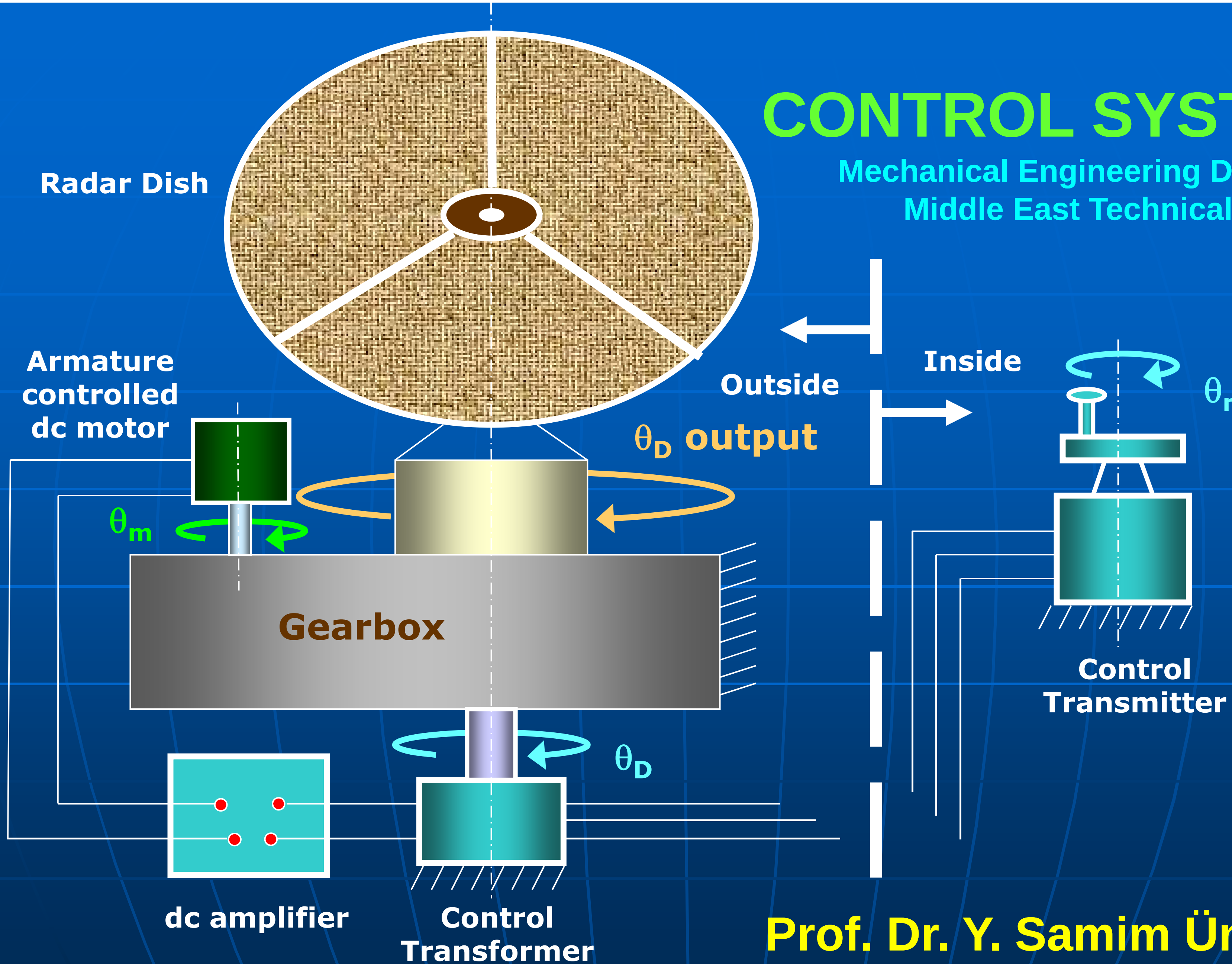
ME 304

CONTROL SYSTEMS

Mechanical Engineering Department,
Middle East Technical University



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CH II

COURSE OUTLINE

I. INTRODUCTION & BASIC CONCEPTS

II. MODELING DYNAMIC SYSTEMS

III. CONTROL SYSTEM COMPONENTS

IV. STABILITY

V. TRANSIENT RESPONSE

VI. STEADY STATE RESPONSE

VII. DISTURBANCE REJECTION

VIII. BASIC CONTROL ACTIONS & CONTROLLERS

IX. FREQUENCY RESPONSE ANALYSIS

X. SENSITIVITY ANALYSIS

XI. ROOT LOCUS ANALYSIS

MODELING DYNAMIC SYSTEMS

OBJECTIVES

Completed !

- **Deriving input-output relations of linear time invariant systems (mechanical, fluid, thermal, and electrical) using elemental and structural equations.**

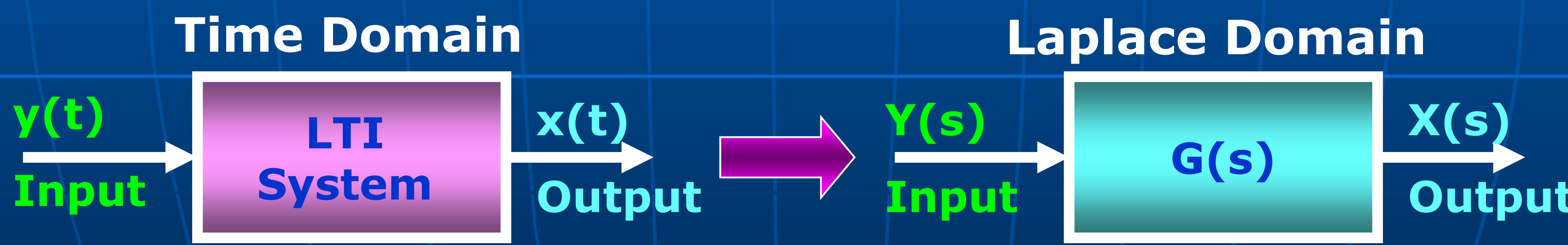
We are here !

- **Obtaining transfer function representation of LTI systems.**
- **Representing control systems with block diagrams.**

TRANSFER FUNCTIONS

- For the analysis of LTI systems, the input-output relationships are usually represented by a **Transfer Function** defined as :

$$G(s) = \left.\frac{\text{Laplace Transform of Output}}{\text{Laplace Transform of Input}}\right|_{\text{zero initial conditions}}$$



- s : Laplace variable.

TRANSFER FUNCTIONS

- **Laplace Transform**
  - Transformation from time to Laplace domain.

$$L\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

  - s : Laplace variable.

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=c\frac{dy}{dt}+ky$$

($m = a_2$, $c = a_1$, $k = a_0$, $c = b_1$, $k = b_0$)

TRANSFER FUNCTIONS

- Let us consider the differential equation of a general LTI system, relating the input and the output, of the general form :

$$a_n\frac{d^nx}{dt^n}+a_{n-1}\frac{d^{n-1}x}{dt^{n-1}}+...+a_1\frac{dx}{dt}+a_0x=b_m\frac{d^my}{dt^m}+b_{m-1}\frac{d^{m-1}y}{dt^{m-1}}+...+b_1\frac{dy}{dt}+b_0y$$

- x : output,
- y : input, and
- n : order of the system and $n \geq m$ for physically realizable systems.

TRANSFER FUNCTIONS

- The transfer function will then be given as :

$$G(s) = \left. \frac{\text{Laplace Transform of Output}}{\text{Laplace Transform of Input}} \right|_{\text{zero initial conditions}}$$

$$G(s) = \left. \frac{L\{x(t)\}}{L\{y(t)\}} \right|_{\text{zero initial conditions}} = \frac{X(s)}{Y(s)}$$

TRANSFER FUNCTIONS

- **Remember that :**

Initial Conditions

$$L\left\{\frac{d^n}{dt^n}f(t)\right\} = s^n F(s) - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}f(0)$$

- **Thus for n=1 and 2 :**

$$L\left(\frac{d}{dt}f(t)\right) = sF(s) - f(0)$$

$$L\left(\frac{d^2}{dt^2}f(t)\right) = s^2F(s) - sf(0) - \dot{f}(0)$$

- **With zero initial conditions :**

$$L\left(\frac{d}{dt}f(t)\right) = sF(s)$$

$$L\left(\frac{d^2}{dt^2}f(t)\right) = s^2F(s)$$

TRANSFER FUNCTIONS

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 x = b_m \frac{d^m y}{dt^m} + b_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \ldots + b_1 \frac{dy}{dt} + b_0 y$$

- **Taking the Laplace transform of every term :**

$$a_n s^n X(s) + a_{n-1} s^{n-1} X(s) + \ldots + a_1 s X(s) + a_0 X(s) =$$
$$b_m s^m Y(s) + b_{m-1} s^{m-1} Y(s) + \ldots + b_1 s Y(s) + b_0 Y(s)$$

taking into common parenthesis and rearranging :

$$\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0\right) X(s) = \left(b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0\right) Y(s)$$

TRANSFER FUNCTIONS

- Finally, the **transfer function** is obtained.

$$G(s)=\frac{X(s)}{Y(s)}=\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}=\frac{N(s)}{D(s)}$$

- n : order of the system ($n \geq m$),
- D(s) : characteristic polynomial.
- Characteristic equation : D(s)=0

$$D(s)=a_n s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0=0$$

$$G(s)=\frac{X(s)}{Y(s)}=\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}=\frac{N(s)}{D(s)}$$

TRANSFER FUNCTIONS

- The roots of the numerator polynomial, i.e.

$$N(s)=b_m s^m + b_{m-1}s^{m-1}+\ldots+b_1 s+b_0=0$$

are called the zeroes of the system.

- The roots of the denominator polynomial

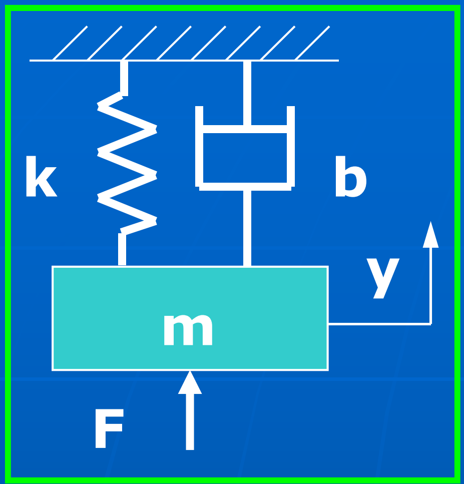
$$D(s)=a_n s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0=0$$

are called the poles of the system.

TRANSFER FUNCTIONS

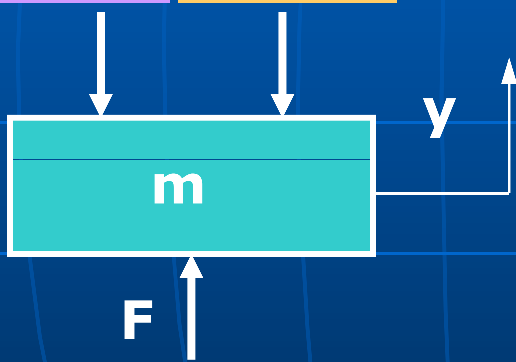
Note that :

- Transfer function is a property of a system and is independent of the input.
- Transfer functions of physically different systems may be identical.
- If the transfer function of a system is known, its dynamic response to various different inputs can be studied.
- Transfer function of a system can be experimentally determined by applying known inputs and examining the resulting input-output relationships.



$f_k = ky$ $f_b = b\dot{y}$

$f_n = m\ddot{y}$



$$m\ddot{y} + b\dot{y} + ky = F$$

EXAMPLE 1

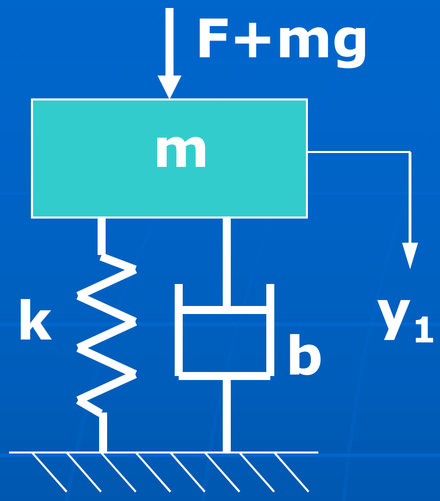
- Take the Laplace transform (with zero initial conditions) of every term.

$L\{m\ddot{y}\} = ms^2Y(s)$ $L\{b\dot{y}\} = bsY(s)$

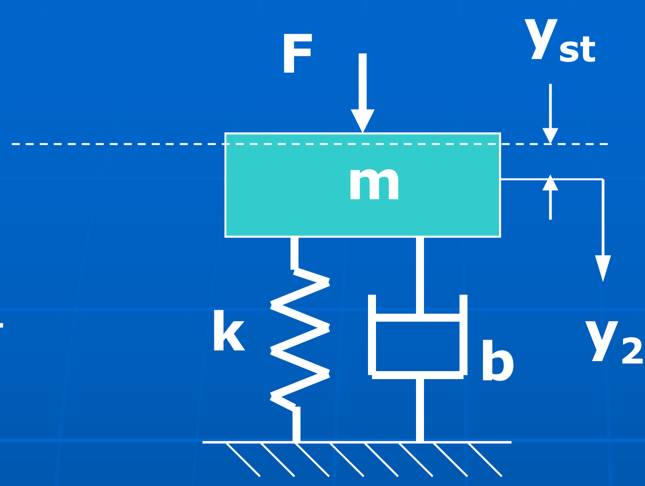
$$\left(ms^2 + bs + k\right)Y(s) = F(s)$$

- The transfer function is given by :

$$G(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$



Undeformed position



Equilibrium position

NOTE

$$y_{st} = \frac{mg}{k}$$

$$y_1 = y_{st} + y_2$$

$$m\ddot{y}_1 + b\dot{y}_1 + ky_1 = F + mg$$

$$y_1 = y_{st} + y_2 \qquad \dot{y}_1 = \dot{y}_2$$

$$\ddot{y}_1 = \ddot{y}_2$$

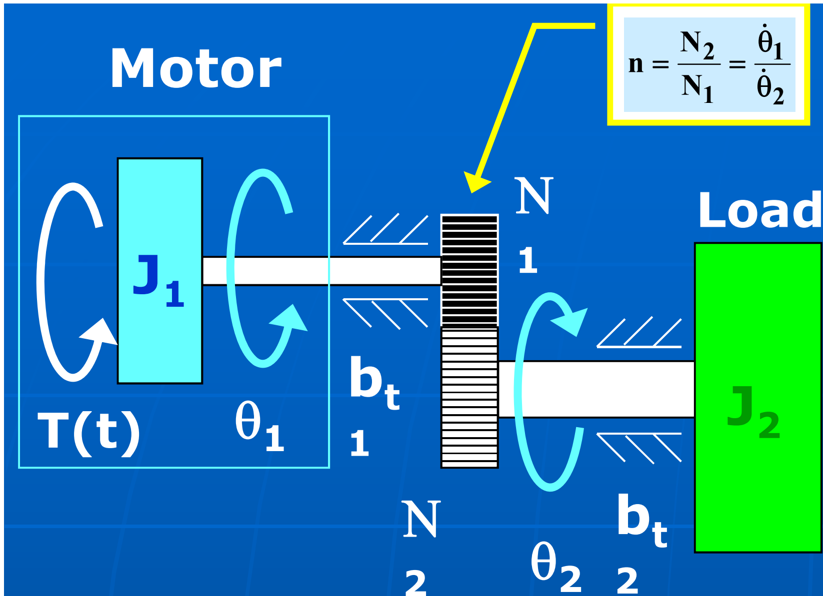
$$m\ddot{y}_2 + b\dot{y}_2 + k\left(y_2 + y_{st}\right) = F + mg$$

$$m\ddot{y}_1 + b\dot{y}_1 + ky_1 = F + mg$$

$$m\ddot{y}_2 + b\dot{y}_2 + k\left(y_2 + \frac{mg}{k}\right) = F + mg$$

$$m\ddot{y}_2 + b\dot{y}_2 + ky_2 = F$$

- If the displacement is measured from the static equilibrium position, gravity force mg can be neglected, since it is balanced by a force ky_{st} in the spring.



EXAMPLE – 2

$$\left(n^2 J_1 + J_2\right)\ddot{\theta}_2 + \left(n^2 b_{t1} + b_{t2}\right)\dot{\theta}_2 = nT(t)$$

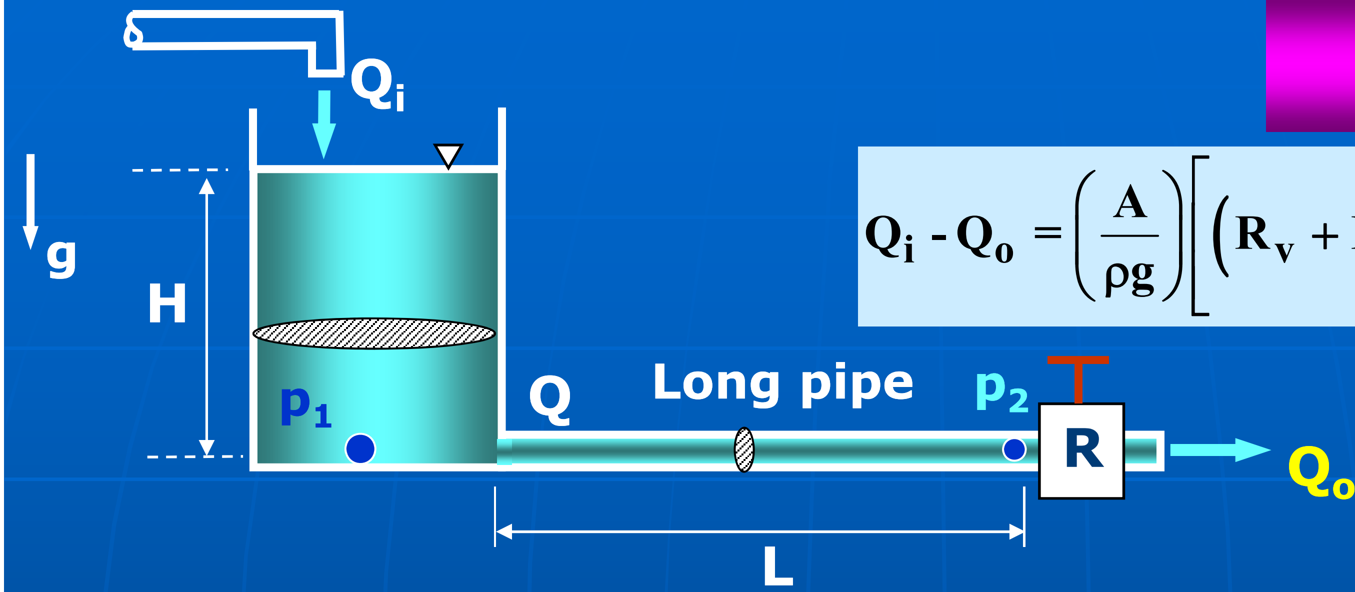
- Take the Laplace transform of all the terms.

$$\left[\left(n^2 J_1 + J_2\right)s^2 + \left(n^2 b_{t1} + b_{t2}\right)s\right]\Theta_2(s) = nT(s)$$

- Obtain the transfer function.

$$G(s) = \frac{\Theta_2(s)}{T(s)} = \frac{n}{\left(n^2 J_1 + J_2\right)s^2 + \left(n^2 b_{t1} + b_{t2}\right)s}$$

EXAMPLE 3



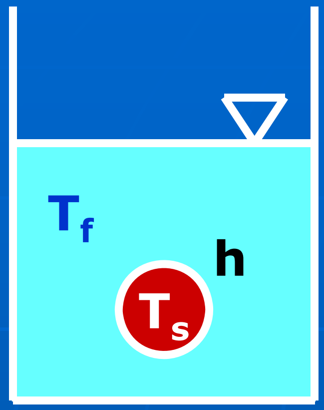
$$Q_i - Q_o = \left(\frac{A}{\rho g}\right)\left[\left(R_v + R_p\right)\frac{dQ_o}{dt} + \left(\frac{\rho L}{a}\right)\frac{d^2Q_o}{dt^2}\right]$$

- **Take the Laplace transform of all the terms.**

$$Q_i(s) - Q_o(s) = \left(\frac{A}{\rho g}\right)\left[\left(R_v + R_p\right)sQ_o(s) + \left(\frac{\rho L}{a}\right)s^2Q_o(s)\right]$$

- **Obtain the transfer function.**

$$G(s) = \frac{Q_o(s)}{Q_i(s)} = \frac{1}{\left(\frac{A}{\rho g}\right)\left[\left(\frac{\rho L}{a}\right)s^2 + \left(R_v + R_p\right)s + \frac{\rho g}{A}\right]}$$



EXAMPLE 4

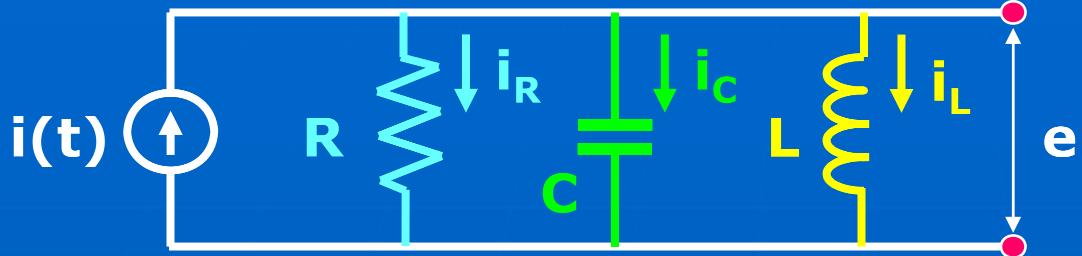
$$mc_p \frac{dT_s}{dt} + hAT_s = hAT_f$$

- Take the Laplace transform of all the terms.

$$mc_p sT_s(s) + hAT_s(s) = hAT_f(s)$$

- Obtain the Laplace transform.

$$G(s) = \frac{T_s(s)}{T_f(s)} = \frac{hA}{mc_p s + hA}$$



EXAMPLE 5

- **Determine the transfer function.**

$$RCL\frac{d^2e}{dt^2} + L\frac{de}{dt} + Re = RL\frac{di(t)}{dt}$$

$$G(s) = \frac{E(s)}{I(s)} = \frac{RLs}{RCLs^2 + Ls + R}$$

MULTI-INPUT MULTI-OUTPUT (MIMO) SYSTEMS

- If a system has more than one input and/or output :

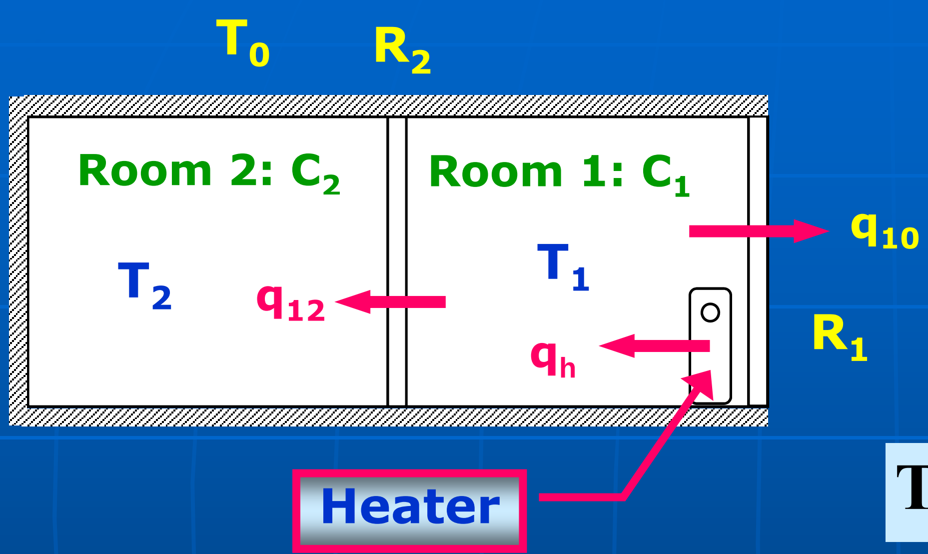
  Then there will be a transfer function relating each output with each input, when all other inputs are assumed to be zero.

$$X_i(s) = G_{ij}(s)Y_j(s) \qquad i = 1, 2, ..., p \qquad j = 1, 2, ..., q$$

- p: number of outputs,
- q : number of inputs.

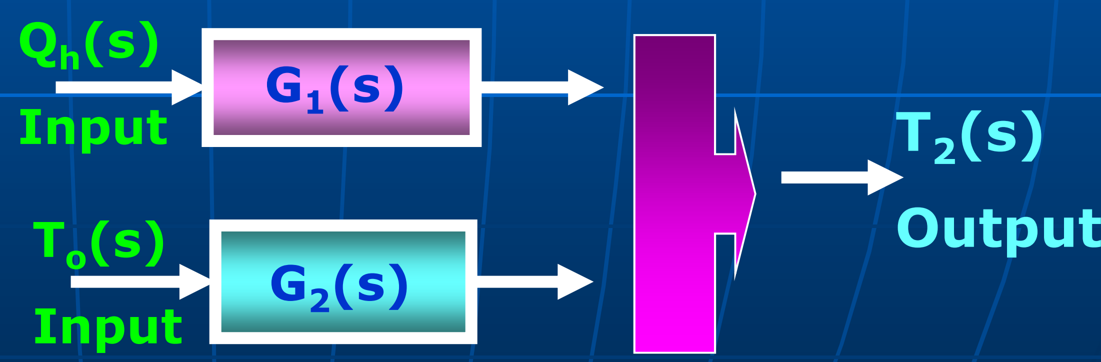
$$(C_1C_2R_1R_2)\frac{d^2T_2}{dt^2}+(C_1R_1+C_2R_1+C_2R_2)\frac{dT_2}{dt}+T_2=R_1q_h+T_0$$

EXAMPLE 6



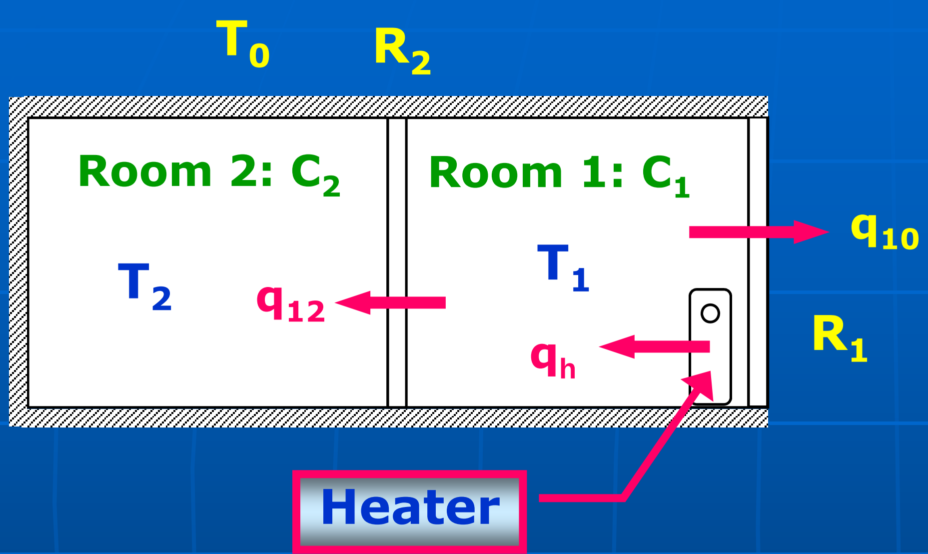
- Note that in this case there are two inputs !

$$T_2(s)=G_1(s)Q_h(s)+G_2(s)T_o(s)$$



$$(C_1C_2R_1R_2)\frac{d^2T_2}{dt^2}+(C_1R_1+C_2R_1+C_2R_2)\frac{dT_2}{dt}+T_2=R_1q_h+T_0$$

EXAMPLE 6



- Note that in this case there are two inputs !

$$T_2(s)=G_1(s)Q_h(s)+G_2(s)T_0(s)$$

$$G_1(s)=\frac{R_1}{(C_1C_2R_1R_2)s^2+(C_1R_1+C_2R_1+C_2R_2)s+1}$$

$$G_2(s)=\frac{1}{(C_1C_2R_1R_2)s^2+(C_1R_1+C_2R_1+C_2R_2)s+1}$$

- Further, the denominator is the same in both transfer functions !

MODELING DYNAMIC SYSTEMS

OBJECTIVES

Completed !

- Deriving input-output relations of linear time invariant systems (mechanical, fluid, thermal, and electrical) using elemental and structural equations.
- Obtaining transfer function representation of LTI systems.
- Representing control systems with block diagrams.

We are here !

BLOCK DIAGRAMS

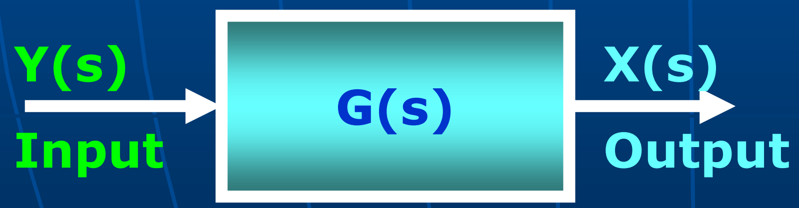
Nise Sections 5.1, 5.2

- Block diagrams are schematic representations of systems, indicating the function of each component and the flow of signals between them.
- A block diagram consists of three elements.
  - Blocks,
  - Summing Points,
  - Branch points.

BLOCK DIAGRAMS

- **Blocks**

  A block shows the operation performed on the input signal to produce the output, i.e. the transfer function, together with the input and output signals.

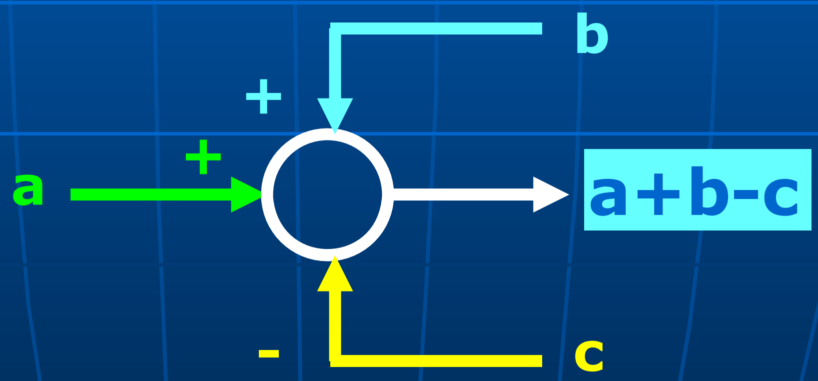


$$X(s) = G(s)Y(s)$$

BLOCK DIAGRAMS

- **Summing points**

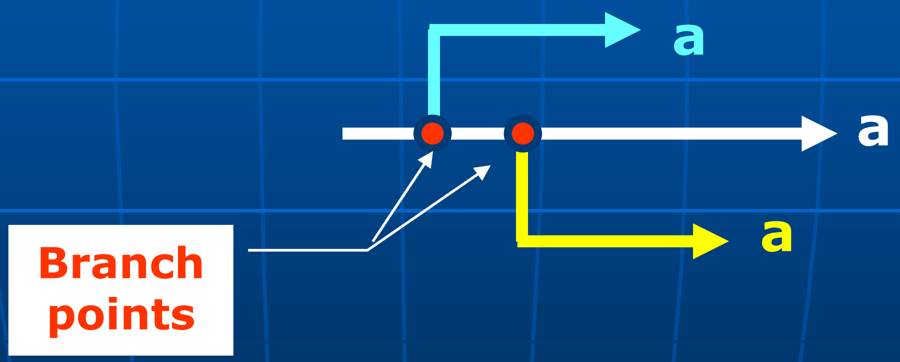
  These indicate a summing operation involving two or more signals. The sign at each arrow-head indicates either summation or subtraction.



BLOCK DIAGRAMS

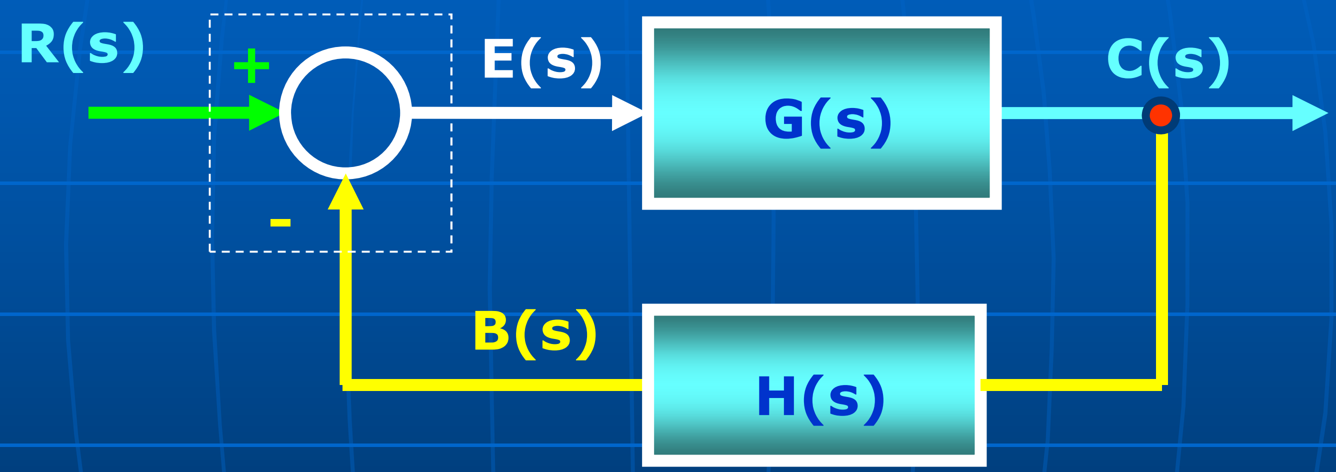
- **Branch points**

  A signal is duplicated to go simultaneously to two or more blocks at a branch point.



BLOCK DIAGRAMS

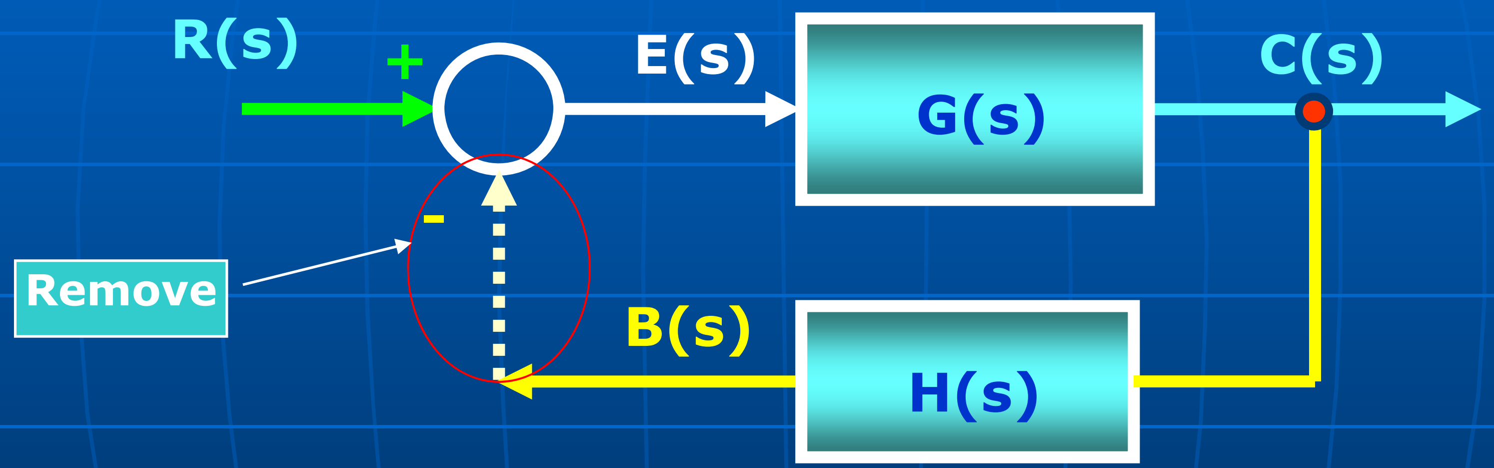
- **Block Diagram of a Closed Loop System**



- **E(s) : Error signal, G(s) : Feedforward TF,**
- **B(s) : Feedback signal, H(s) : Feedback TF.**

BLOCK DIAGRAMS

- **Open Loop Transfer Function of a Closed Loop System**

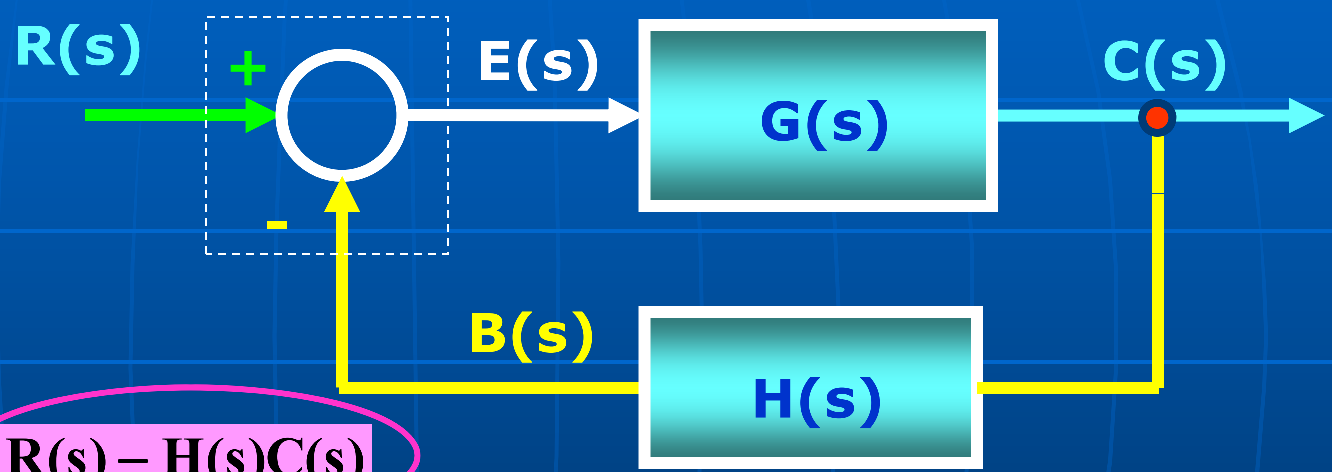


$R(s) \Rightarrow B(s)$

$$\text{Open Loop Transfer Function} = G(s)H(s)$$

BLOCK DIAGRAMS

- **Block Diagram of a Closed Loop System**



$$E(s) = R(s) - B(s) = R(s) - H(s)C(s)$$

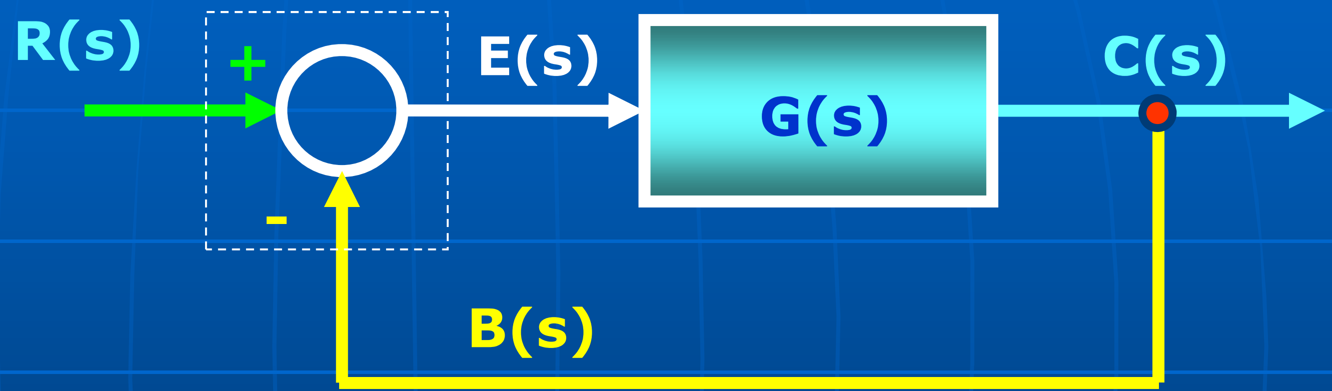
$$C(s) = G(s)E(s) = G(s)R(s) - G(s)H(s)C(s)$$

$$\left[1 + G(s)H(s)\right]C(s) = G(s)R(s)$$

$$\text{Closed Loop Transfer Function} = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

BLOCK DIAGRAMS

- Block Diagram of a **Unity Feedback** System



$$E(s) = R(s) - B(s) = R(s) - C(s)$$

$$C(s) = G(s)E(s) = G(s)R(s) - G(s)C(s)$$

$$\left[1 + G(s)\right]C(s) = G(s)R(s)$$

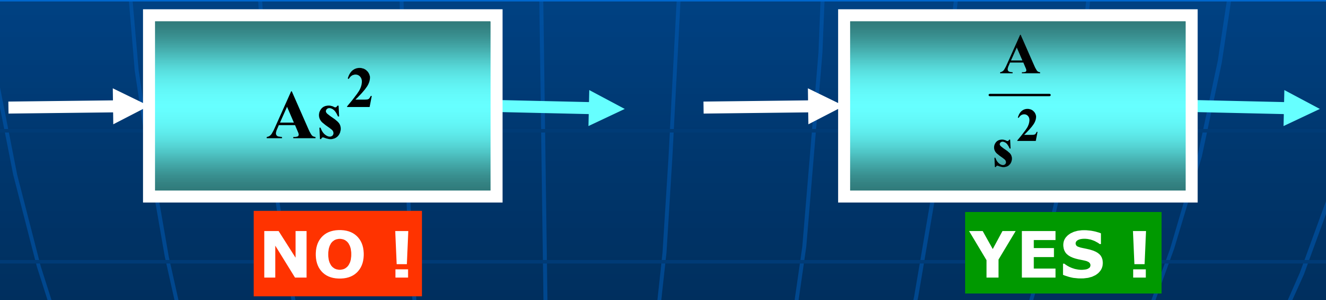
$$\text{Closed Loop Transfer Function} = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

BLOCK DIAGRAMS

- **Procedure for drawing block diagrams :**
  - **Write down the elemental and structural equations.**
  - **Obtain the transfer function for each element in the system using elemental equations.**
  - **Start with the system output and work back; inserting blocks, summing points, and branch points towards the system input.**

BLOCK DIAGRAMS

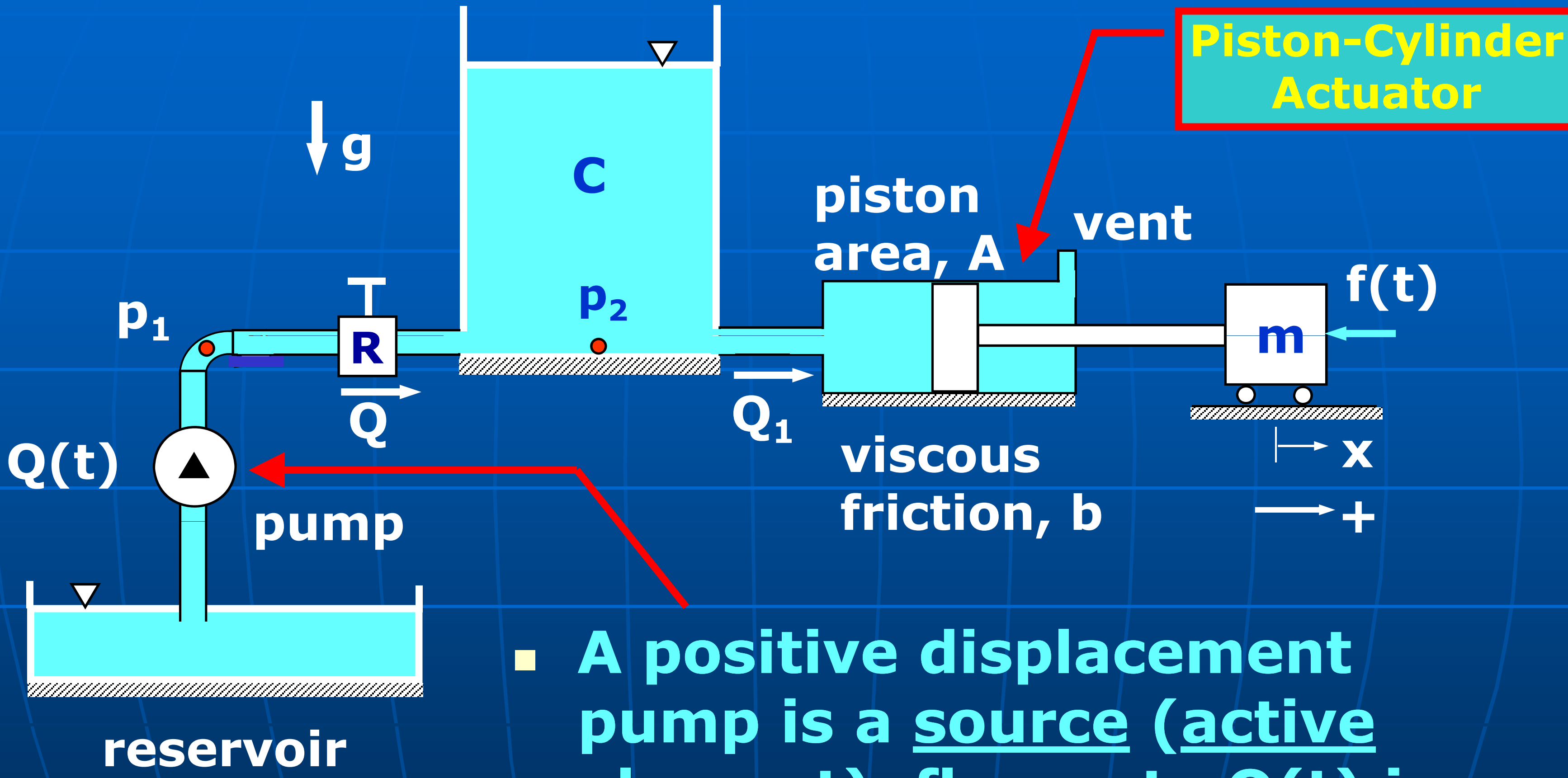
- **NOTE**
  - **Use each elemental transfer function only once.**
  - **Write elemental transfer functions such that terms containing s and its powers are in the denominator unless it is unavoidable.**



BLOCK DIAGRAMS

- **WARNING**
  - **Do not use overall transfer functions or manipulated equations in drawing the block diagrams !**
  - **You use the block diagrams to obtain the overall transfer functions not vice versa.**

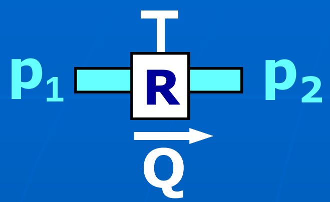
BLOCK DIAGRAMS - EXAMPLE



- A positive displacement pump is a source (active element), flow rate Q(t) is specified.

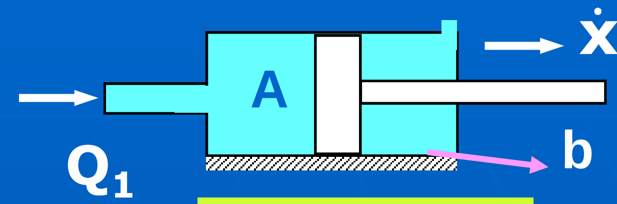
BLOCK DIAGRAMS EXAMPLE

Valve



$$p_1 - p_2 = RQ$$

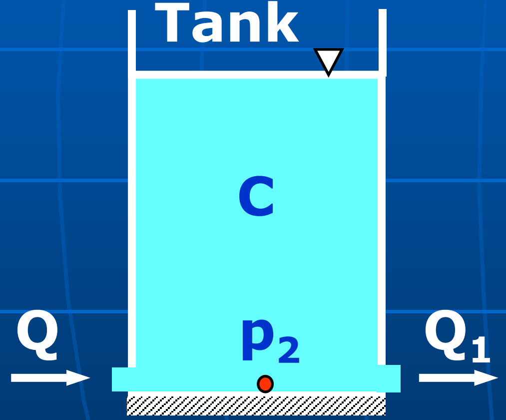
Piston+cylinder



Viscous friction $f_b = b\dot{x}$

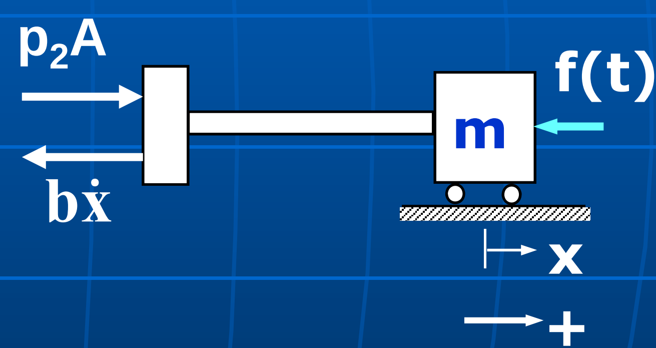
$$Q_1 = A\frac{dx}{dt}$$

Tank



$$Q - Q_1 = C\frac{dp_2}{dt}$$

Piston+mass



Mass

$$f_n = m\ddot{x}$$

$$Ap_2 - b\frac{dx}{dt} - f = m\frac{d^2x}{dt^2}$$

BLOCK DIAGRAMS - EXAMPLE

- Inputs : Q(t), f(t)
- Output : x(t)
-

	Equations :	Laplace transforms :
Valve :	$p_1 - p_2 = RQ$	$P_1(s) - P_2(s) = RQ(s)$
Actuator :	$Q_1 = A\dfrac{dx}{dt}$	$Q_1(s) = AsX(s)$
Tank :	$Q - Q_1 = C\dfrac{dp_2}{dt}$	$Q(s) - Q_1(s) = CsP_2(s)$
Mass :	$Ap_2 - b\dfrac{dx}{dt} - f = m\dfrac{d^2x}{dt^2}$	$AP_2(s) - bsX(s) - F(s) = ms^2X(s)$

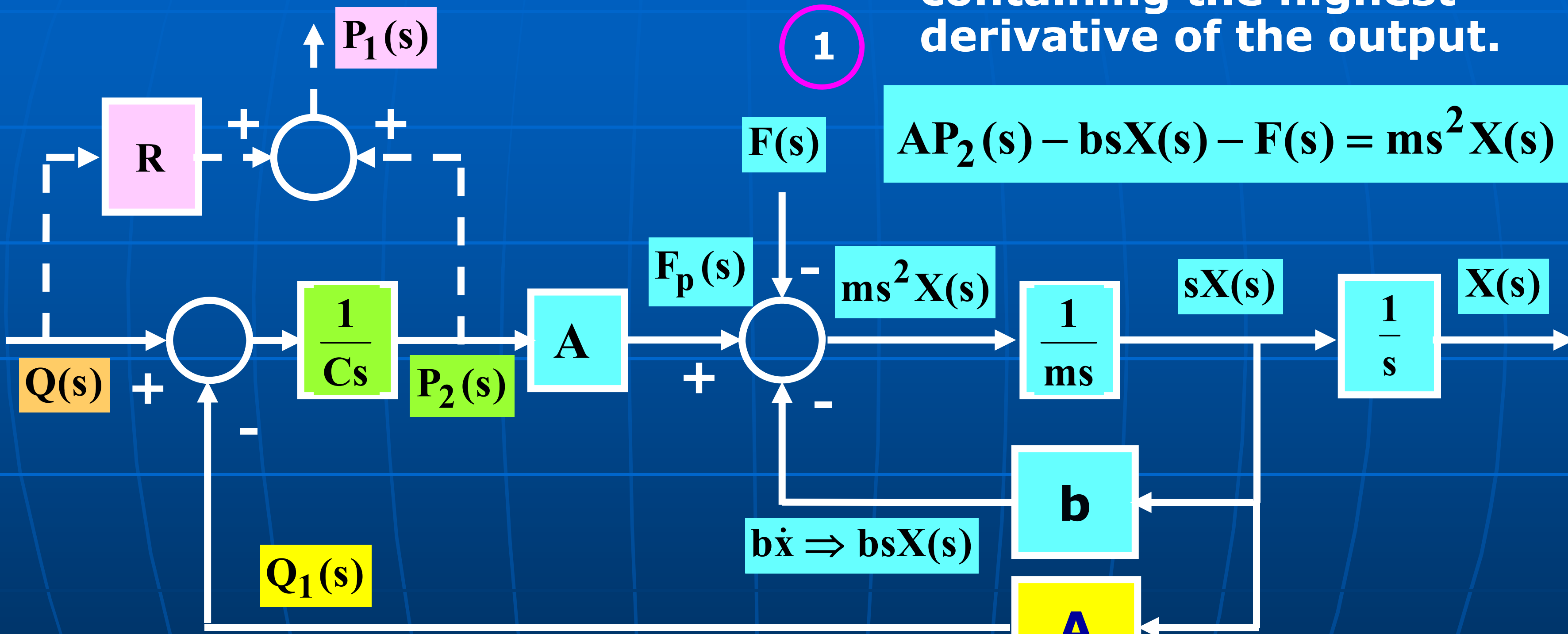
BLOCK DIAGRAMS - EXAMPLE

- Draw the block diagram.

$$P_1(s) - P_2(s) = RQ(s)$$ (4)

Start with the equation containing the highest derivative of the output. (1)

$$AP_2(s) - bsX(s) - F(s) = ms^2X(s)$$

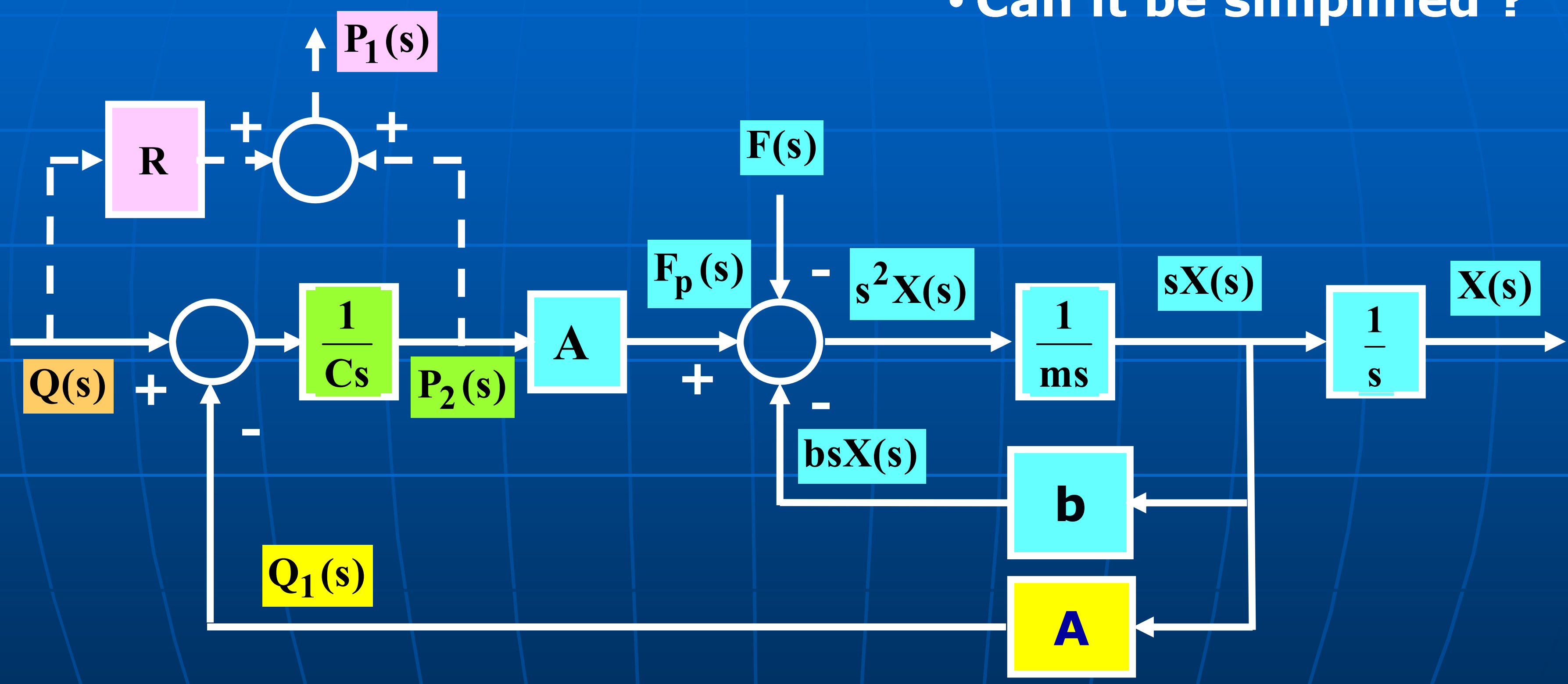


$$Q(s) - Q_1(s) = CsP_2(s)$$ (3)

$$Q_1(s) = AsX(s)$$ (2)

BLOCK DIAGRAMS - EXAMPLE

- **Block diagram.**
  - **Can it be simplified ?**



BLOCK DIAGRAMS - EXAMPLE

$AP_2(s) - bsX(s) - F(s) = ms^2X(s)$

$Q(s) - Q_1(s) = CsP_2(s)$

$Q_1(s) = AsX(s)$

$P_1(s) - P_2(s) = RQ(s)$

Not Used !

- **To find the overall transfer function, eliminate all variables except the inputs Q(s), F(s) and the output X(s).**

$$Q_1(s) = AsX(s)$$

$$Q(s) - Q_1(s) = CsP_2(s)$$

$$Q(s) - AsX(s) = CsP_2(s)$$

$$AP_2(s) - bsX(s) - F(s) = ms^2X(s)$$

$$\left(ms^2 + bs + \frac{A^2}{C}\right)X(s) = \frac{A}{Cs}Q(s) - F(s)$$

BLOCK DIAGRAMS - EXAMPLE

- **Transfer Function Representation**

$$\left(ms^2 + bs + \frac{A^2}{C}\right)X(s) = \frac{A}{Cs}Q\left(s\right) - F(s)$$

$$X(s) = \frac{\frac{A}{s}}{\left(Cms^2 + Cbs + A^2\right)}Q\left(s\right) - \frac{C}{\left(Cms^2 + Cbs + A^2\right)}F(s)$$

BLOCK DIAGRAM REDUCTION

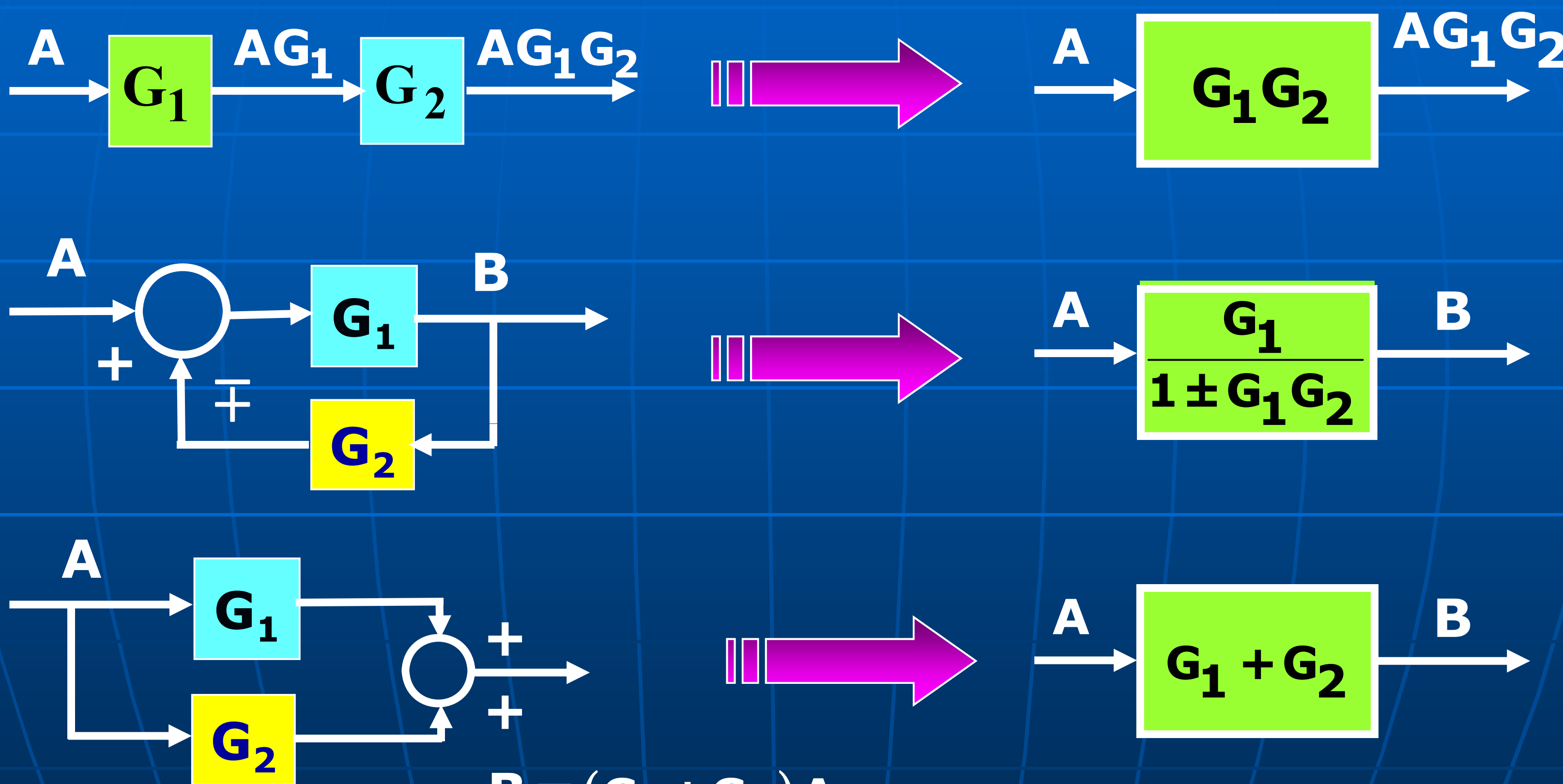
Nise Section 5.2

- **A complicated block diagram can be simplified down to a single block containing the overall transfer function of the system.**
- **This can be performed by manipulating**
  - **the elements of the block diagram (Block Diagram Algebra), or**
  - **the equations for the blocks and summing points to eliminate intermediate variables.**

BLOCK DIAGRAM REDUCTION

Dorf&Bishop Example 2.7, Table 2.6

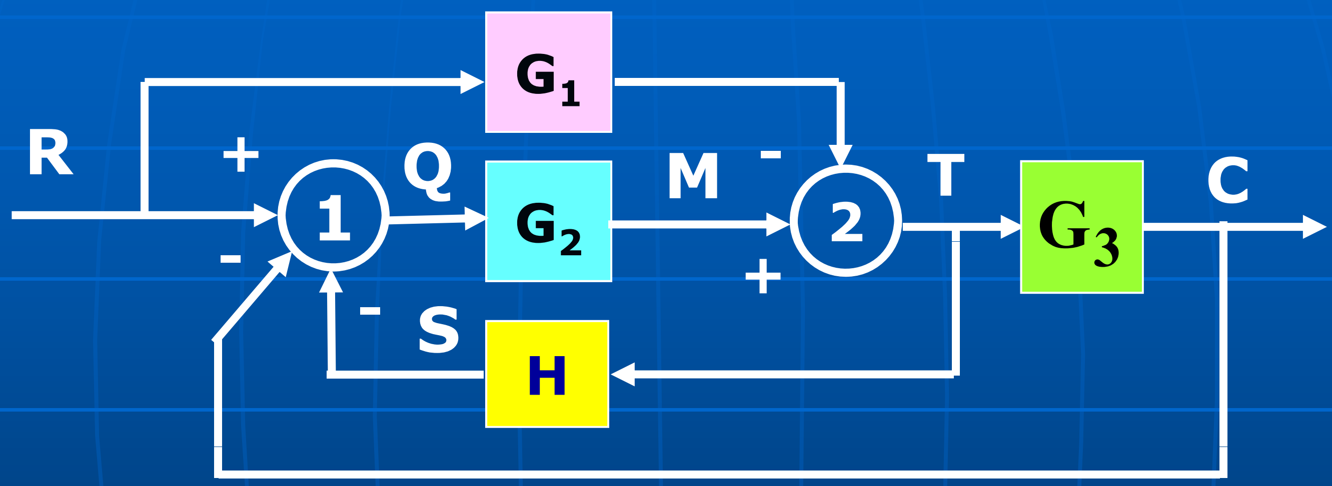
- **Block Diagram Algebra – some commonly used rules :**

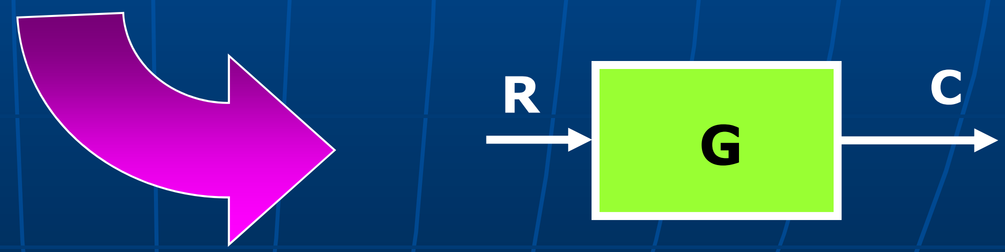


$$B = (G_1 + G_2)A$$

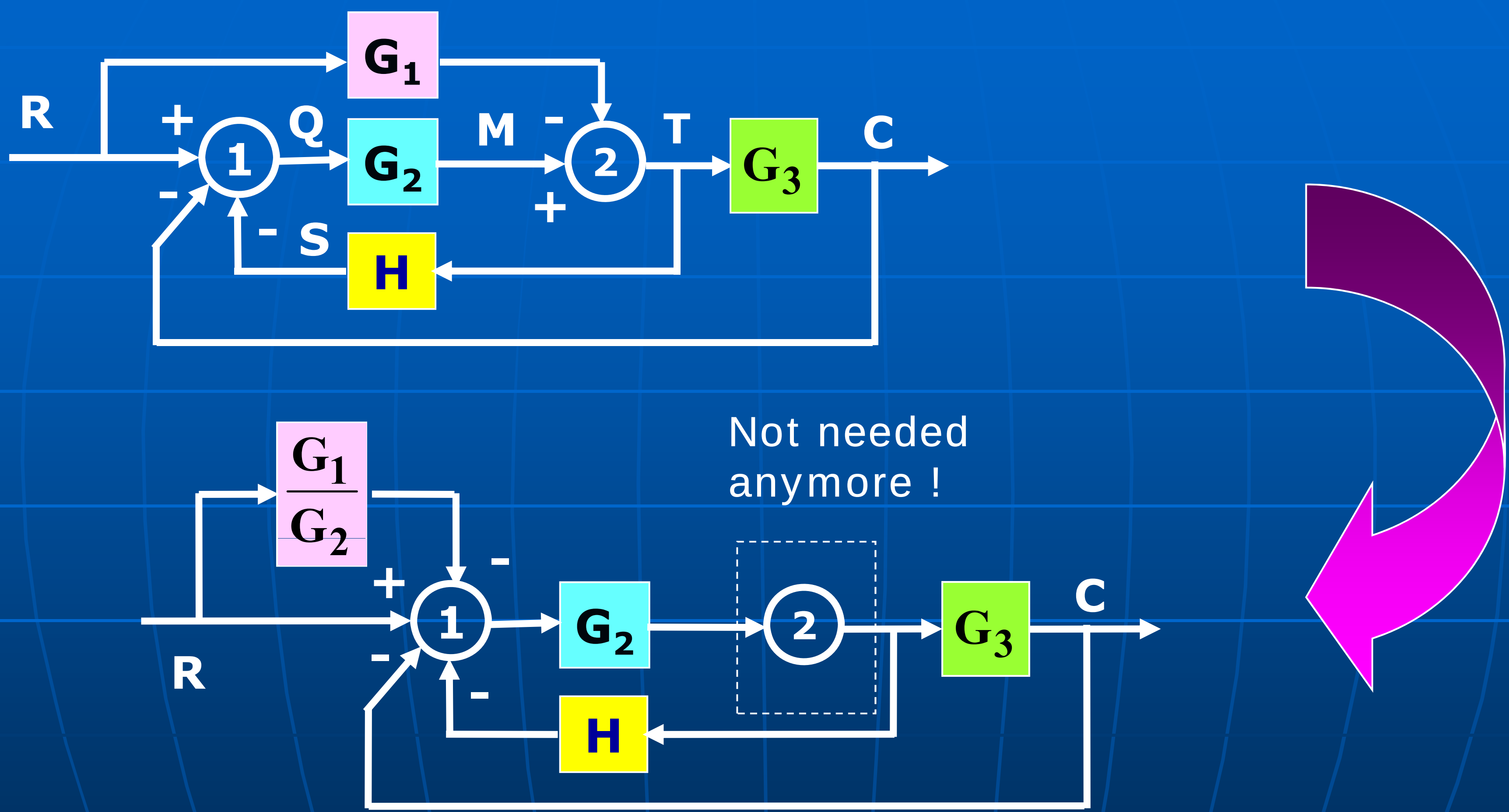
BLOCK DIAGRAM REDUCTION - EXAMPLE

- Simplify the block diagram to get TF.

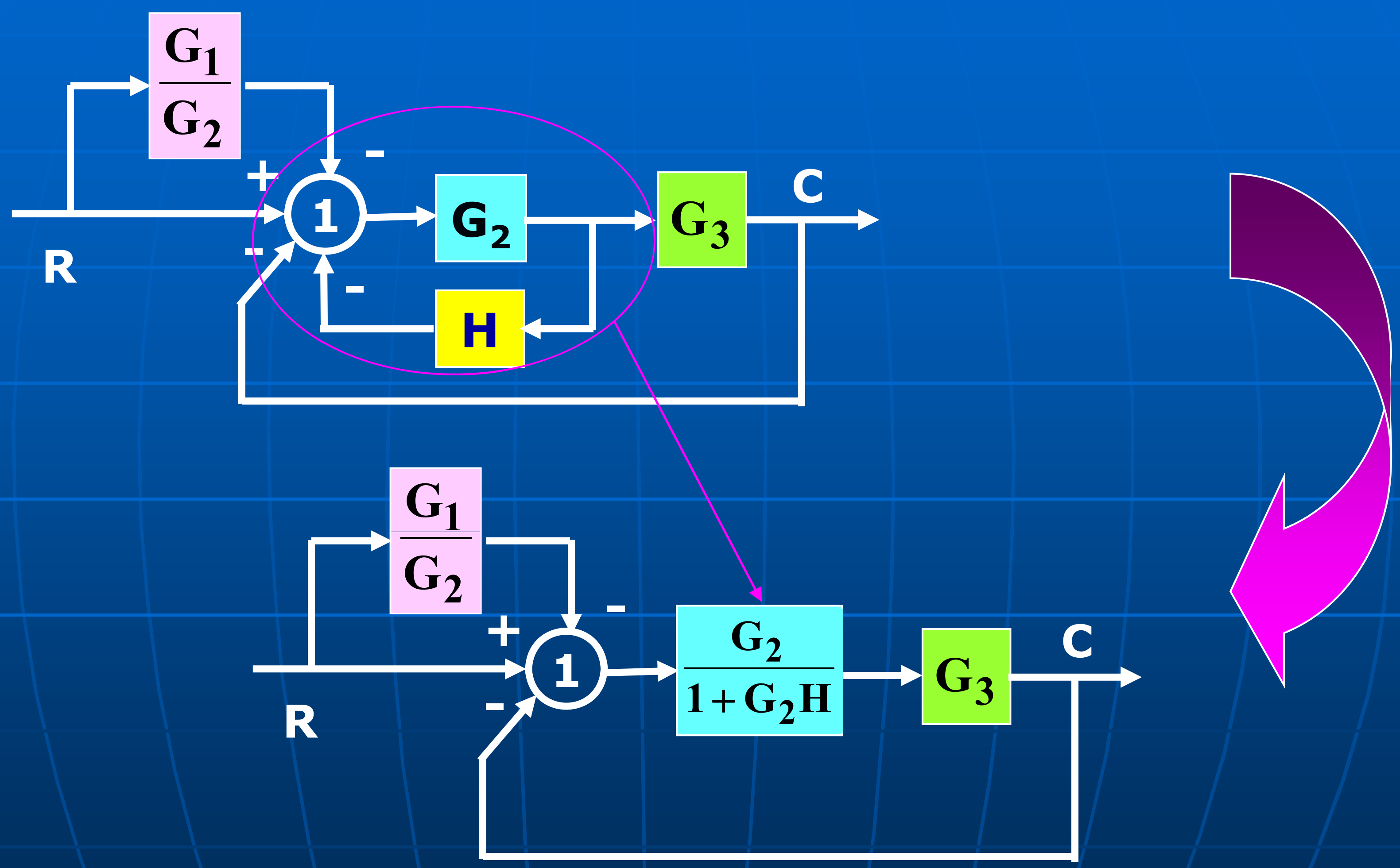




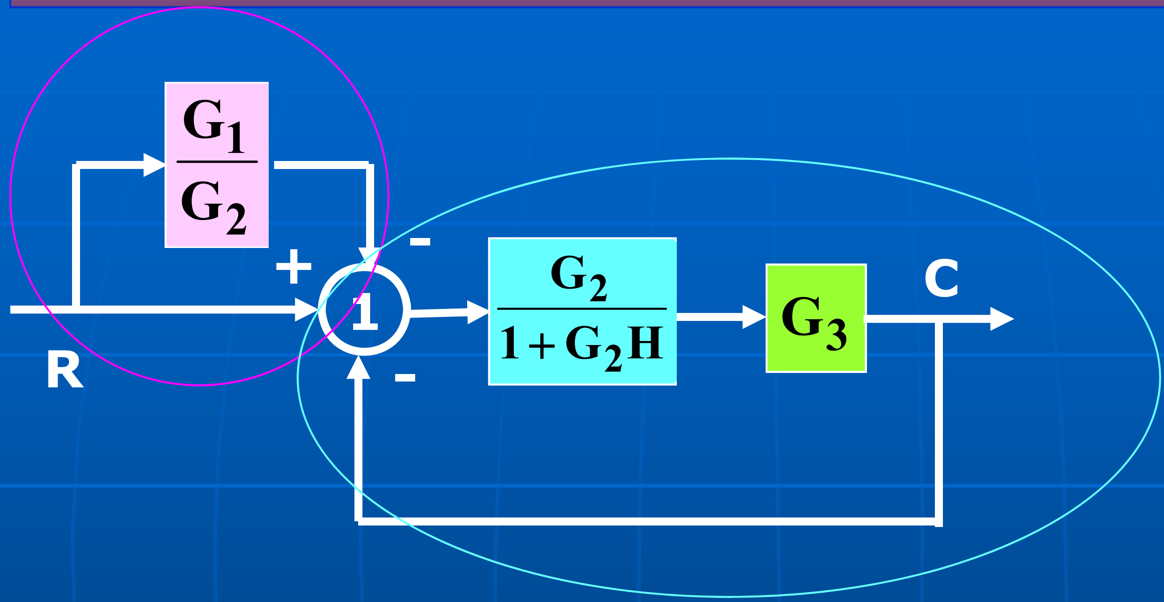
BLOCK DIAGRAM REDUCTION - EXAMPLE

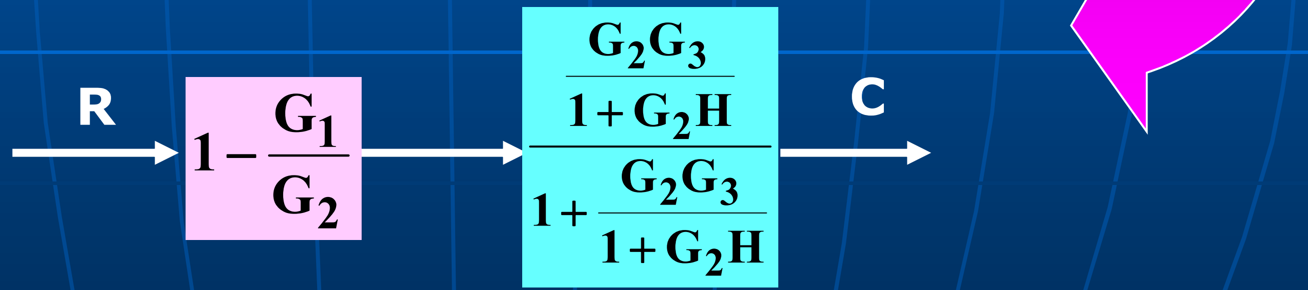


BLOCK DIAGRAM REDUCTION - EXAMPLE

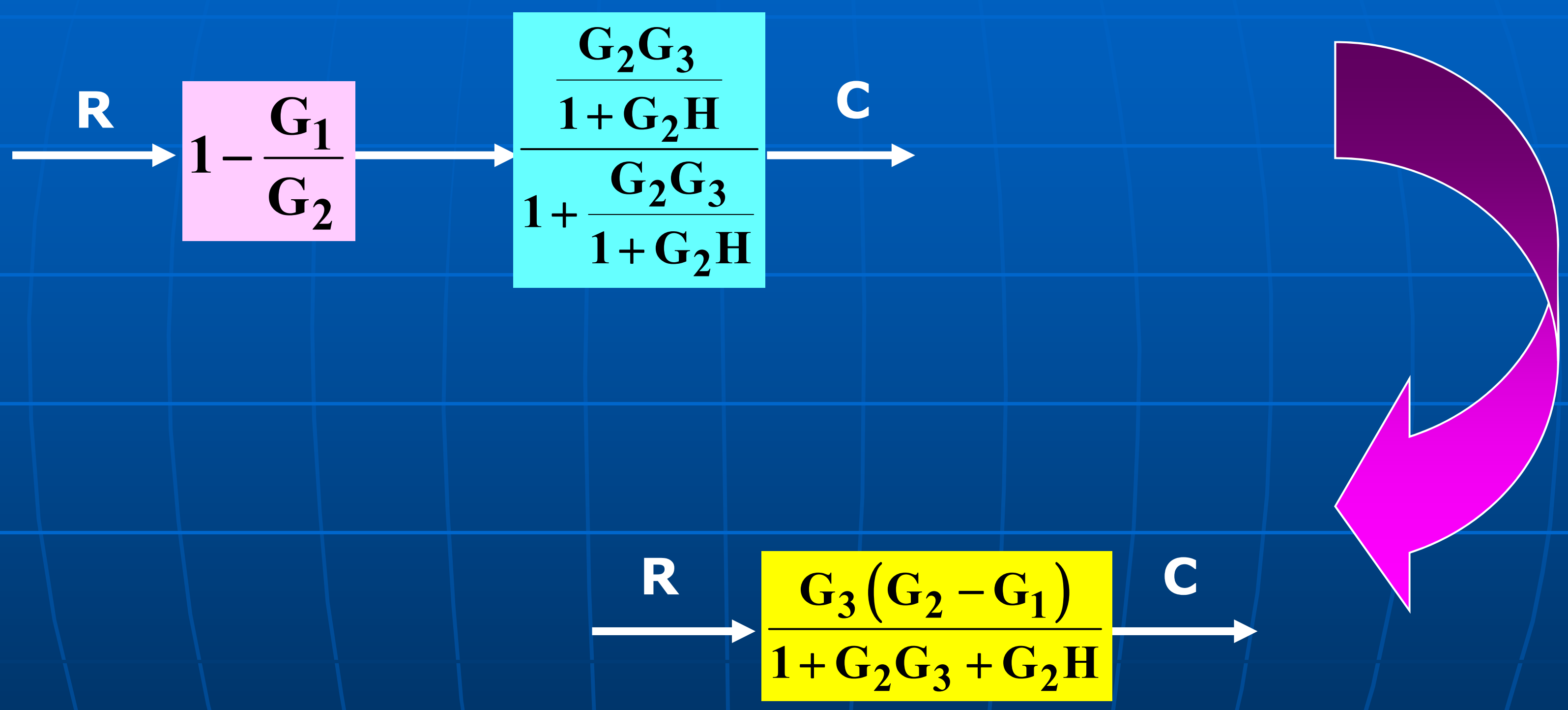


BLOCK DIAGRAM REDUCTION - EXAMPLE



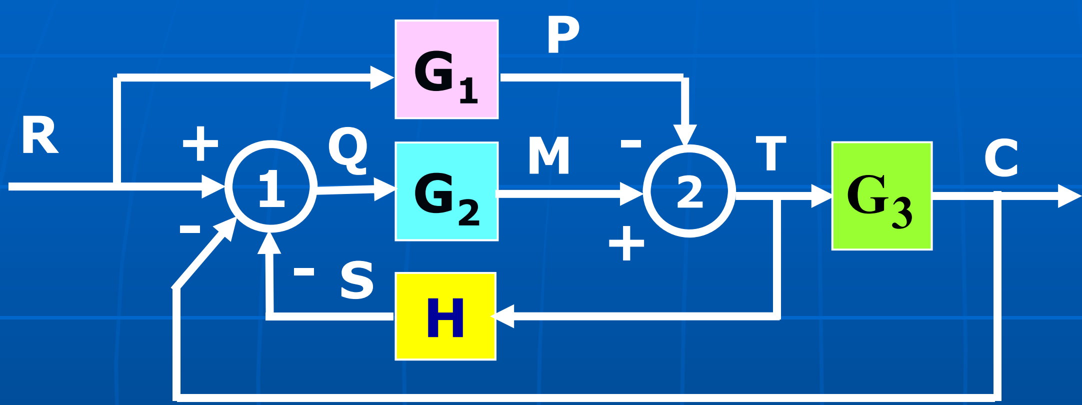


BLOCK DIAGRAM REDUCTION - EXAMPLE



BLOCK DIAGRAM REDUCTION - EXAMPLE

- Alternative Approach to block diagram simplification.



$P = G_1 R$

$Q = R - S - C$

$M = G_2 Q$

$T = M - P$

$S = HT$

$C = G_3 T$

Warning : T appears twice on the right hand side !

- Eliminate intermediate variables M, P, Q, S, T using the equations.

BLOCK DIAGRAM REDUCTION - EXAMPLE

- Start with T, which appears twice on the right hand side and eliminate it first.

$$M = G_2 Q \qquad P = G_1 R \qquad Q = R - S - C \qquad S = HT$$

$$T = M - P \rightarrow T = G_2 Q - G_1 R \rightarrow T = G_2 (R - S - C) - G_1 R$$

$$T = (G_2 - G_1) R - G_2 H T - G_2 C$$

$$(1 + G_2 H) T = (G_2 - G_1) R - G_2 C$$

$$T = \frac{(G_2 - G_1)}{(1 + G_2 H)} R - \frac{G_2}{(1 + G_2 H)} C \rightarrow C = G_3 T$$

Insert this in the last equation.

BLOCK DIAGRAM REDUCTION - EXAMPLE

- All the equations have been used.

$$C = G_3 T$$

$$T = \frac{(G_2 - G_1)}{(1 + G_2 H)} R - \frac{G_2}{(1 + G_2 H)} C$$

$$C = \frac{G_3 (G_2 - G_1)}{(1 + G_2 H)} R - \frac{G_2 G_3}{(1 + G_2 H)} C$$

$$\left[1 + \frac{G_2 G_3}{(1 + G_2 H)}\right] C = \frac{G_3 (G_2 - G_1)}{(1 + G_2 H)} R$$

$$\frac{C}{R} = \frac{G_3 (G_2 - G_1)}{(1 + G_2 G_3 + G_2 H)}$$

READING

- **Nise**
  - **Sections 2.1, 2.2, 2.3, 5.1, 5.2**
- **Ogata**
  - **Sections 2-3, 2-4, 4-2, 4-4, 4-5,**