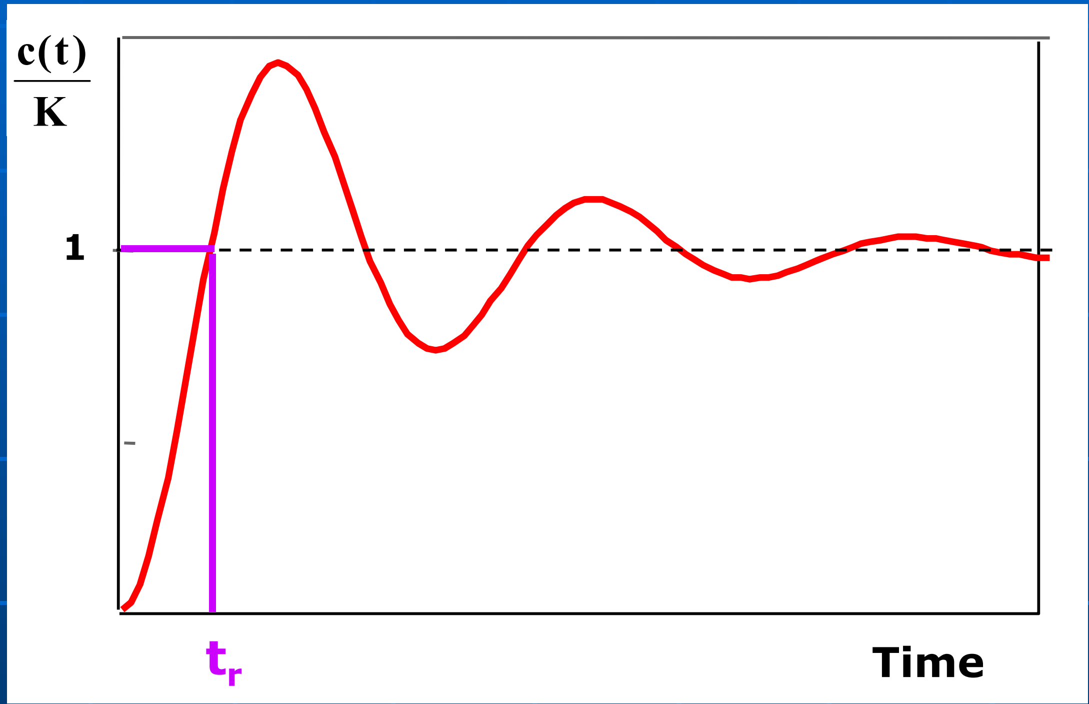
$$\frac{c(t=t_r)}{K}=1$$

which gives

$$t_r=\frac{\pi-\beta}{\omega_d}$$

where

$$\beta=\tan^{-1}\frac{\omega_d}{\xi\omega_n}$$

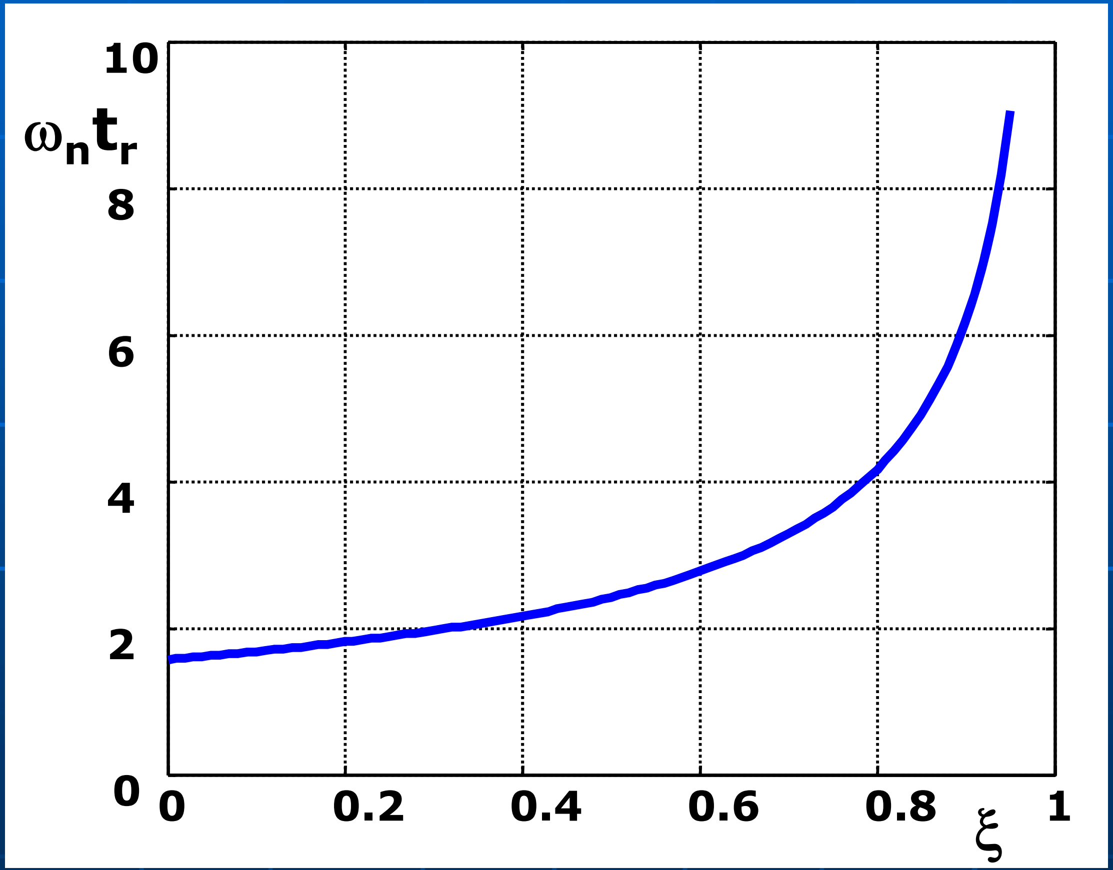


TRANSIENT RESPONSE SPECIFICATIONS

- **Rise Time (t_r)**

Note that to minimize t_r :

1. **ξ must be as small as possible for a given ω_n.**
2. **For a given ξ, ω_n must be as high as possible.**



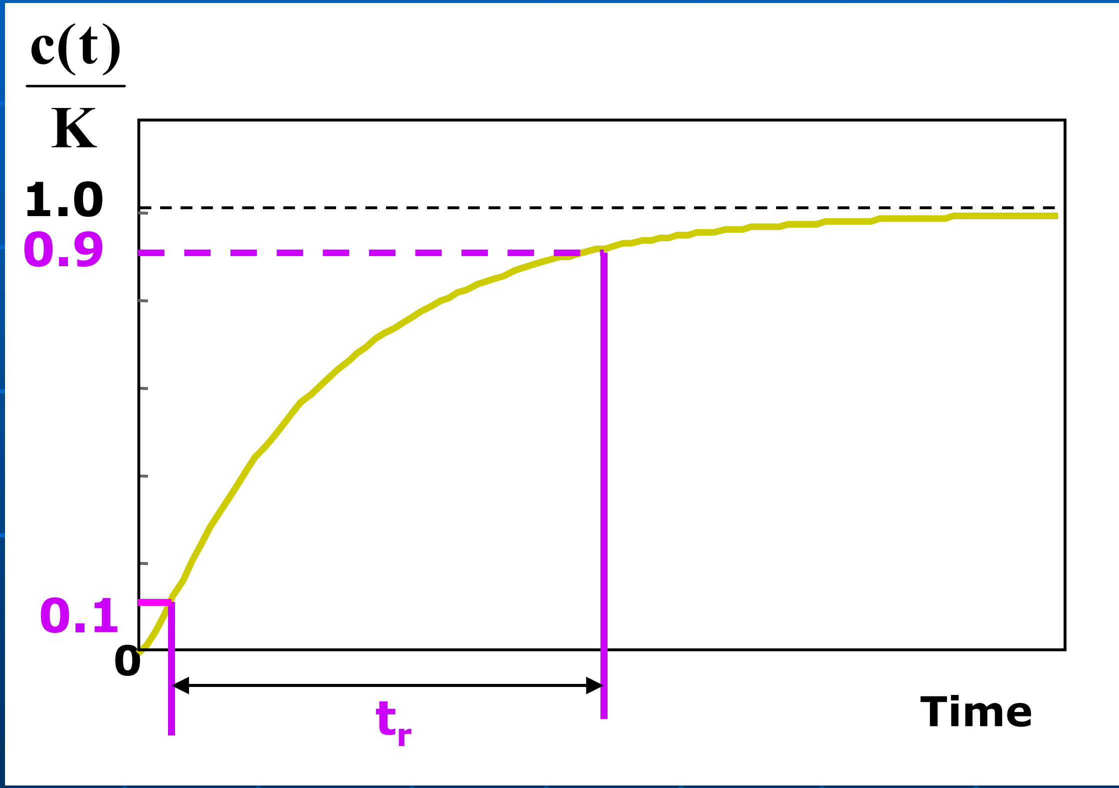
TRANSIENT RESPONSE SPECIFICATIONS

- **Rise Time (t_r)**

For overdamped systems :

Time required for the unit step response to reach :

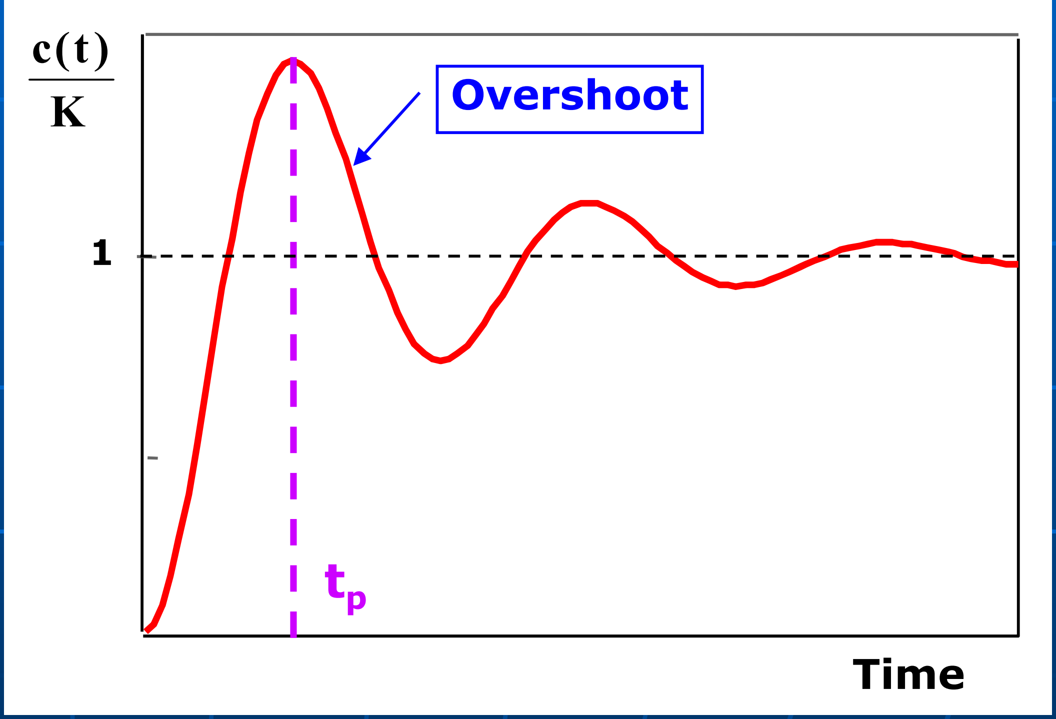
from 10 to 90 % of the final value.



TRANSIENT RESPONSE SPECIFICATIONS

- **Peak Time (t_p)**

  Time required for the unit step response to reach the peak of the first overshoot.



TRANSIENT RESPONSE SPECIFICATIONS

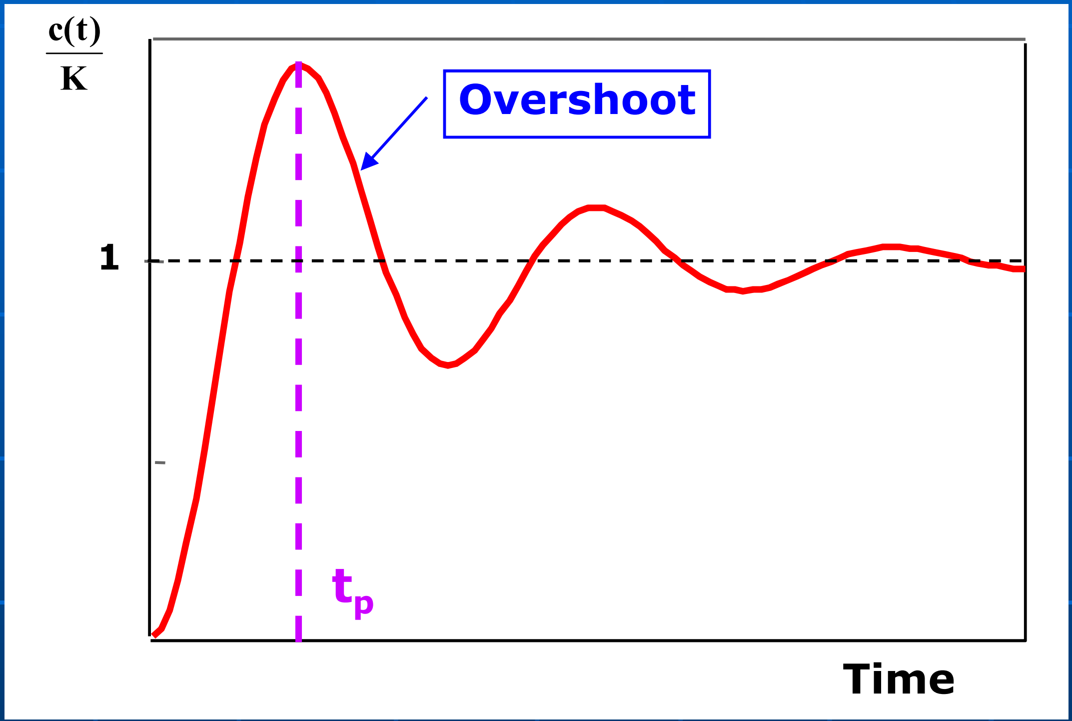
- **Peak Time (t_p)**

To find the value of t_p, set :

$$\left.\frac{dc(t)}{dt}\right|_{(t=t_p)} = 0$$

which gives

$$t_p = \frac{\pi}{\omega_d}$$



$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

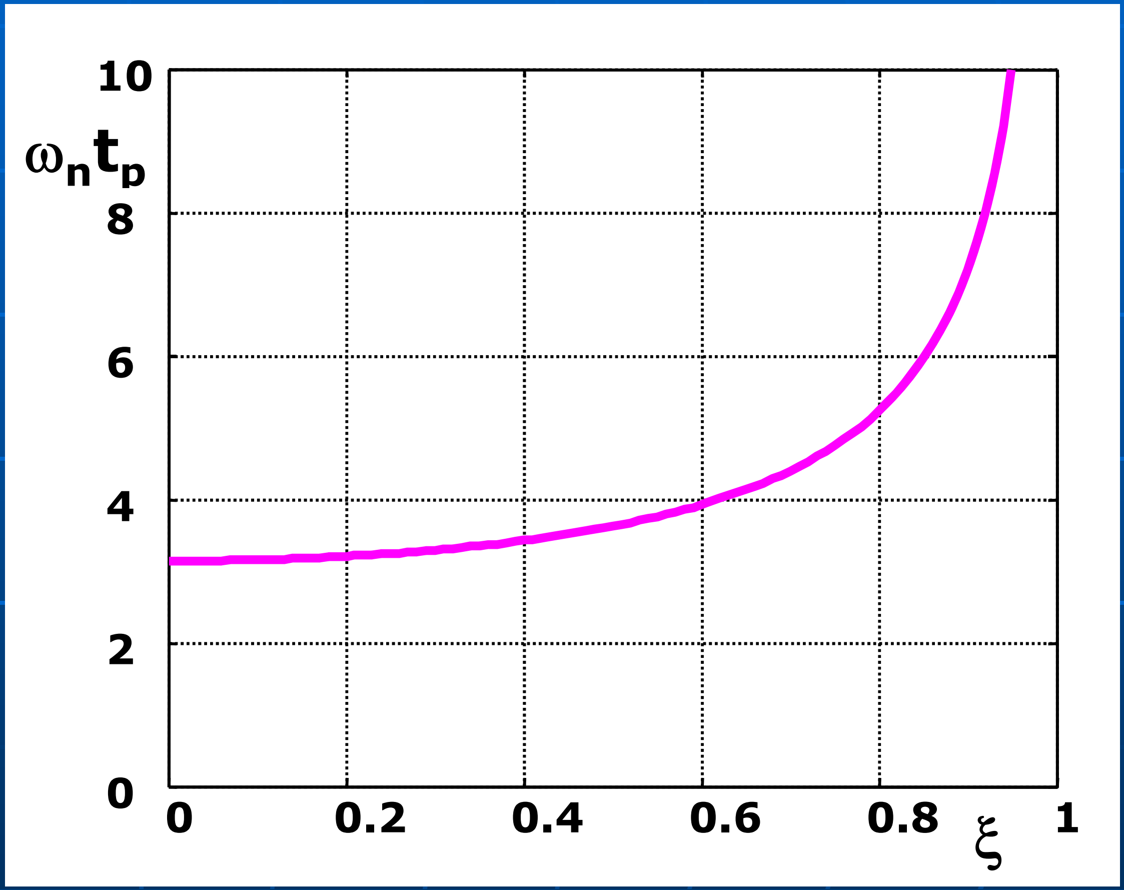
TRANSIENT RESPONSE SPECIFICATIONS

- **Peak Time (t_p)**

Note that to minimize t_p :

1. ξ must be as small as possible for a given ω_n.

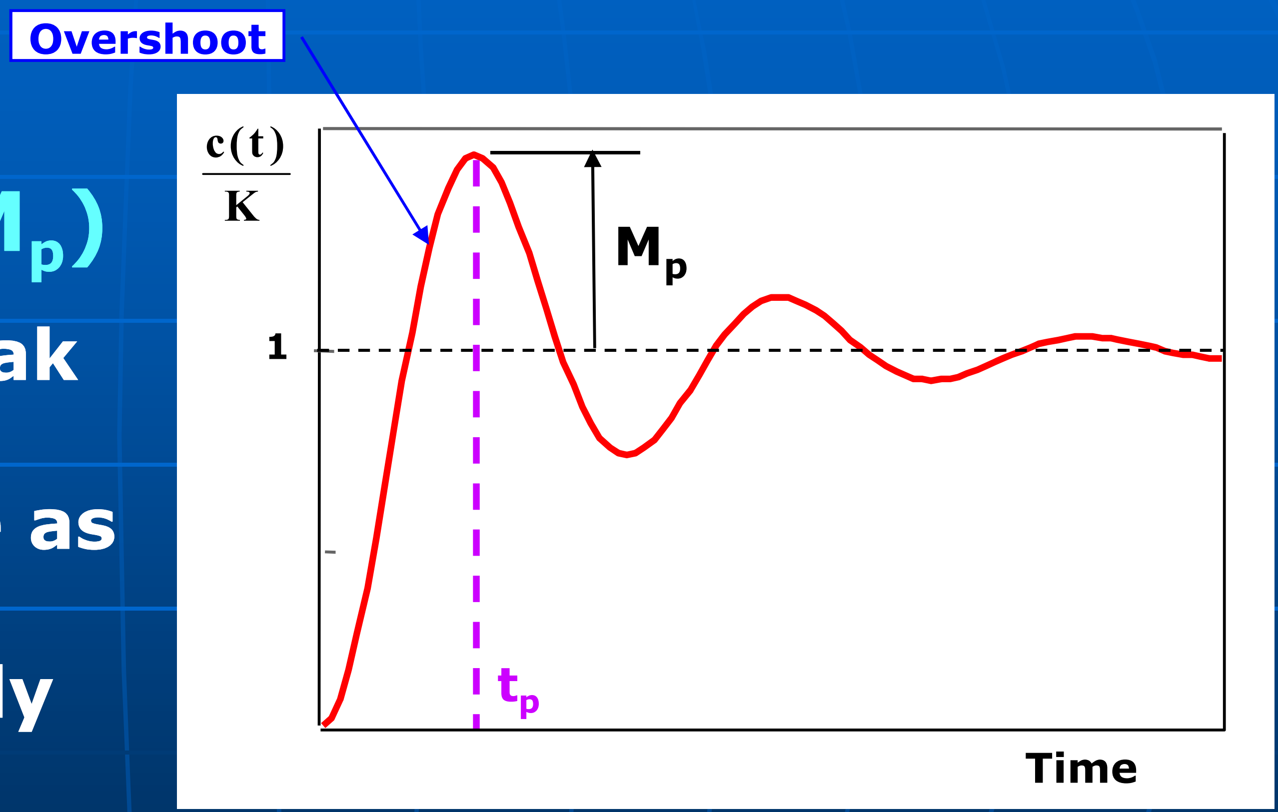
2. For a given ξ, ω_n must be as high as possible.



TRANSIENT RESPONSE SPECIFICATIONS

- **Maximum Overshoot (M_p)**

  The highest peak value of the response curve as measured from the final (steady state) value.



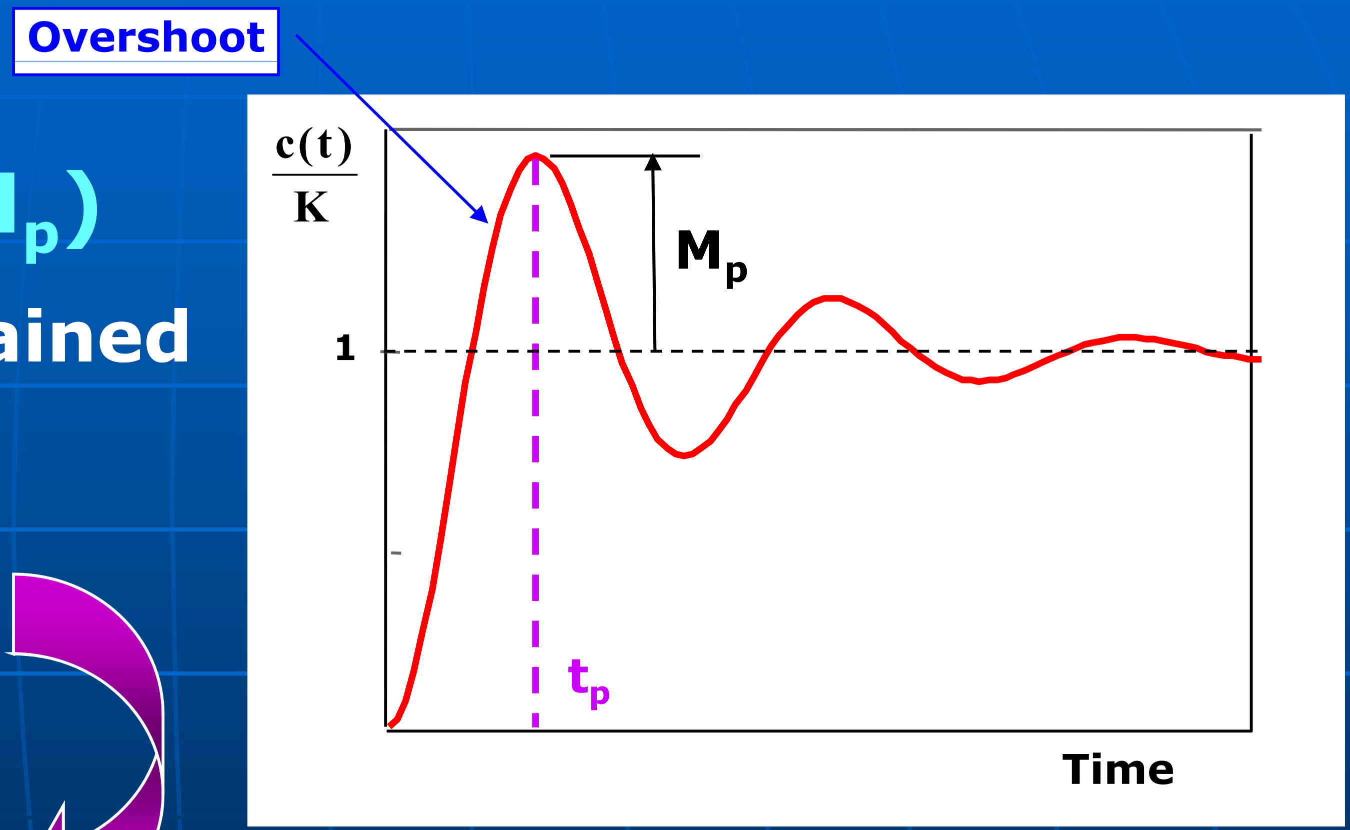
TRANSIENT RESPONSE SPECIFICATIONS

- **Maximum Overshoot (M_p)**

Since M_p is obtained at t_p :

$$M_p = \frac{c(t_p)}{K} - 1$$

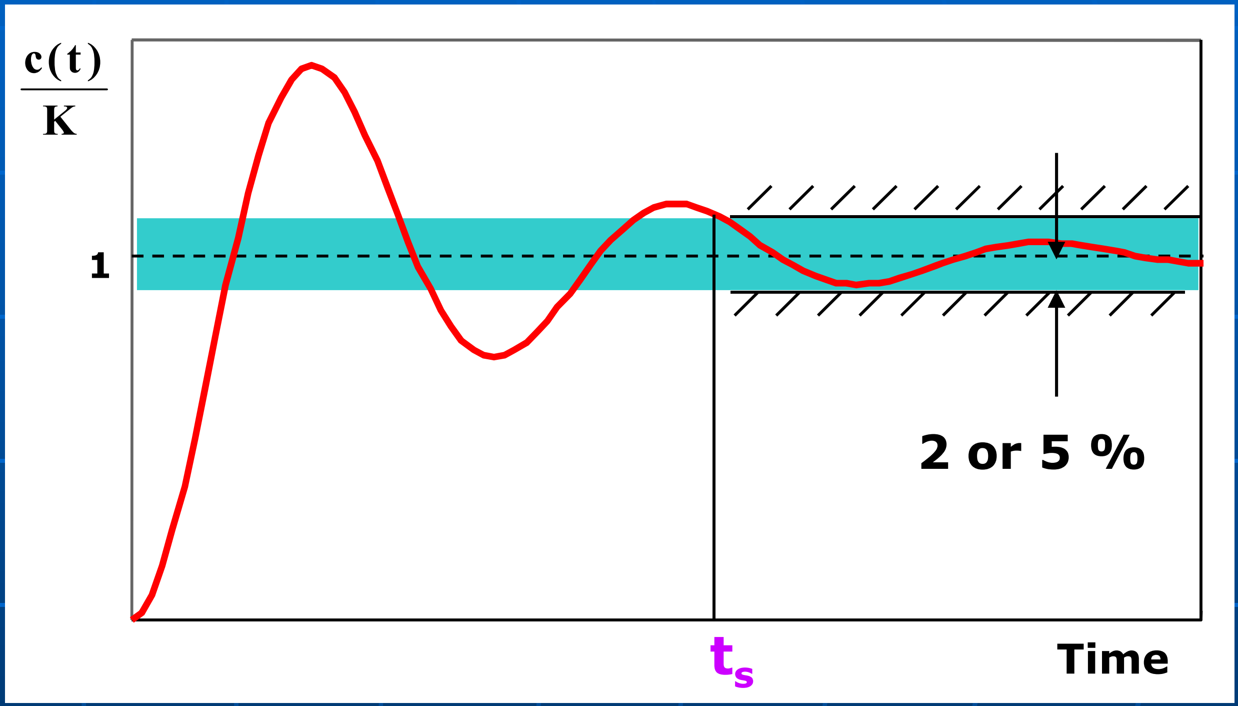
$$M_p = e^{-\pi\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)}$$



TRANSIENT RESPONSE SPECIFICATIONS

- **Settling Time (t_s)**

  Time required for the response to reach and stay within a range (either 2 or 5 %) about the final value.



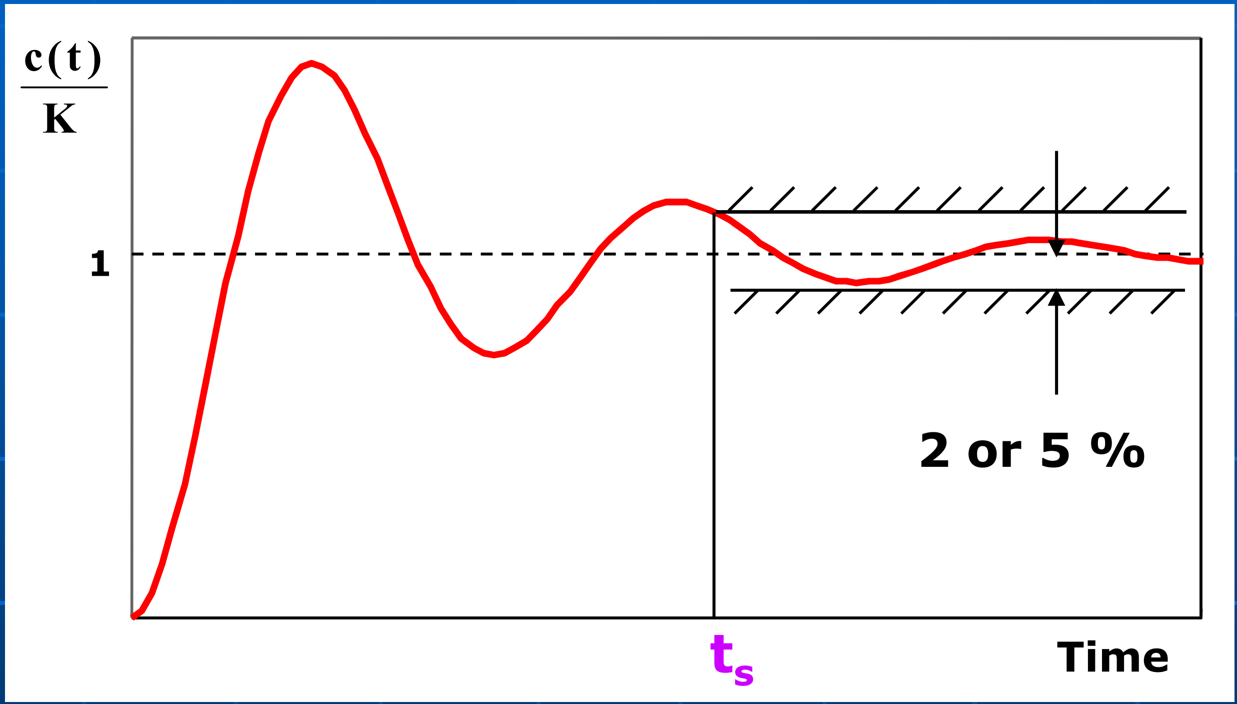
TRANSIENT RESPONSE SPECIFICATIONS

- **Settling Time (t_s)**

Approximate expressions are given by :

$$t_s(2\%) = \frac{4}{\xi\omega_n}$$

$$t_s(5\%) = \frac{3}{\xi\omega_n}$$



TRANSIENT RESPONSE SPECIFICATIONS for Systems With Zeros

- If a more general transfer function where numerator dynamics exist, the general form of the transfer function takes the following form.

$$G(s) = \frac{C(s)}{R(s)} = \frac{K\left(\eta\omega_n s + \omega_n^2\right)}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- In this case, the expressions for the calculation of relevant transient response specifications are somewhat more involved.

TRANSIENT RESPONSE SPECIFICATIONS for Systems With Zeros

- **Rise Time (t_r) (0-100%)**

$$t_r = \frac{\varphi + \frac{\pi}{2}}{\omega_d}$$

where

$$\varphi = \tan^{-1}\left(\frac{\xi - \eta}{\sqrt{1-\xi^2}}\right)$$

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

TRANSIENT RESPONSE SPECIFICATIONS for Systems With Zeros

- **Peak Time (t_p)**

$$t_p = \frac{\varphi + \beta + \frac{\pi}{2}}{\omega_d}$$

where

$$\varphi = \tan^{-1}\left(\frac{\xi - \eta}{\sqrt{1-\xi^2}}\right)$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\beta = \sin^{-1}\sqrt{1-\xi^2}$$

TRANSIENT RESPONSE SPECIFICATIONS for Systems With Zeros

- **Settling Time (t_s)**

$$t_s = \frac{1}{\xi \omega_n} \ln \frac{a_0}{\varepsilon_s}$$

where ε_s is either 0.02 or 0.05 and

$$a_0 = \frac{\sqrt{\eta^2 - 2\xi\eta + 1}}{\sqrt{1 - \xi^2}}$$

TRANSIENT RESPONSE SPECIFICATIONS for Systems With Zeros

- **Maximum Overshoot (M_p)**

$$M_p = a_0 \sin\beta \exp\left[-\frac{\xi}{\sqrt{1-\xi^2}} \left(\varphi + \beta + \frac{\pi}{2} \right) \right]$$

where

$$a_0 = \frac{\sqrt{\eta^2 - 2\xi\eta + 1}}{\sqrt{1-\xi^2}}$$

$$\varphi = \tan^{-1}\left(\frac{\xi - \eta}{\sqrt{1-\xi^2}} \right)$$

$$\beta = \sin^{-1}\sqrt{1-\xi^2}$$

TRANSIENT RESPONSE SPECIFICATIONS for Systems With Zeros

- **Note that when**

$$\eta=0$$

these expressions will reduce to the forms given earlier for the general system with no numerator dynamics.

TRANSIENT RESPONSE OBJECTIVES

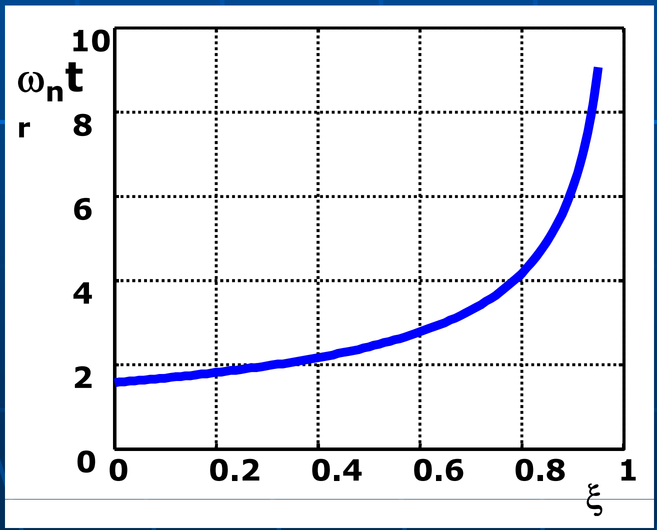
In this chapter :

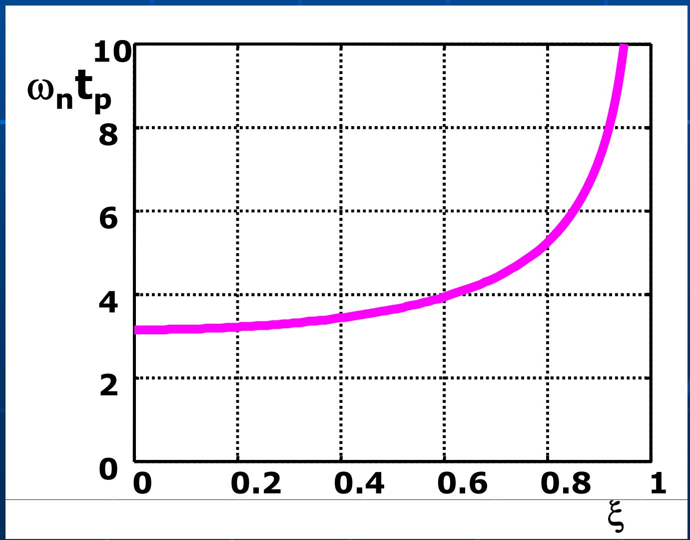
- Time response of general first and second order systems to standard test inputs will be obtained.
- Specification of transient response as performance characteristics for control systems will be examined.
- The selection of controller parameters to meet transient response specifications will be explored.

Now we are here !

TRANSIENT RESPONSE SPECIFICATIONS

- It is noted that for **fast response** which requires **small t_d, t_r,** and **t_p**:
  - ω_n must be as large as possible, and
  - ξ must be as small as possible.





TRANSIENT RESPONSE SPECIFICATIONS

- **On the other hand, for better relative stability (indicated by M_p and t_s), a higher value of ξ is desired.**
- **In view of the conflicting requirements for fast response and better relative stability, a compromise value for ξ is required, i.e. İt is not possible to reduce, say, both rise time and maximum overshoot.**

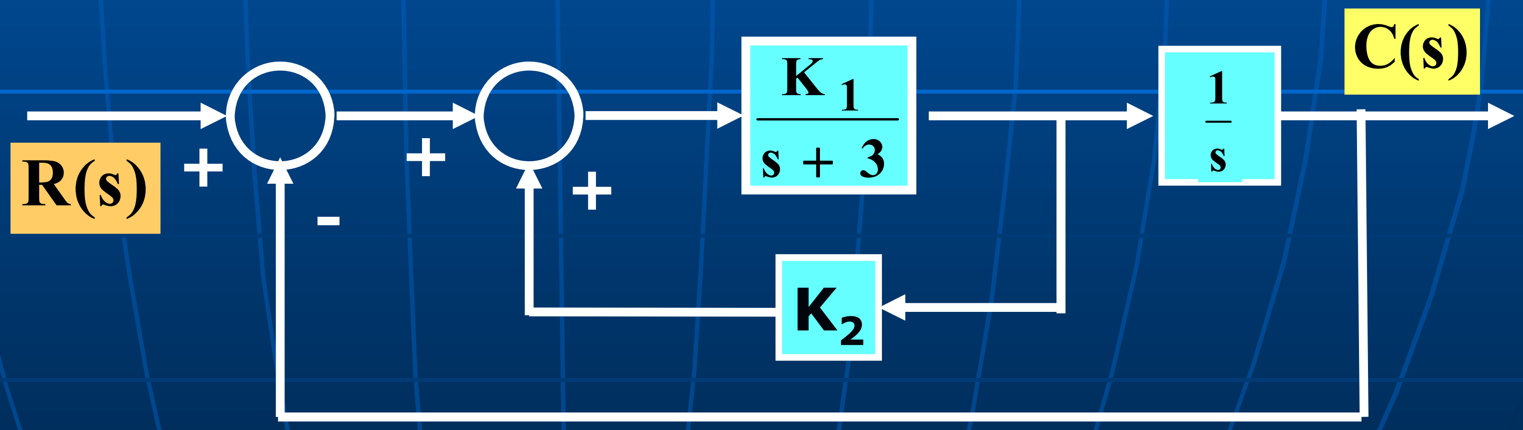
TRANSIENT RESPONSE SPECIFICATIONS

- **Therefore, a value within the range 0.5 to 0.8 (0.7 most common) is commonly used.**
- **For certain applications, such as robotic manipulators, where overshoot is unacceptable, critical damping ($\xi = 1$) is used.**

TRANSIENT RESPONSE SPECIFICATIONS – Example 1a

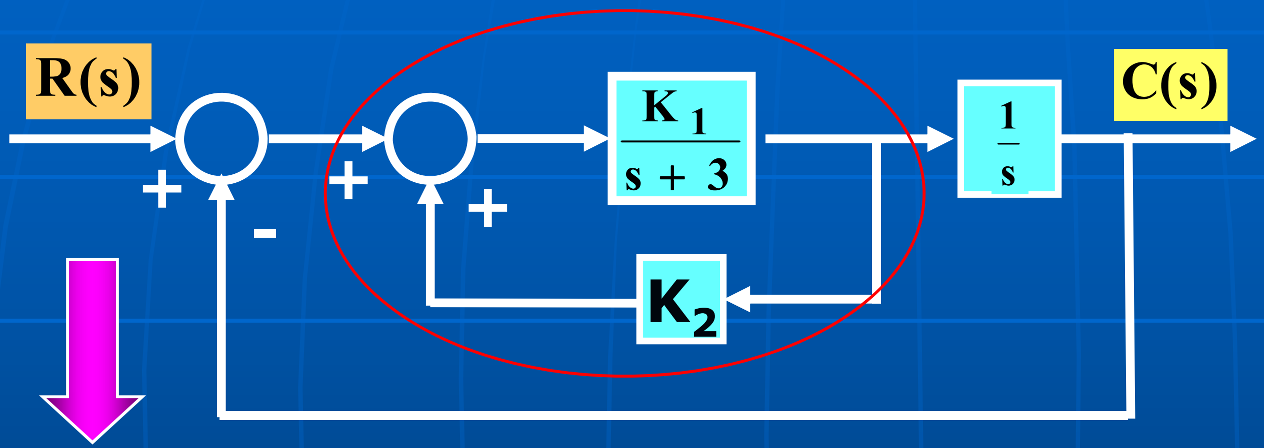
- **Consider a control system represented by the block diagram shown. Determine the values of K_1 and K_2 such that the**
  - **maximum overshoot is <u>to be</u> 6 %, and the**
  - **5 % settling time is to be <u>at most</u> 4 seconds**

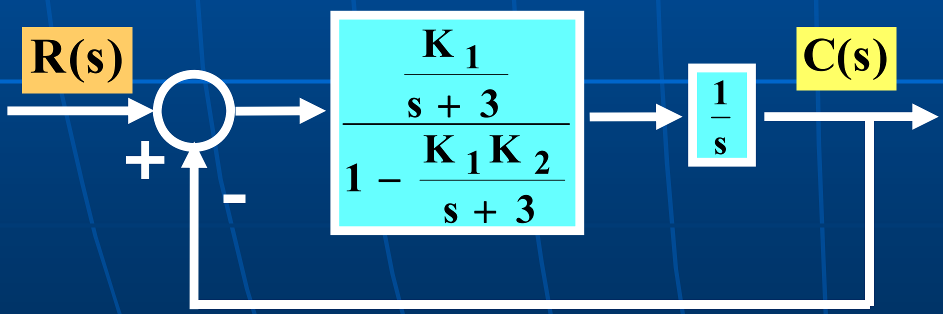
for a unit step input.

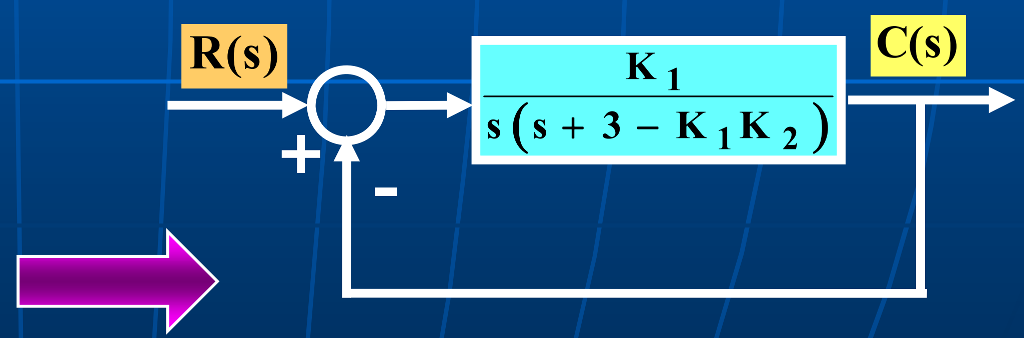


TRANSIENT RESPONSE SPECIFICATIONS – Example 1b

- **First obtain the overall transfer function.**







TRANSIENT RESPONSE SPECIFICATIONS – Example 1c

- First obtain the overall transfer function.

R(s) → C(s)

$$\frac{K_1}{s^2+(3-K_1K_2)s+K_1}$$

$$G(s)=\frac{K\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$G(s)=\frac{K_1}{s^2+(3-K_1K_2)s+K_1}$$

- Comparing with the standard form :

$$\omega_n^2=K_1$$

$$K=1$$

$$2\xi\omega_n=3-K_1K_2$$

$$K_2=\frac{3-2\xi\omega_n}{K_1}$$

TRANSIENT RESPONSE SPECIFICATIONS – Example 1d

- **Maximum overshoot is to be 6 %.**

$$M_p = e^{-\pi\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)} = 0.06$$

$$-\pi\left(\frac{\xi}{\sqrt{1-\xi^2}}\right) = -2.813$$

$$\frac{\xi}{\sqrt{1-\xi^2}} = 0.896$$

$$\xi = 0.67$$

TRANSIENT RESPONSE SPECIFICATIONS – Example 1e

- **5 % settling time is specified to be at most 4 seconds.**

$$t_s\left(5\%\right) \cong \frac{3}{\xi\omega_n} \le 4$$

$$\xi = 0.67$$

$$\omega_n \ge \frac{3}{4\xi} = \frac{3}{4\left(0.67\right)} \cong 1.12\left[\mathrm{rad/s}\right]$$

$$\omega_n \ge 1.12\left[\mathrm{rad/s}\right]$$

TRANSIENT RESPONSE SPECIFICATIONS – Example 1f

- **Find the range for K_1.**

$$\xi = 0.67$$

$$\omega_n \geq 1.12 [rad/s]$$

$$K_1 = \omega_n^2 \geq (1.12)^2 \cong 1.25$$

$$K_1 \geq 1.25$$

TRANSIENT RESPONSE SPECIFICATIONS – Example 1g

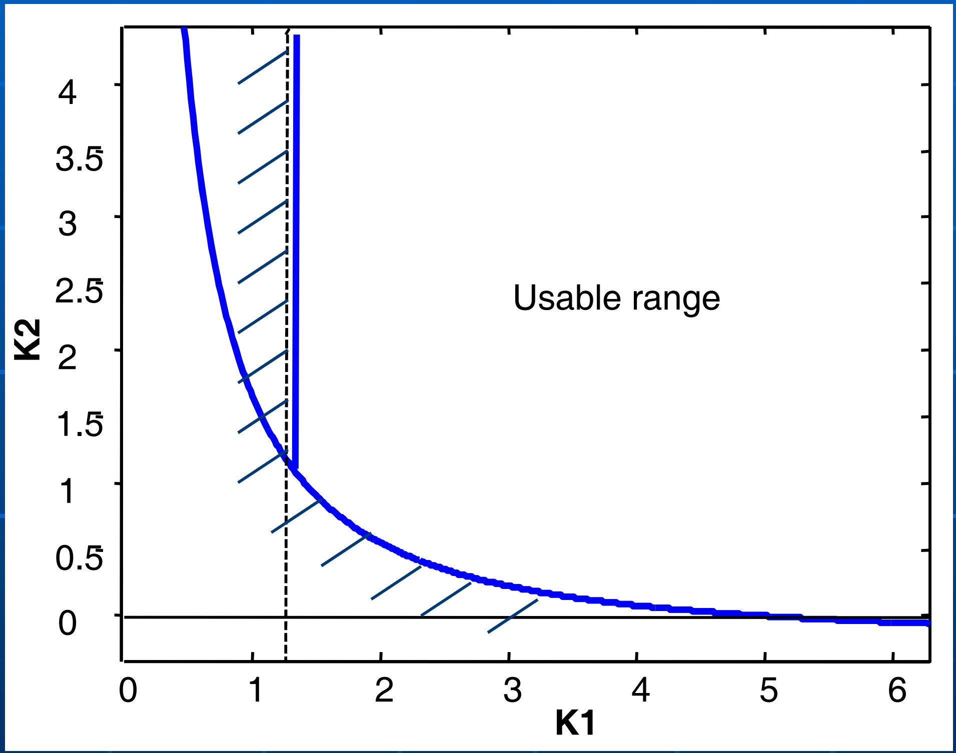
- **Find the range for K_2.**

$\xi = 0.67$ $\omega_n \geq 1.12\,[\text{rad/s}]$

$K_1 \geq 1.25$

$$K_2 = \frac{3 - 2\xi\omega_n}{K_1}$$

$$K_2 = \frac{3 - 2(0.67)\sqrt{K_1}}{K_1}$$



TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS

Nise Sect. 4.7, Dorf & Bishop 5.4,pp. 258, Ogata pp. 240-242

- **The transfer function of an nth order system, with no multiple poles, is given by :**

$$G(s) = \frac{K \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{q} (s + p_j) \prod_{k=1}^{r} \left(s^2 + 2\xi_k \omega_k s + \omega_k^2\right)}$$

where

- **m : number of zeros ($-z_i$),**
- **q : number of distinct poles ($-p_j$),**
- **r : number of complex conjugate pair poles.**

TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS

- As an example, remember the overdamped second order system (ξ>1).

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{K}{\left(s + \xi\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)} \frac{\omega_n^2}{\left(s + \xi\omega_n - \omega_n\sqrt{\xi^2 - 1}\right)}$$

- Notice that it is composed of two first order systems.

$$G(s) = \frac{K_1}{(s + p_1)} \frac{K_2}{(s + p_2)}$$

TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS

- **The unit step time response of the system will be :**

$$y(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} \left[A_k e^{-\xi_k \omega_k t} \cos\left(\omega_k \sqrt{1-\xi_k^2}\, t - \phi_k \right) \right]$$

Exponential decays **Decaying Oscillations**

- **The step response of a higher order system is composed of the step responses of a number of first and second order underdamped systems.**

TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS

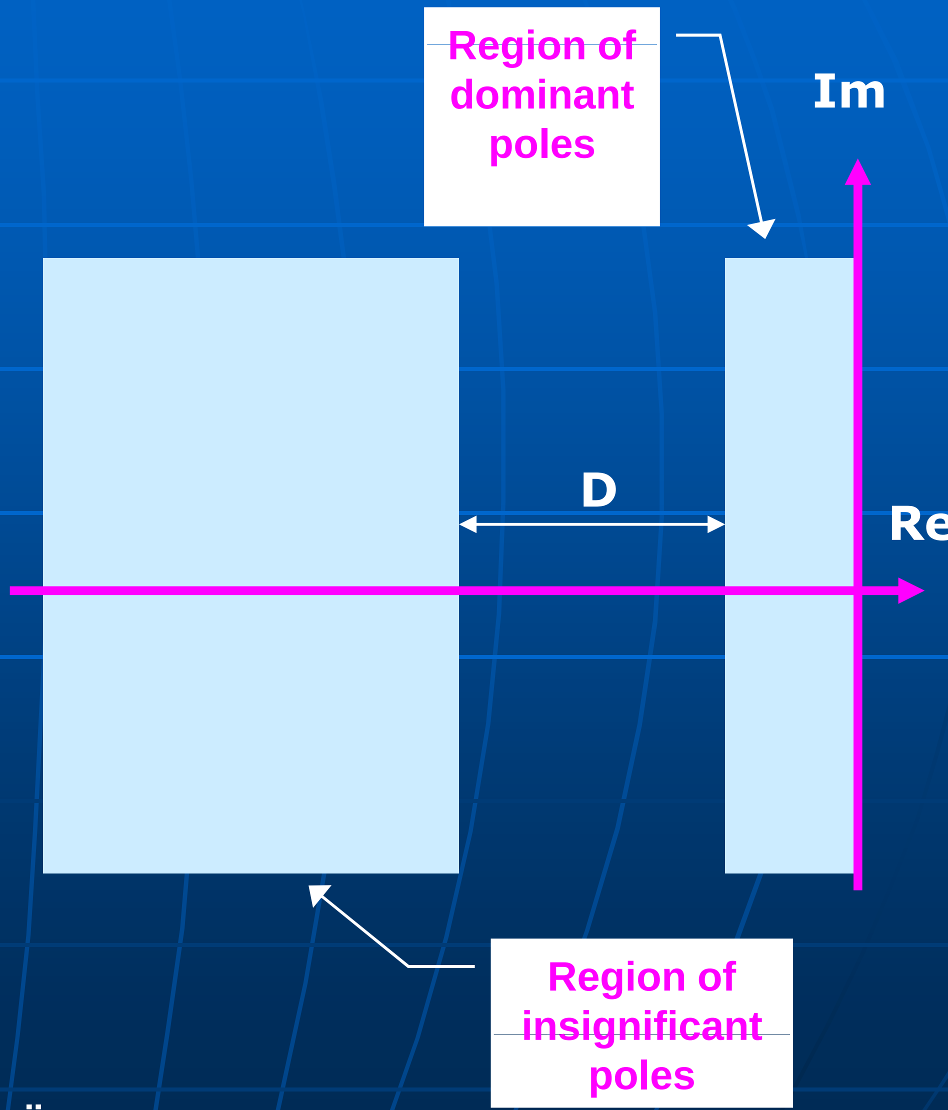
- **Evidently the larger p_j and $\xi_k\omega_k$ are, the faster will be the decay of the response due to these poles.**
- **Coefficients a_j and A_k depend not only on poles but also on zeros of the transfer function.**

$$y(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} \left[A_k e^{-\xi_k \omega_k t} \cos\left(\omega_k \sqrt{1-\xi_k^2}\, t - \phi_k \right) \right]$$

TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS
Dorf&Bishop, pp. 253-258, Ogata pp. 240-242

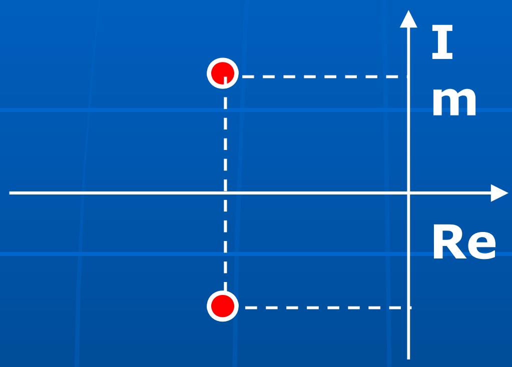
Two points to note :

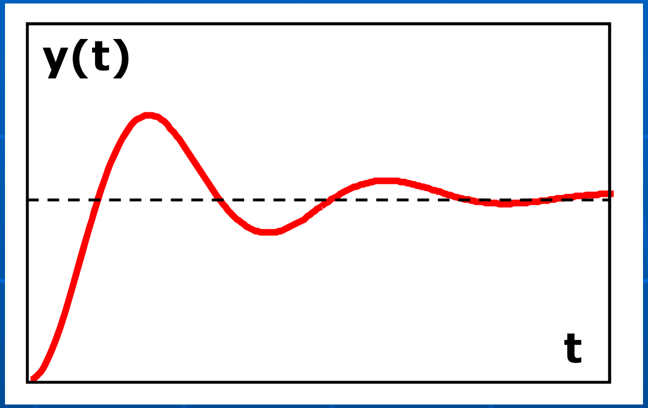
- **If the ratio of the real parts of two poles exceeds five (5) and there are no nearby zeros to cancel those poles, the pole nearest to the imaginary axis will be dominant.**
- **If the value of a zero is very close to that of a pole, then the effect of the pole will be cancelled.**



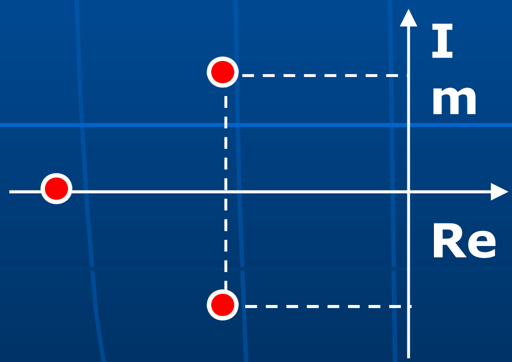
TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS

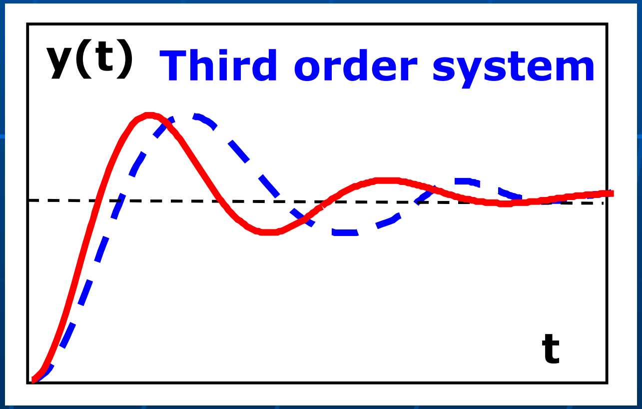
Second order system





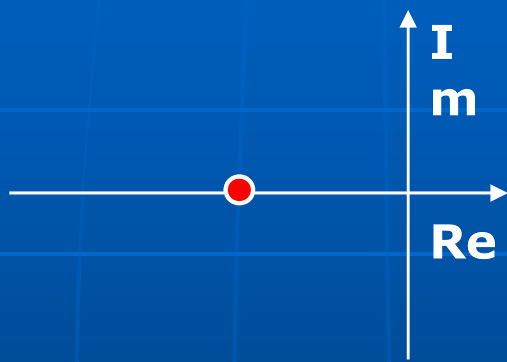
Third order system

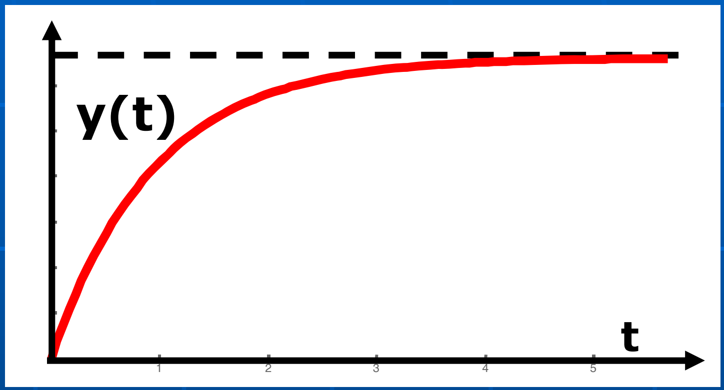




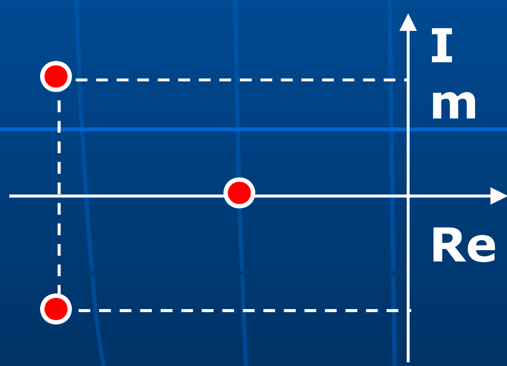
TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS

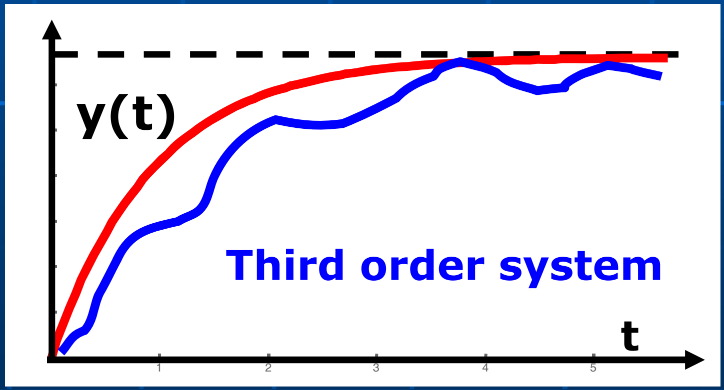
First order system



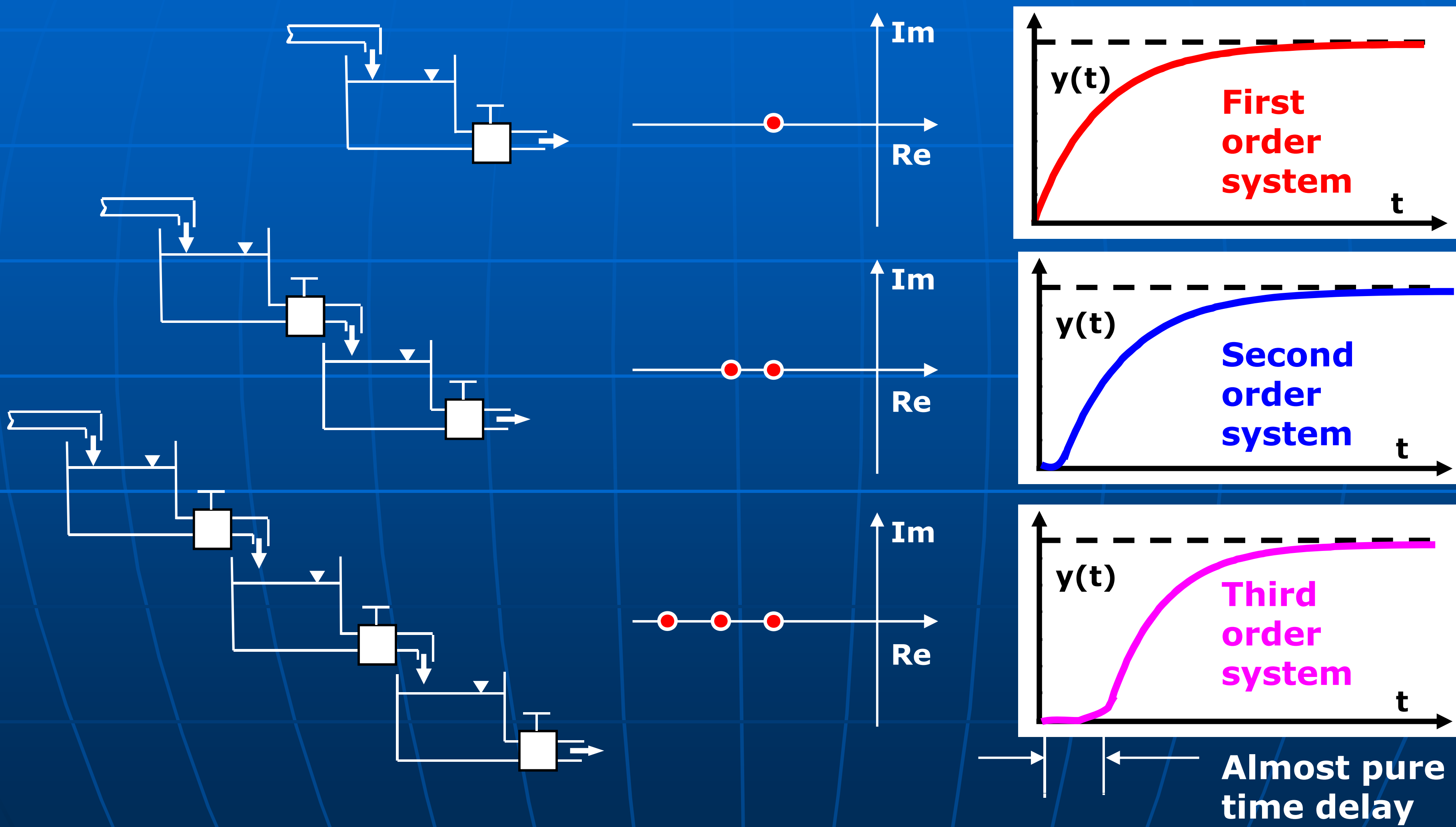


Third order system





TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS



TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS – Example a

- A control system is represented by the following closed loop transfer function.

$$G(s) = \frac{10s + 29}{\left(s^2 + 2s + 2\right)\left(s^2 + 28s + 75\right)}$$

- Estimate the time after which the output y(t) remains within 2% of the final value.

$$G(s) = \frac{10s + 29}{\left(s^2 + 2s + 2\right)\left(s^2 + 28s + 75\right)}$$

TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS

Example b

- **System is fourth order and the poles and zeros are :**

$$s^2 + 2s + 2 = 0 \quad \Rightarrow \quad p_1 = -1 + j \ , \ p_2 = -1 - j$$

$$s^2 + 28s + 75 = 0 \quad \Rightarrow \quad p_3 = -25 \quad , \ p_4 = -3$$

$$10s + 29 = 0 \quad \Rightarrow \quad z = -2.9$$

$$G(s) = \frac{10\left(s + 2.9\right)}{\left(s + 1 - 1j\right)\left(s + 1 + 1j\right)\left(s + 25\right)\left(s + 3\right)}$$

TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS – Example c

$$G(s)=\frac{10(s+2.9)}{(s+1-j)(s+1+j)(s+25)(s+3)}$$

- As the zero (-2.9) is very near the fourth pole (-3.0), one can cancel these two.
- The third pole (-25.0) is more than five times away from the real part of the complex poles, thus it can be neglected.

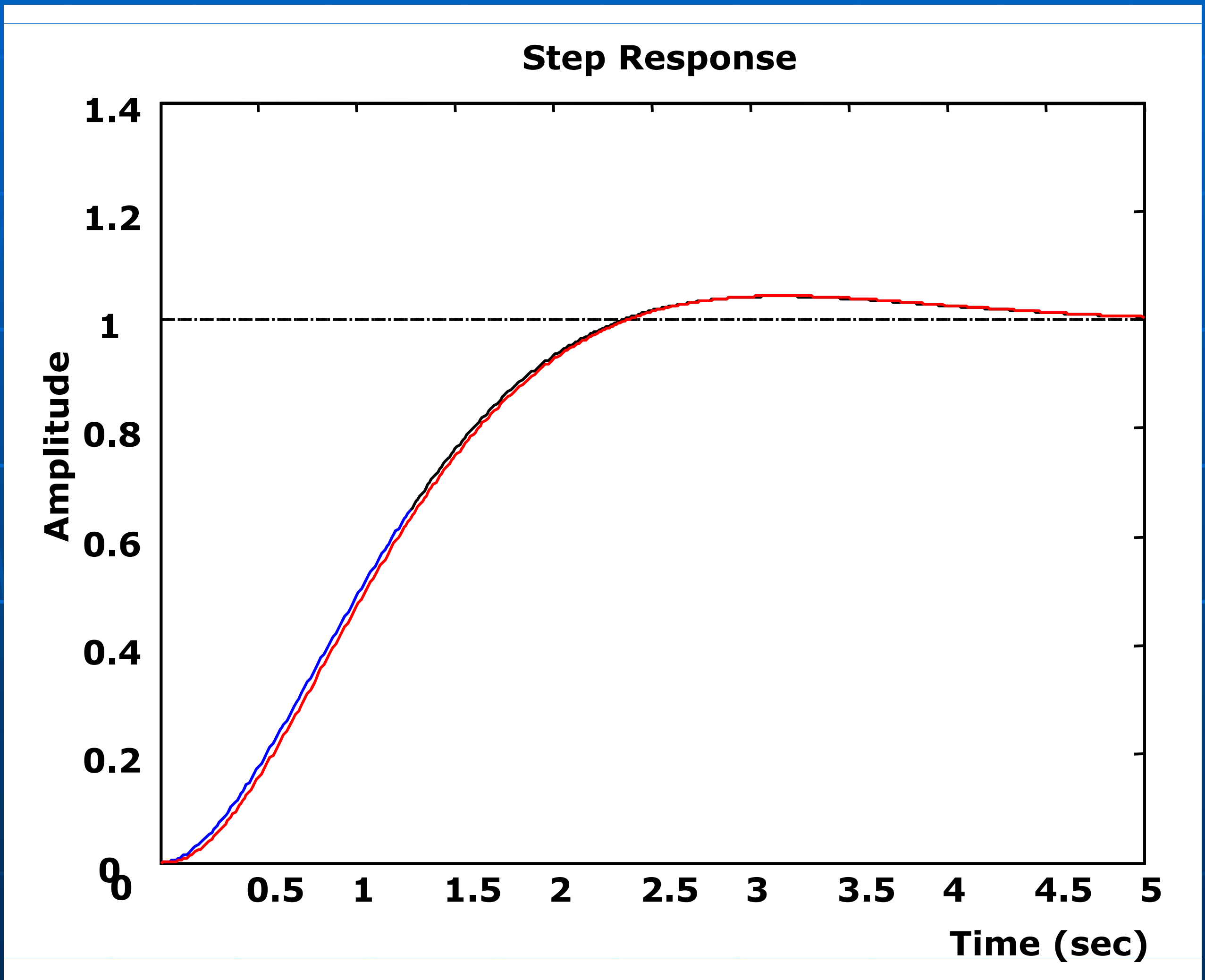
$$G(s)=\frac{10}{(s+1-j)(s+1+j)}=\frac{10}{s^2+2s+2}$$

TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS – Example d

$$G(s) = \frac{10}{s^2 + 2s + 2}$$

- **It is left to the student to show that the 2 % settling time of the simplified system is around 4 seconds.**

TRANSIENT RESPONSE OF HIGHER ORDER SYSTEMS – Example e



- The unit step responses of the fourth order and the simplified second order system are almost indistinguishable.

MIDTERM I

	2005	2006	2007	2008
Average	71.6	63.3	66.0	56.5
St. Dev.	18.4	24.2	22.1	20.5
Max.	99	100	100	99
Min.	10	12	01	12