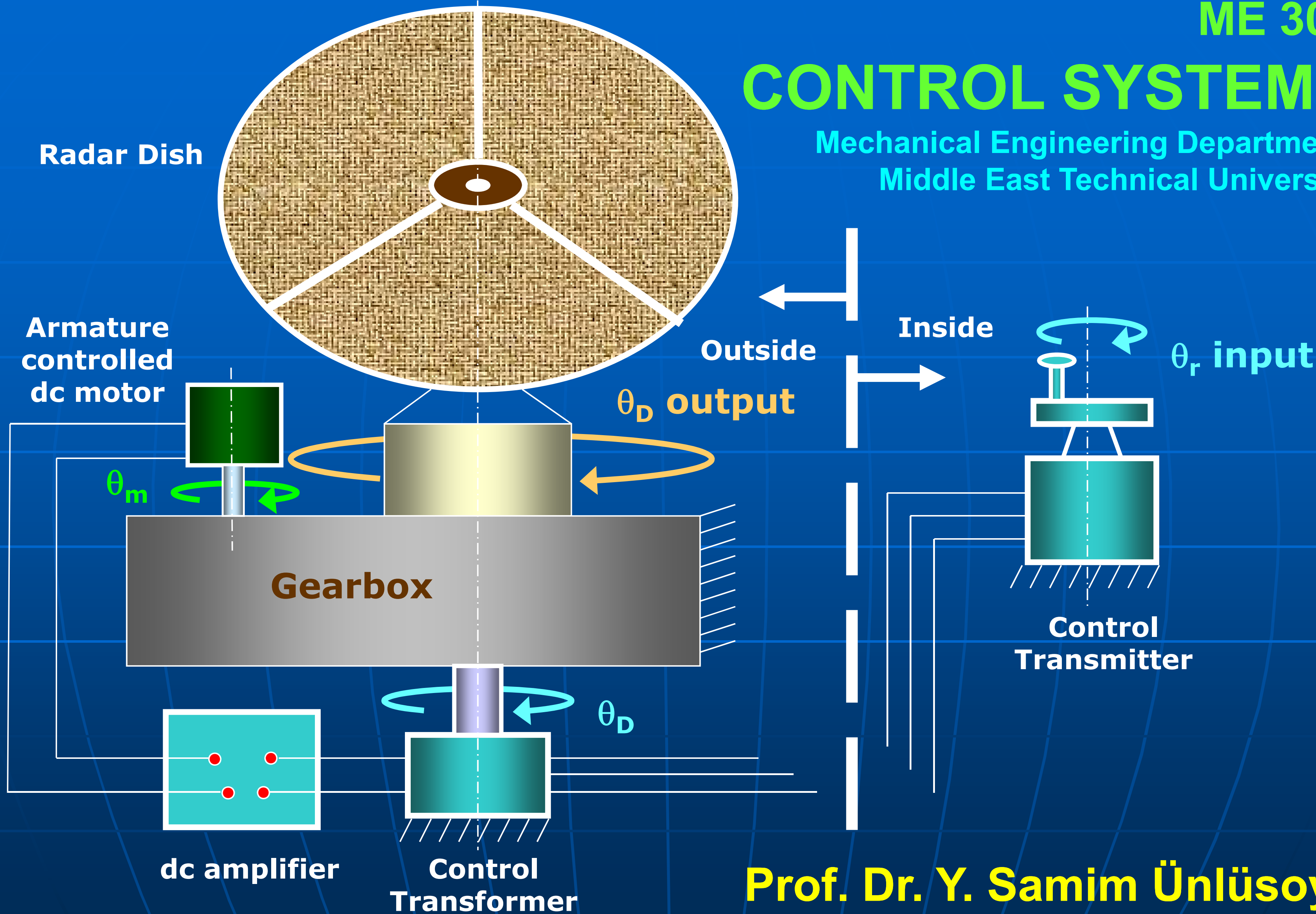
ME 304
CONTROL SYSTEMS

Mechanical Engineering Department,
Middle East Technical University



Prof. Dr. Y. Samim Ünlüsoy

CH II

COURSE OUTLINE

MODELING DYNAMIC SYSTEMS

OBJECTIVES

- Deriving input-output relations of linear time invariant (LTI) systems (mechanical, fluid, thermal, and electrical) using elemental and structural equations.
- Obtaining transfer function representation of LTI systems.
- Representing control systems with block diagrams.

MODELING DYNAMIC SYSTEMS

- A **mathematical model** of a physical system is a description of its dynamic behaviour in terms of mathematical equations.
- Once a mathematical model of a physical system is available, it is possible to study its dynamic properties in detail.

MODELING DYNAMIC SYSTEMS

- **In developing a mathematical model, a compromise between <u>simplicity</u> of the model and the <u>accuracy</u> of the results must be reached.**
- **Model should be as simple as possible while providing all the required characteristics of the system in hand.**

MODELING DYNAMIC SYSTEMS

- Mathematical models may be classified as :
  - **Lumped** or distributed parameter,
  - **Linear** or nonlinear,
  - **Time invariant** or time varying,
  - **Deterministic** or stochastic, and
  - **Continuous time** or discrete time.

MODELING DYNAMIC SYSTEMS

- **In this course, Linear Time Invariant (LTI) systems will be considered.**
- **Further they will be of lumped parameter, deterministic and continuous time systems.**
- **These systems will have input-output relations described by linear ordinary differential equations with constant coefficients.**

NONLINEAR SYSTEMS

Nise Ch. 2.10-11

- It should be noted that, in general, all physical systems are essentially **nonlinear**.
- They behave linearly, however, within some limited range of the variables of interest.
- Thus, it is possible to treat most physical systems as linear if the variations of variables are restricted to somewhat narrow ranges.

LINEARIZATION

- Let us consider a nonlinear relationship between two variables y(t) and x(t).

$$y(t) = f\{x(t)\}$$

- If we consider some operating point $x = x_o$ and write a Taylor series expansion of the nonlinear function about the operating point :

$$y = f(x) = f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o} \frac{(x - x_o)}{1!} + \left.\frac{d^2f}{dx^2}\right|_{x=x_o} \frac{(x - x_o)^2}{2!} + \dots$$

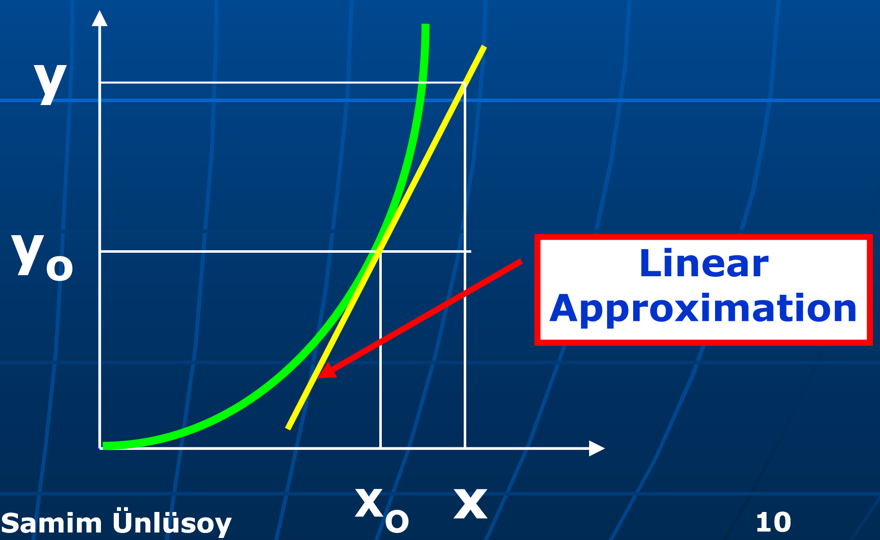
LINEARIZATION

- Taking only the first two terms of the expansion, a linear approximation to the original function around the operating point can be obtained.

$$y = f(x) = f(x_o) + \left.\frac{df}{dx}\right|_{x=x_o} (x - x_o) = y_o + m(x - x_o)$$

- Thus the linear relation takes the form :

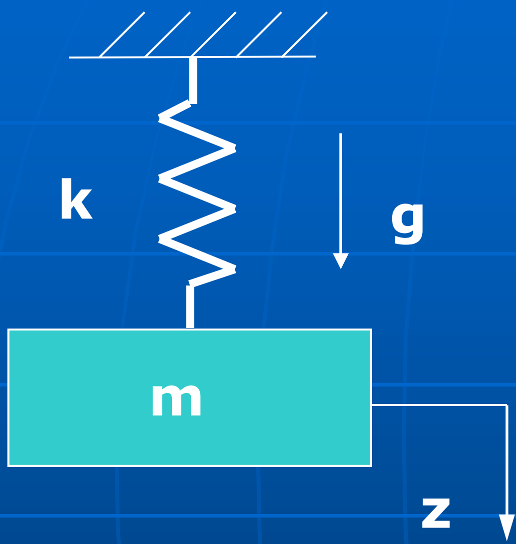
$$y - y_o = m(x - x_o)$$



LINEARIZATION

- **Obviously the linear approximation agrees well with the nonlinear relation in a narrow range about the operating point (xo, yo).**

LINEARIZATION - Example



- Consider a mass carried by a spring as illustrated in the figure.
- The force-deflection characteristic of the so-called stiffening spring is expressed as :

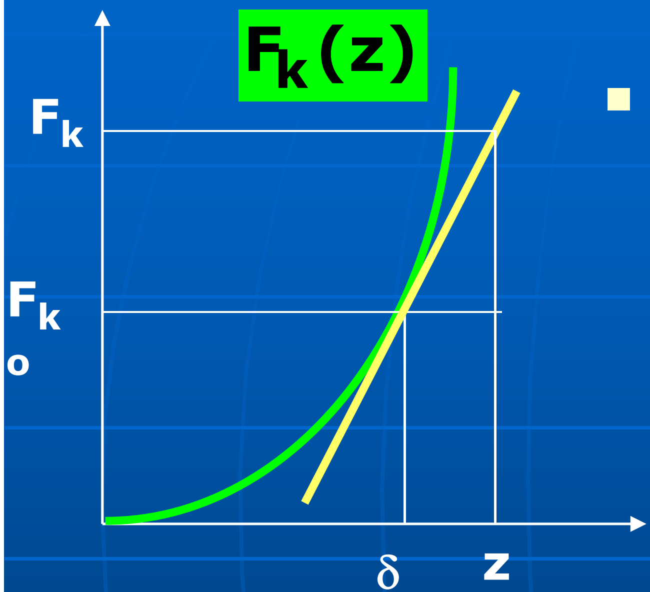
$$F_k = az^2$$

$$F_k = a\delta^2 = mg$$

$$\delta = \sqrt{\frac{mg}{a}}$$

- In the static equilibrium configuration, the deflection δ of the spring will be given by :

LINEARIZATION - Example



- Thus if the spring characteristic is linearized around the static equilibrium position :

$$F_k - F_{ko} = \left.\frac{dF_k}{dz}\right|_{z=\delta}(z-\delta) = 2a\delta(z-\delta)$$

- The linearized spring characteristic, valid around the static equilibrium position, will then be given by :

$$\Delta F_k = (2a\delta)\Delta z$$

Stiffness

MODELING DYNAMIC SYSTEMS

- **In deriving input-output relations for a system, the procedure detailed below may be followed.**
  - **Define the input and output.**
  - **Break the system into its elements.**
  - **Write the elemental equations.**
  - **Write the structural equations.**
  - **Combine elemental and structural equations to relate input and output.**

MODELING DYNAMIC SYSTEMS

- **Elemental** equations can be written for each element of a system irrespective of the way they are connected to each other.
- **Structural** equations describe how these elements are connected to form the system.

ELEMENTAL EQUATIONS
Translational Mechanical Elements

- **Ideal Mass** (lumped, point mass m)

$$f_n = ma$$

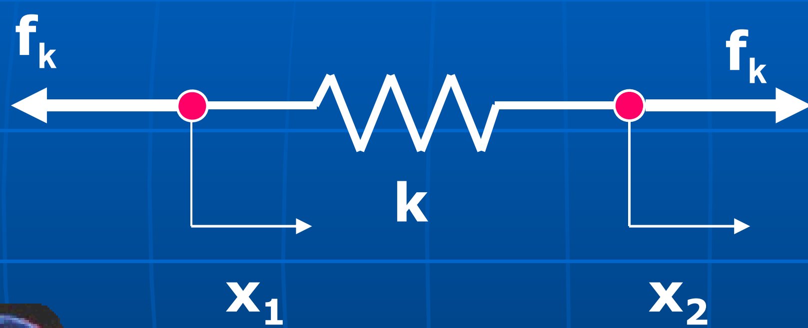
- f_n = net external force,
  a = acceleration.

Translational Mechanical Elements

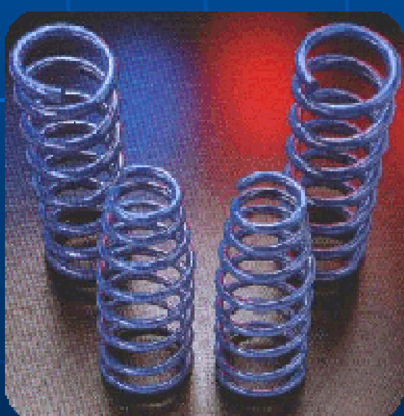
- **Ideal Linear Spring**

k : spring constant

- f_k : force applied by the spring



$$f_k = k\left(x_1 - x_2\right)$$



If $x_1 > x_2 \Rightarrow$ spring is in compression

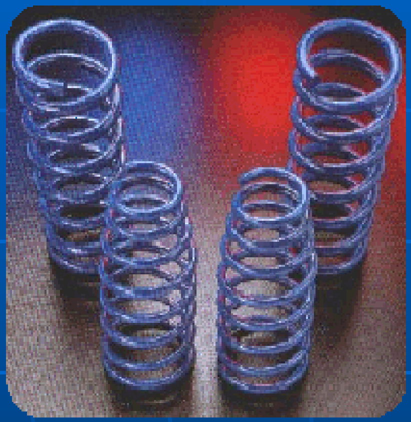
If $x_1 < x_2 \Rightarrow$ spring is in tension

Translational Mechanical Elements

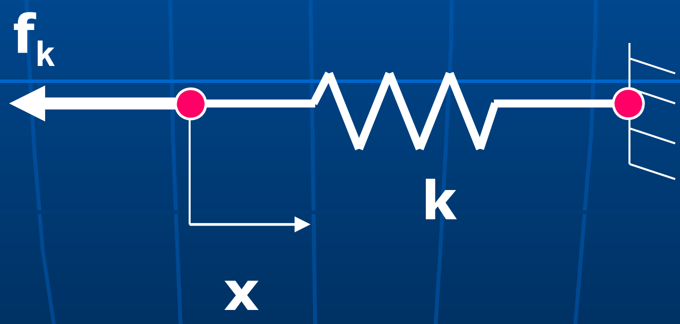
- **Ideal Linear Spring**

k : spring constant

- f_k : force applied **by** the spring



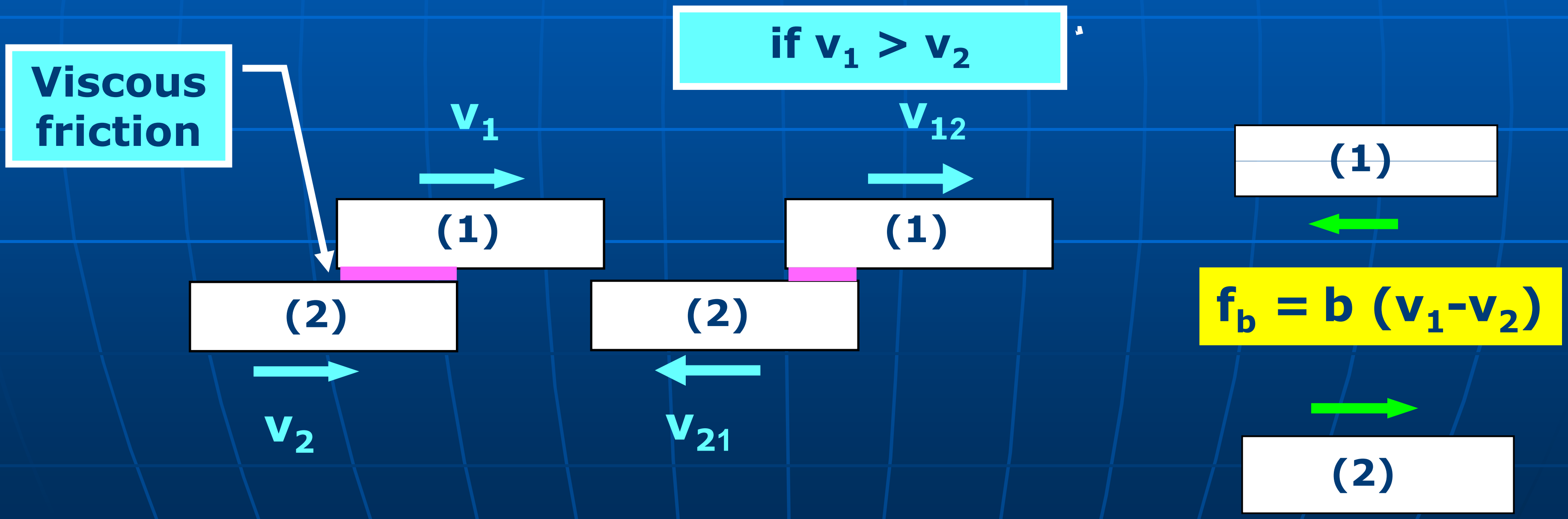
- Ideal spring with one end fixed:



$$f_k = kx$$

Translational Mechanical Elements

- **Viscous Friction** - Direction of friction force always opposes relative motion.
- b : coefficient of viscous friction,
- v_{ij} : relative velocity of body i with respect to body j.



Translational Mechanical Elements

- **Viscous Friction** - Direction of friction force always opposes relative motion.
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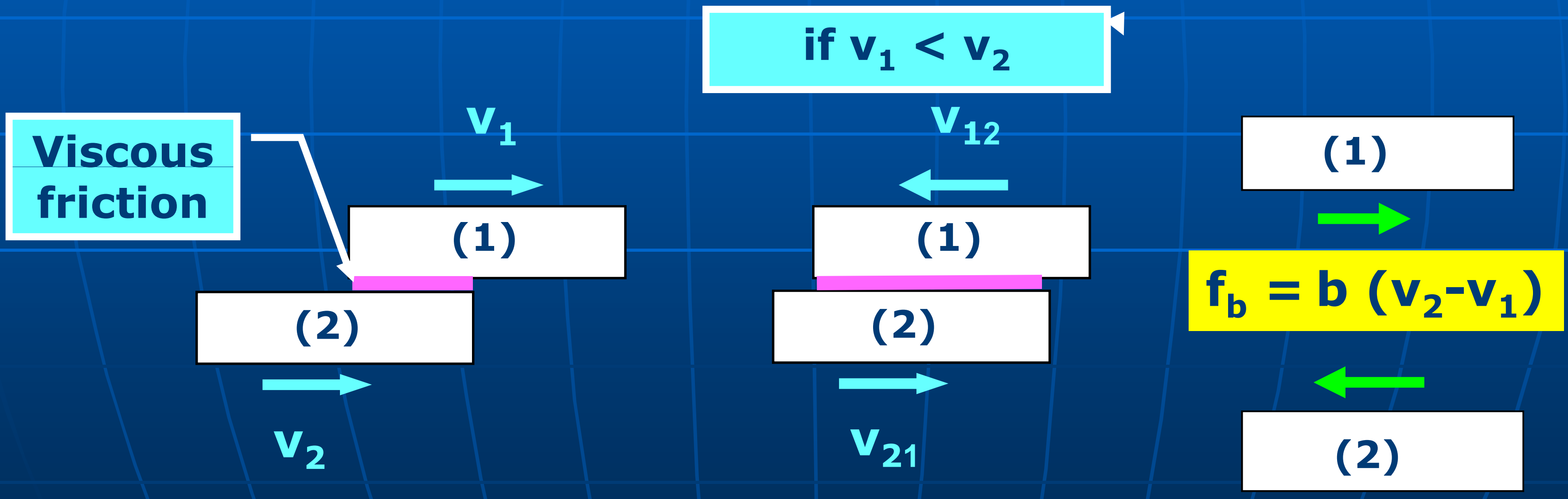
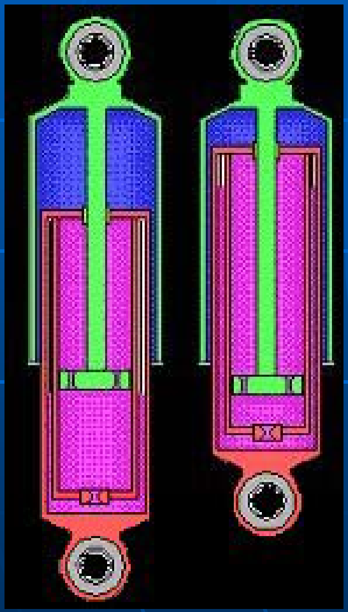
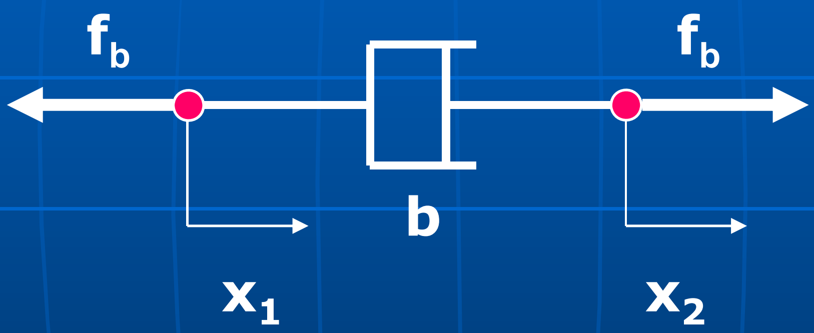


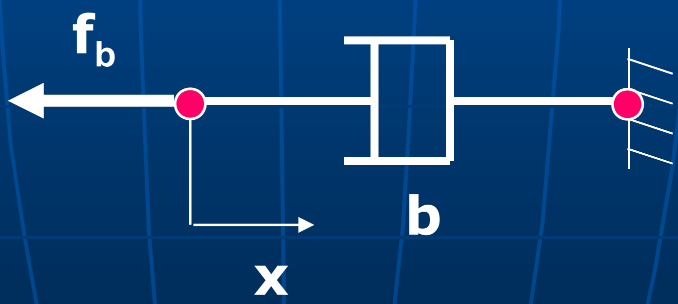
Translational Mechanical Elements

- **Ideal Viscous Damper**
  - b : coefficient of viscous damping
    - f_b : force applied by the damper





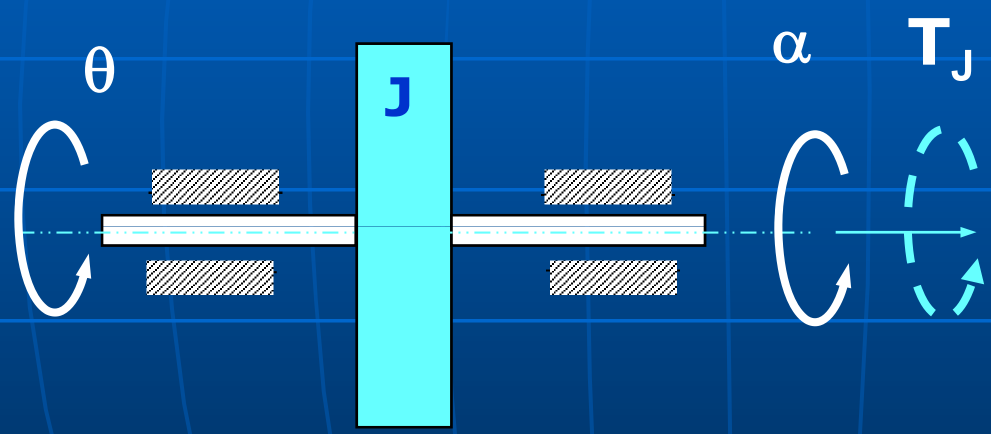
$$f_b = b\left(\dot{x}_1 - \dot{x}_2\right)$$



$$f_b = b\dot{x}$$

ELEMENTAL EQUATIONS
Rotational Mechanical Elements

- **Inertia (Rotating Mass)**
- **J : Mass moment of inertia**



$$T_J = J\alpha = J\ddot{\theta}$$

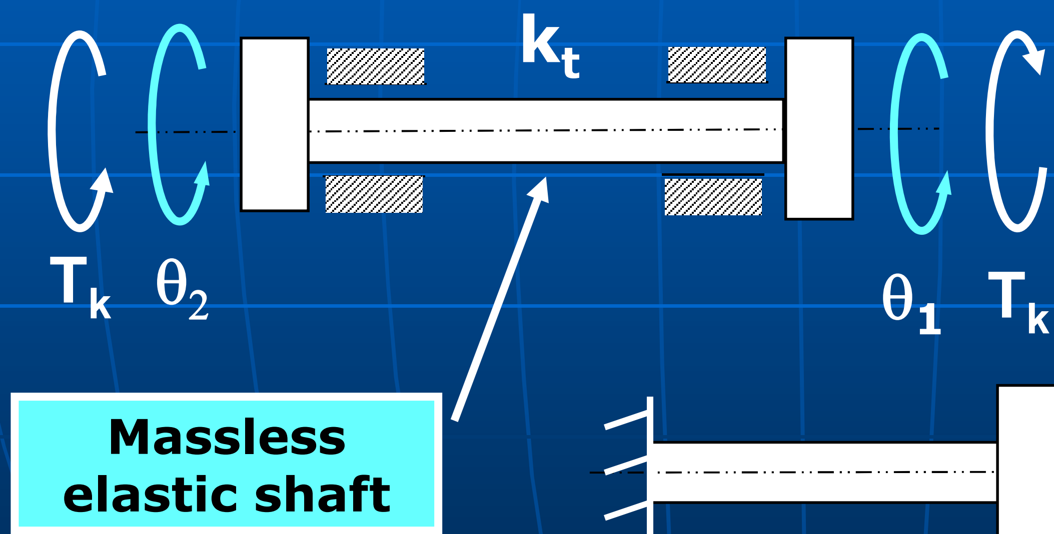
- T_J = Net external torque,
- α = Angular acceleration.

Rotational Mechanical Elements

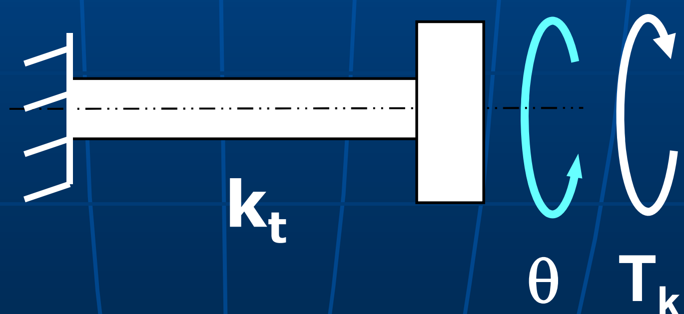
- **Ideal Torsional Spring (massless elastic shaft)**

k_t : Torsional spring constant

T_k : Torque applied by the torsional spring



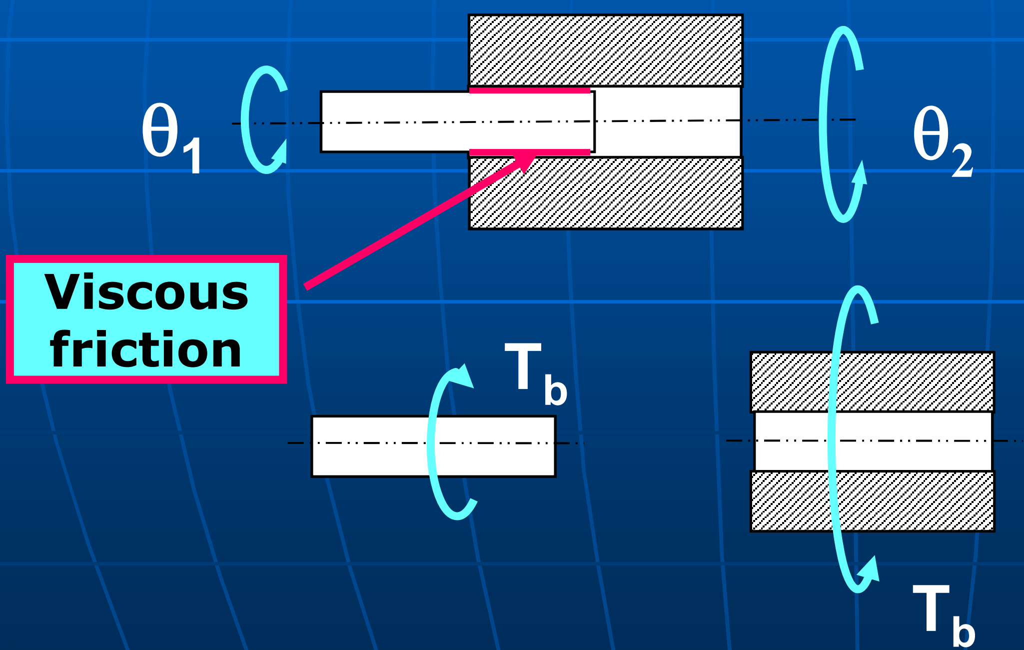
$$T_k = k_t(\theta_1 - \theta_2)$$



$$T_k = k_t\theta$$

Rotational Mechanical Elements

- **Viscous Friction**
  - b_t : coefficient of viscous friction



$$T_b = b_t\left(\dot{\theta}_1 - \dot{\theta}_2\right)$$

- Which is correct here ?

$$\dot{\theta}_1 < \dot{\theta}_2 \qquad ?$$

$$\dot{\theta}_1 > \dot{\theta}_2 \qquad ?$$

STRUCTURAL EQUATIONS

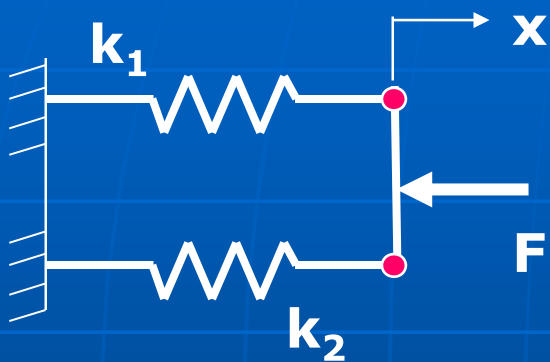
- **Structural equations describe how the elements are connected to form the system.**
- **There are two types of structural equations.**
  - **Continuity equations. In mechanical systems, these correspond to force (or torque) balance at a node or on an element.**
  - **Compatibility equations. In mechanical systems, these correspond to kinematic position, velocity, and acceleration relations.**

STRUCTURAL EQUATIONS

- **After gaining some experience, one may draw and annotate the free body diagrams such that the structural equations are automatically implemented and do not have to be written separately.**

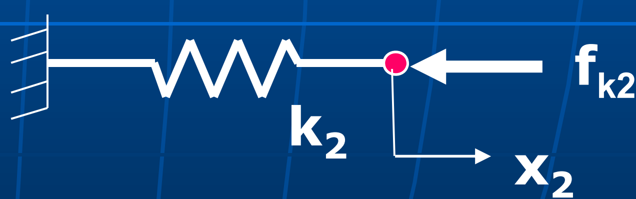
STRUCTURAL EQUATIONS – Example

See solved example A-3-14 in Ogata !



Write the input(x)-output (F) relation for the system and determine the equivalent stiffness.

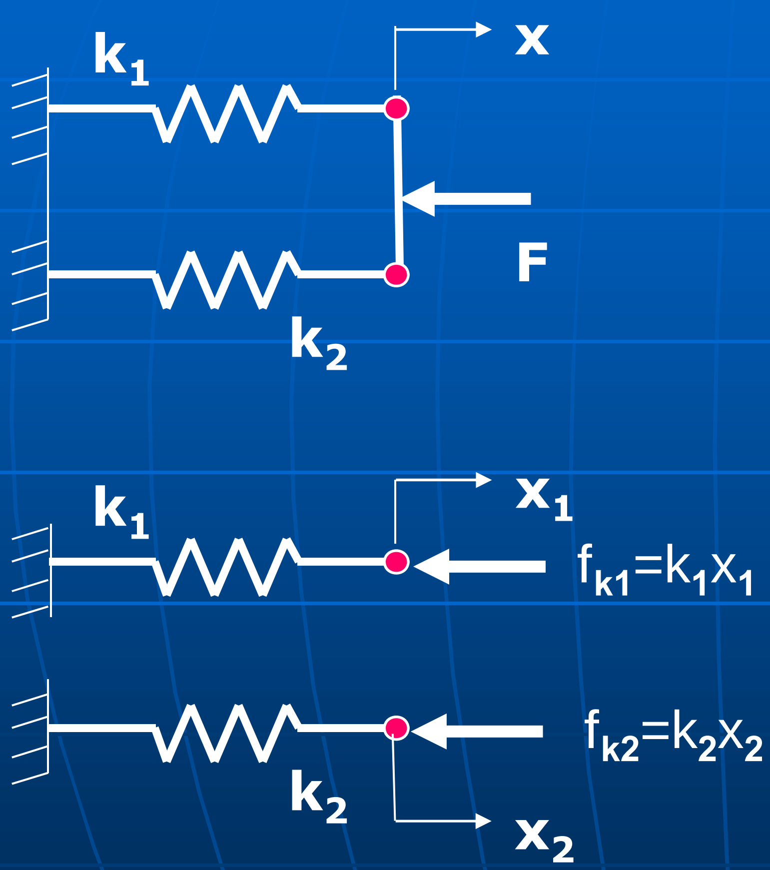
- Identify elements and write the elemental equations.



$$f_{k1} = k_1 x_1$$

$$f_{k2} = k_2 x_2$$

STRUCTURAL EQUATIONS – Example



- **Continuity equation :**

$$F = f_{k1} + f_{k2}$$

- **Compatibility equation :**

$$x = x_1 = x_2$$

- **Input-output relation :**

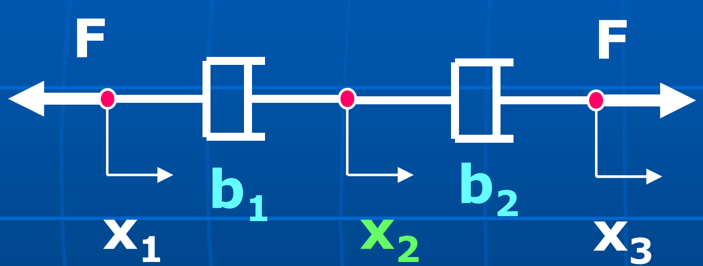
$$F = (k_1 + k_2)x$$

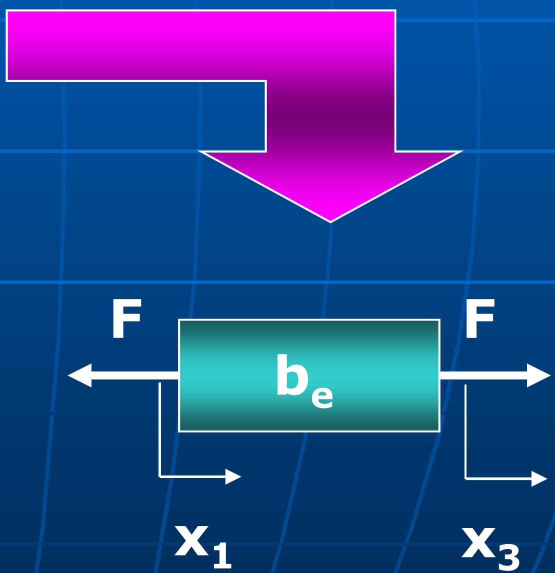
$$F = k_{eq}x \qquad k_{eq} = k_1 + k_2$$

STRUCTURAL EQUATIONS – Example

See solved example A-3-15 in Ogata !

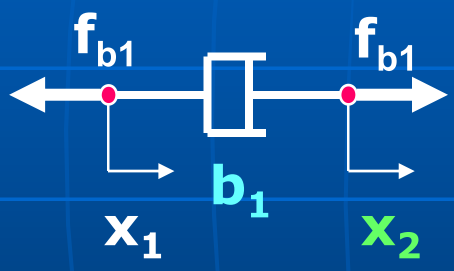
- Replace the two dampers in series with a single equivalent damper.

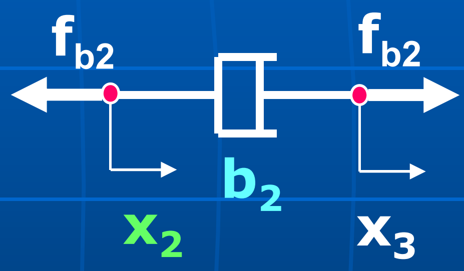




STRUCTURAL EQUATIONS – Example

- **Identify the elements and write the elemental equations :**

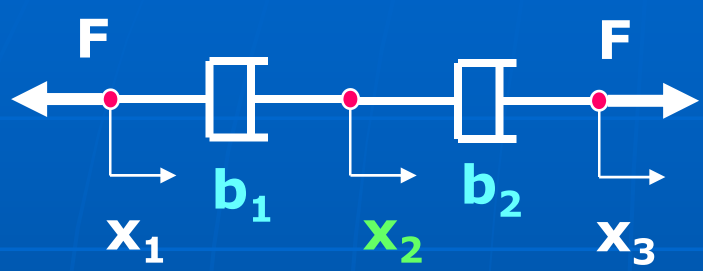


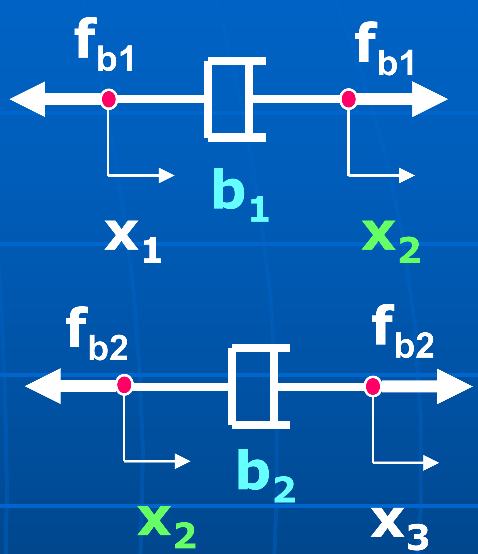


$$f_{b1} = b_1\left(\dot{x}_1 - \dot{x}_2\right)$$

$$f_{b2} = b_2\left(\dot{x}_2 - \dot{x}_3\right)$$

STRUCTURAL EQUATIONS - Example





- **Continuity equation :**

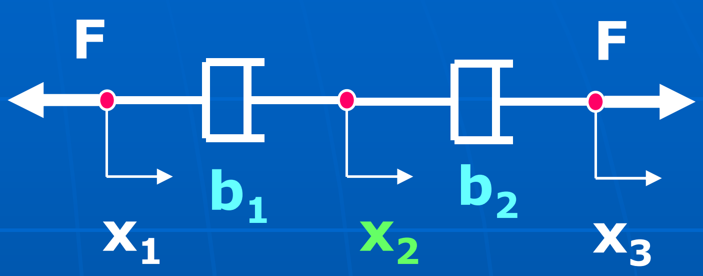
$$F = f_{b1} = f_{b2}$$

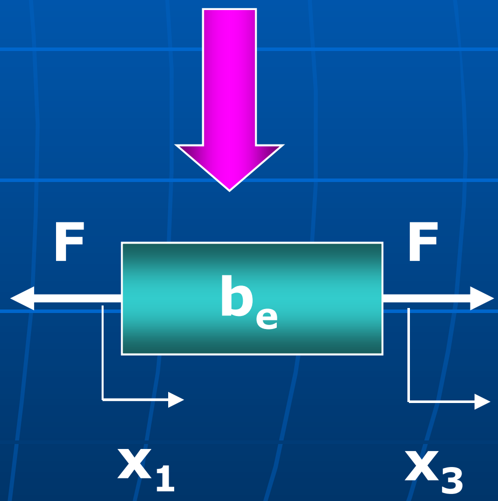
- **Insert elemental equations :**

$$F = b_1\left(\dot{x}_1 - \dot{x}_2\right) = b_2\left(\dot{x}_2 - \dot{x}_3\right)$$

STRUCTURAL EQUATIONS - Example

- From the continuity equation obtain the expression for the intermediate velocity.



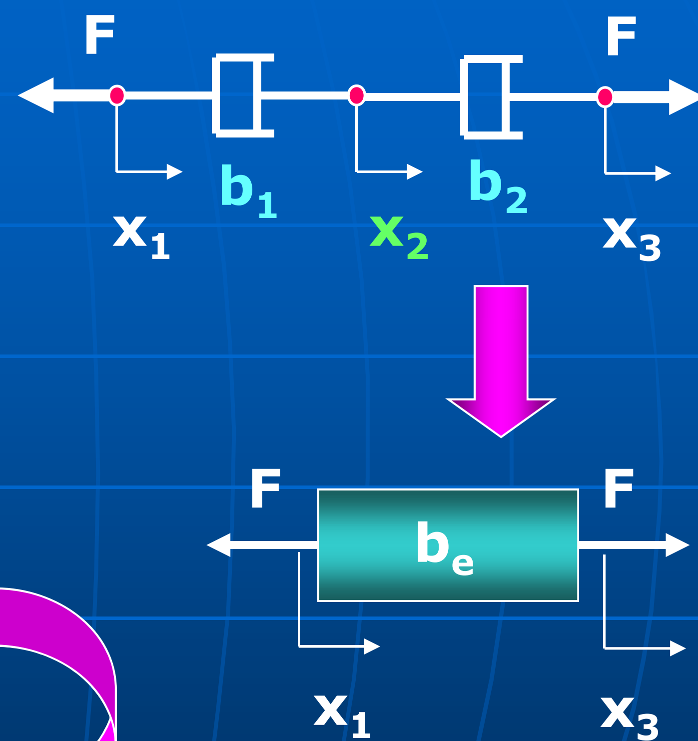


$$F = b_1\left(\dot{x}_1 - \dot{x}_2\right) = b_2\left(\dot{x}_2 - \dot{x}_3\right)$$

$$\dot{x}_2 = \left(\frac{b_1}{b_1 + b_2}\right)\dot{x}_1 + \left(\frac{b_2}{b_1 + b_2}\right)\dot{x}_3$$

STRUCTURAL EQUATIONS - Example

- **Eliminate intermediate velocity to obtain the input-output relation for a single equivalent damper.**



$$\dot{x}_2 = \left(\frac{b_1}{b_1+b_2}\right)\dot{x}_1 + \left(\frac{b_2}{b_1+b_2}\right)\dot{x}_3$$

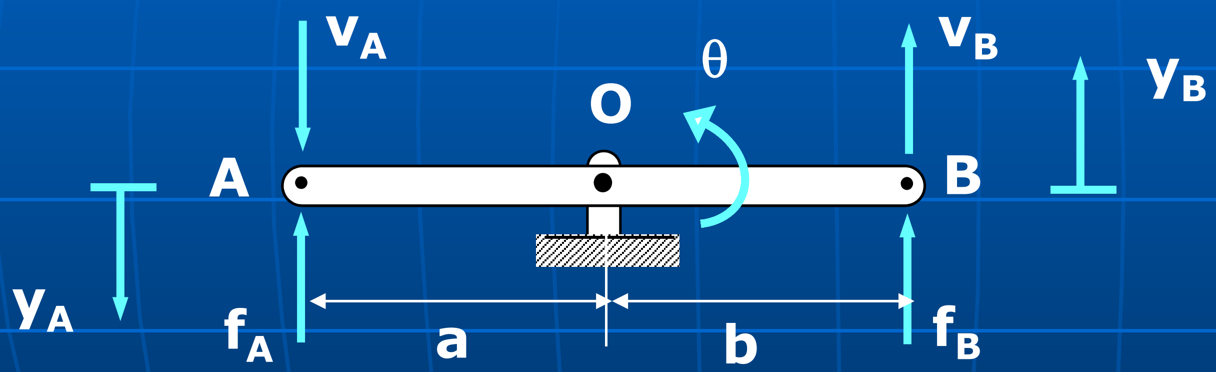
$$F = b_1\left(\dot{x}_1 - \dot{x}_2\right)$$

$$F = \frac{b_1 b_2}{b_1+b_2}\left(\dot{x}_1 - \dot{x}_3\right) = b_{eq}\left(\dot{x}_1 - \dot{x}_3\right)$$

$$b_{eq} = \frac{b_1 b_2}{b_1+b_2}$$

STRUCTURAL EQUATIONS
Transforming Elements - Example

- **Lever (massless)**



Compatibility

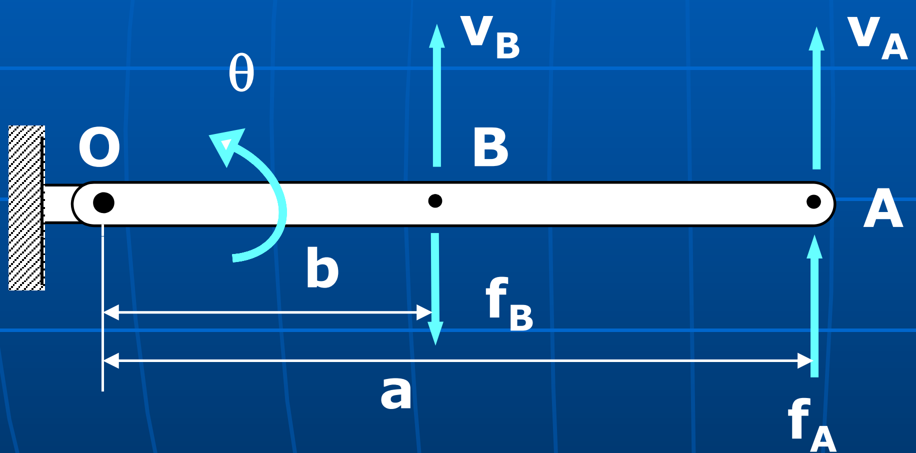
$$\frac{v_A}{v_B} = \frac{a}{b} = \frac{y_A}{y_B}$$

Continuity

$$\frac{f_A}{f_B} = \frac{b}{a}$$

STRUCTURAL EQUATIONS - Example

- **Lever (massless)**



Compatibility

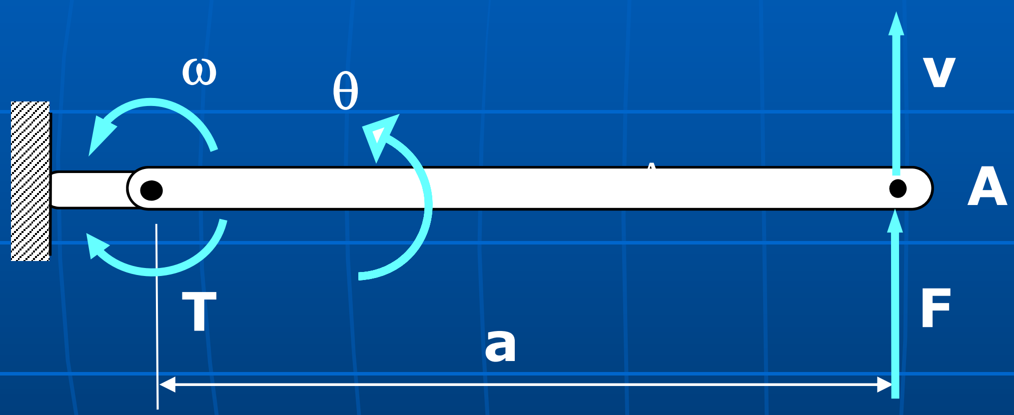
$$\frac{v_A}{v_B} = \frac{a}{b} = \frac{y_A}{y_B}$$

$$\frac{f_A}{f_B} = \frac{b}{a}$$

Continuity

STRUCTURAL EQUATIONS - Example

- **Lever (massless)**



Compatibility

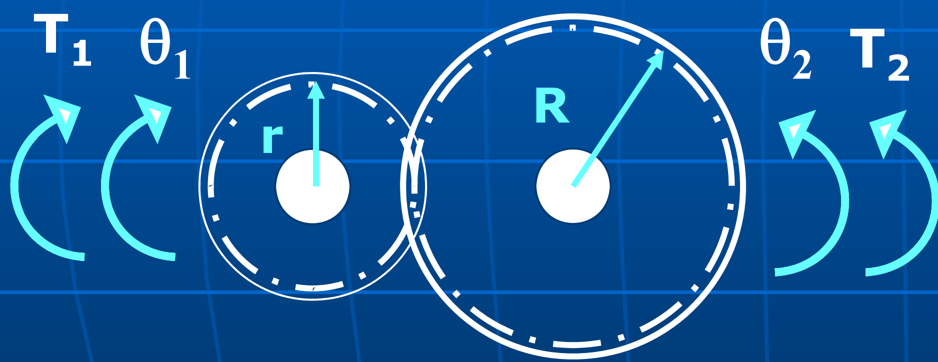
$$\frac{v}{\omega} = \frac{v}{\dot{\theta}} = a$$

$$\frac{F}{T} = \frac{1}{a}$$

Continuity

STRUCTURAL EQUATIONS - Example

- **Geared Systems** n : reduction ratio



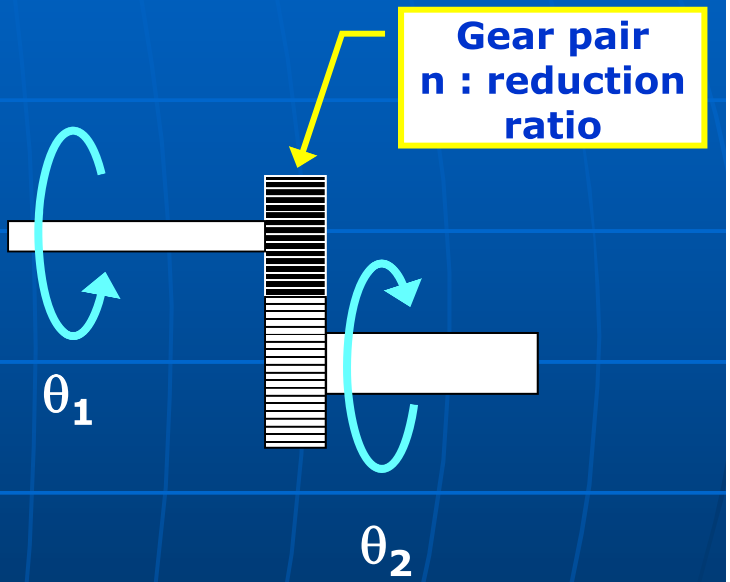
$$n = \frac{\theta_1}{\theta_2} = \frac{R}{r} = \frac{N_2}{N_1}$$

$$\frac{\theta_1}{\theta_2} = \frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{\ddot{\theta}_1}{\ddot{\theta}_2} = \frac{T_2}{T_1} = n$$

- Gear inertias are neglected !

STRUCTURAL EQUATIONS - Example

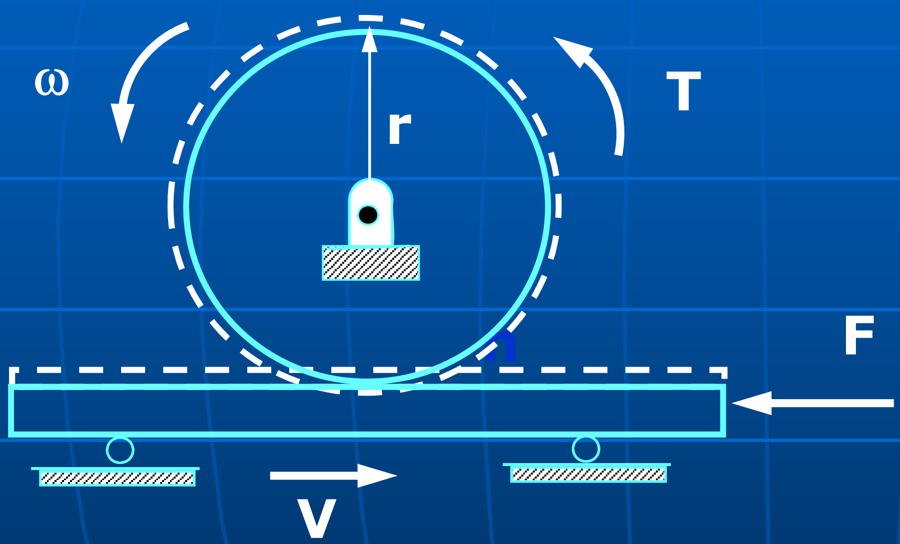
- Write the **compatibility equation**.
- Here the **compatibility equation** establishes the relation between the angular positions (and their derivatives) of the two shafts.



$$\theta_1 = n\theta_2 \qquad \dot{\theta}_1 = n\dot{\theta}_2 \qquad \ddot{\theta}_1 = n\ddot{\theta}_2$$

STRUCTURAL EQUATIONS - Example

- **Rack and Pinion**



$$\frac{V}{\omega} = \frac{T}{F} = r$$

- **Inertias are neglected !**

INPUT-OUTPUT RELATION

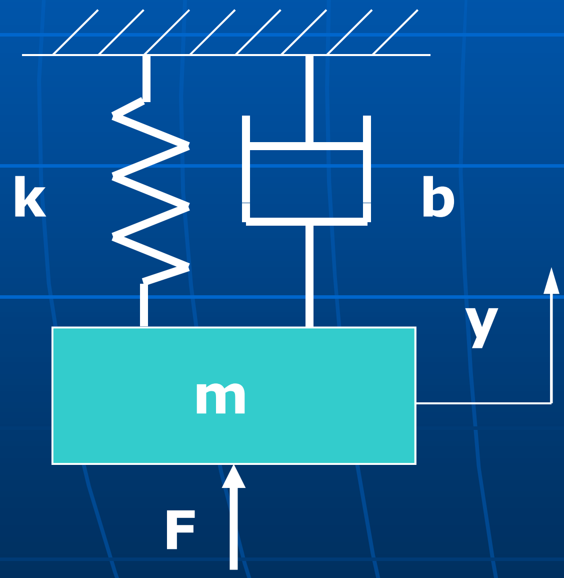
To obtain the input-output relation (equation of motion) for a system :

1. First define input and output.
2. Identify the elements and write the elemental equations.
3. Write the structural equations.
4. Substitute the elemental equations into the continuity equations.
5. Use the compatibility equations to eliminate all variables to leave only the input and output.

EXAMPLE – 1a

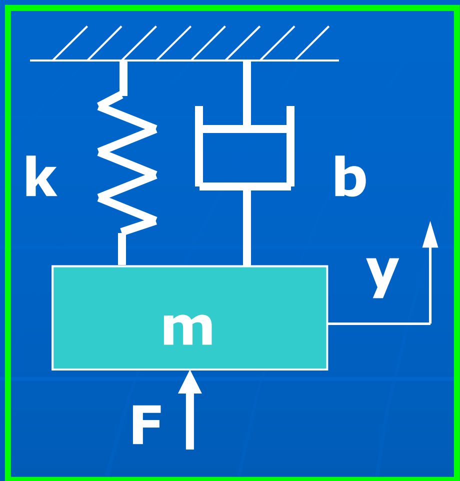
See example 2.16 in Nise !

- Obtain the input-output relation (equation of motion) for the one degree of freedom system shown in the figure.



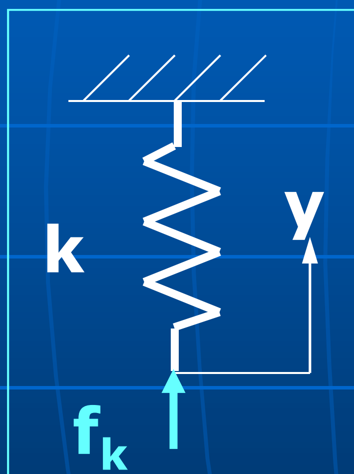
- First define input and output.
  - **Input :** force applied to mass, F
  - **Output :** Displacement of mass, y.

EXAMPLE – 1b



- **Identify the elements of the system and write the elemental equations.**

$$f_k = ky$$

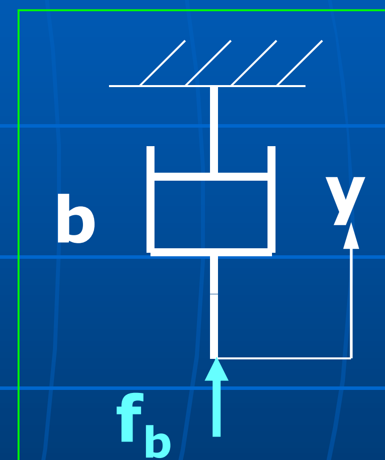


Ideal Spring

Lumped Mass

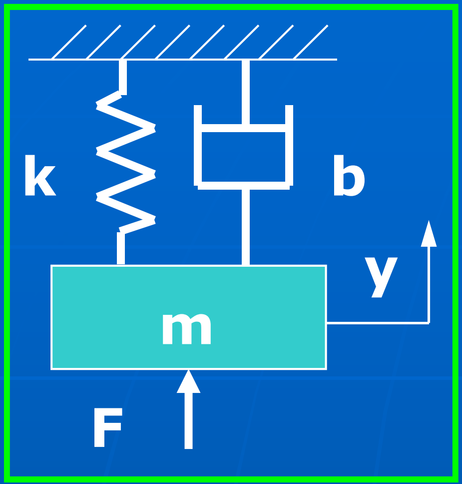
$$f_n = m\ddot{y}$$

$$f_b = b\dot{y}$$

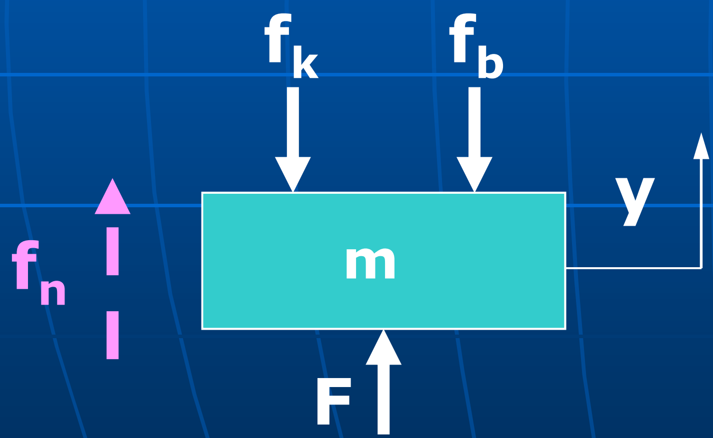


Ideal Damper

EXAMPLE – 1c



- In this system there is one **structural equation**, which is the **continuity equation** represented by the **force balance on the mass**.



$$f_n = F - f_b - f_k$$

$f_n = F - f_b - f_k$

EXAMPLE – 1d

- **Now insert the elemental equations into the structural equation, eliminate f_n, f_k, and f_b to obtain the equation of motion for the system.**

$f_n = m\ddot{y}$

$f_b = b\dot{y}$

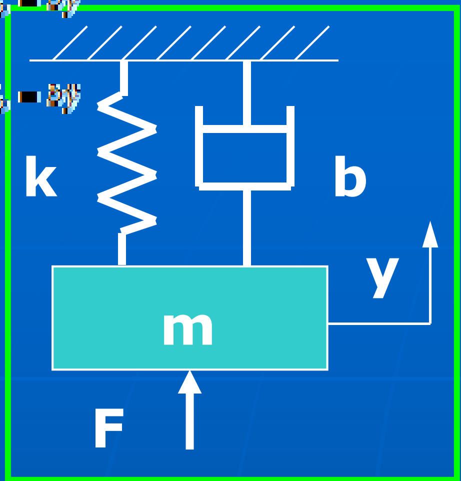
$f_k = ky$

$$m\ddot{y} = F - f_b - f_k$$

$$m\ddot{y} = F - b\dot{y} - ky$$

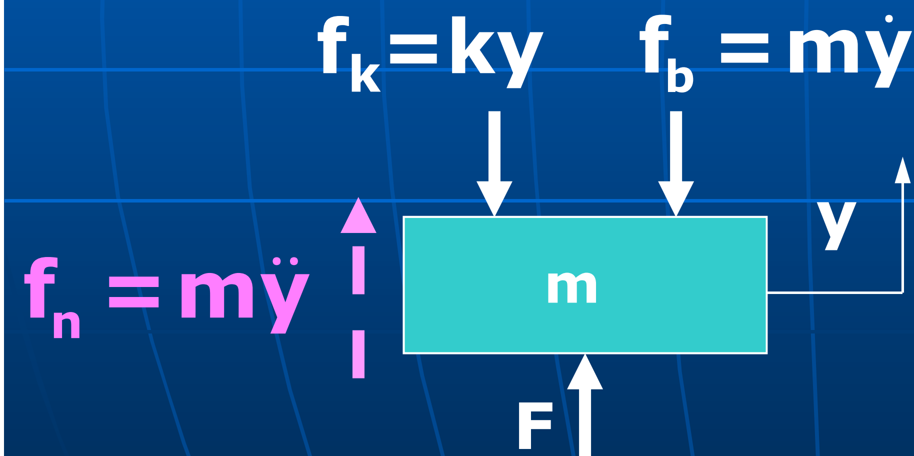
- **Thus, the input-output relation is obtained in the form :**

$$m\ddot{y} + b\dot{y} + ky = F$$



EXAMPLE – 1e

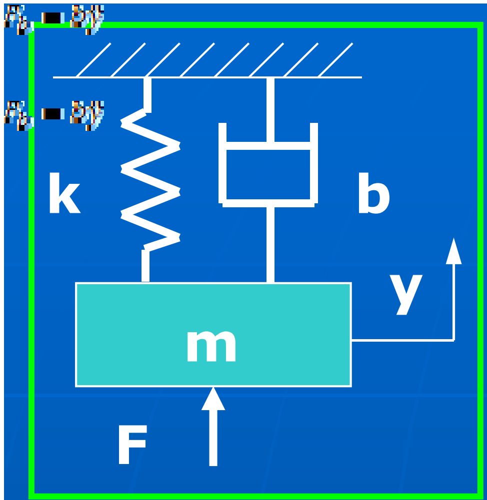
- For this simple example, you can reach the last stage at one step and write the equation directly by inspection.



$$m\ddot{y} = F - b\dot{y} - ky$$

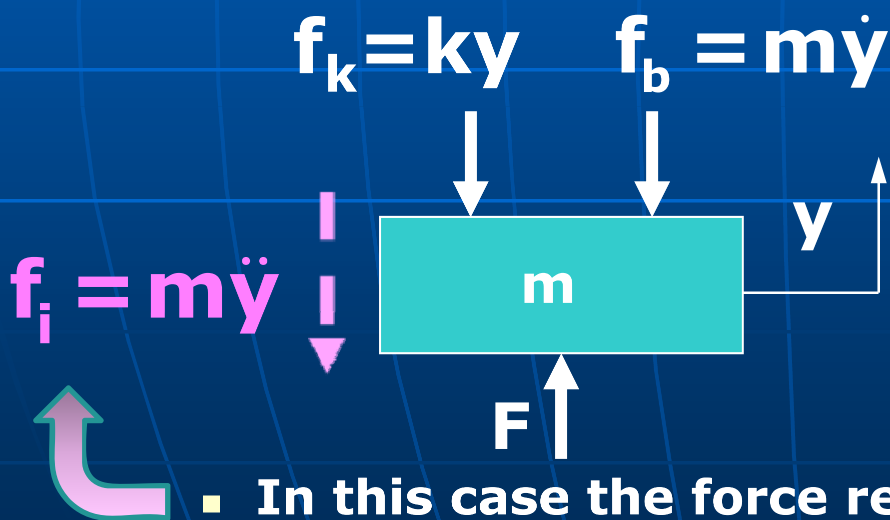
or

$$m\ddot{y} + b\dot{y} + ky = F$$



EXAMPLE – 1e

- Or you may simply argue that the force balance requires the applied force F causing the motion to be equal to the sum of all resistances.

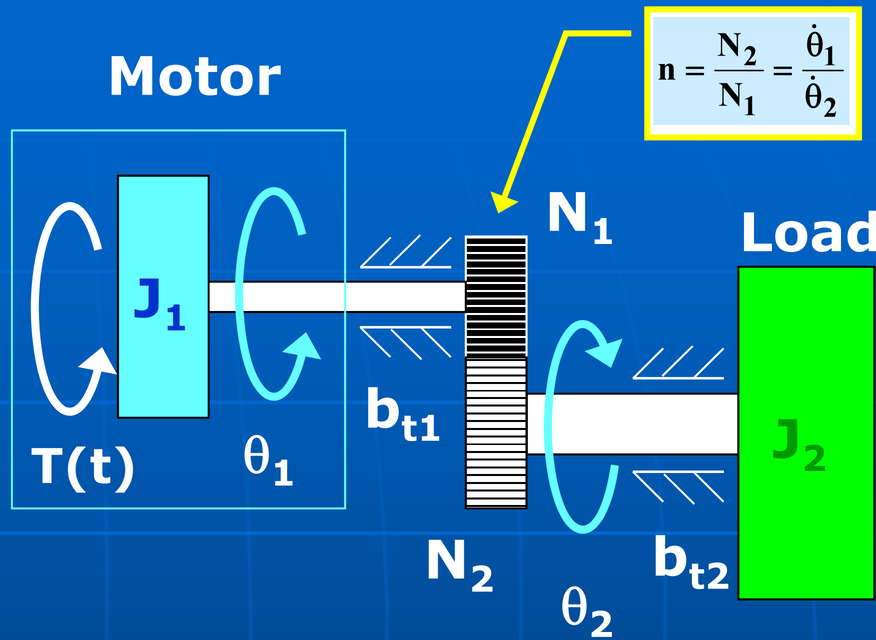


$$m\ddot{y} + b\dot{y} + ky = F$$

- In this case the force required to accelerate the mass is considered to be a resistance – note change of direction !

EXAMPLE – 2a

- A motor drives a load inertia through a gear pair and massless rigid shafts. Viscous damping is assumed at the bearings.
- Obtain the relation between **motor torque** (input) and **load shaft position** (output).



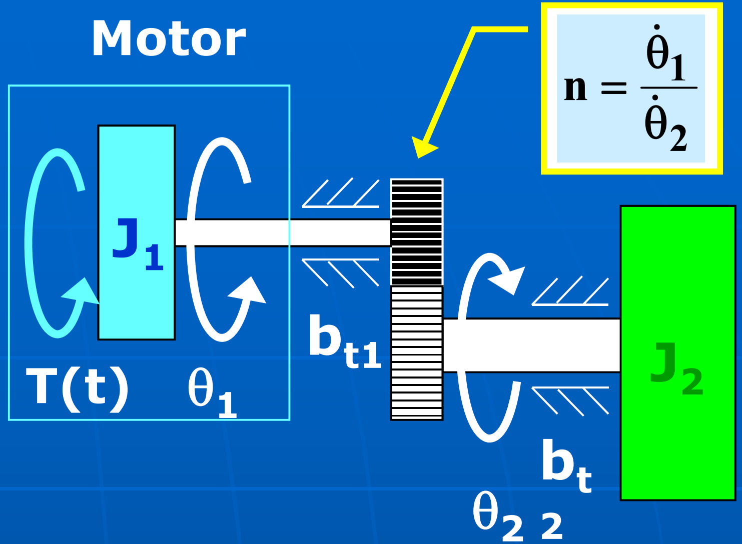
See section 2.7 in Nise and example problem A-5-3 in Ogata for different versions.

J_i : mass moment of inertia,
b_{ti} : viscous friction at bearing i,
$T(t)$: motor torque,
θ_i : rotational position of shaft i,
n : reduction ratio,
N : number of teeth on gear .

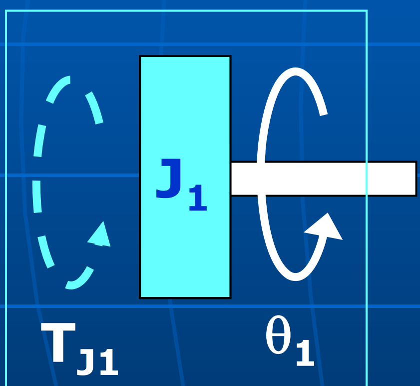
EXAMPLE – 2b

- **Identify the elements and write the elemental equations.**

$$n = \frac{\dot{\theta}_1}{\dot{\theta}_2}$$

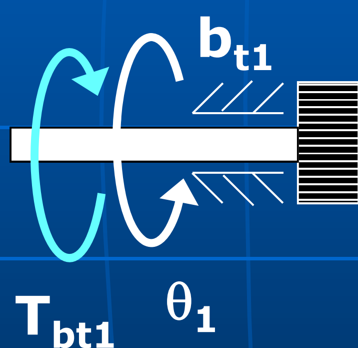


Motor Inertia



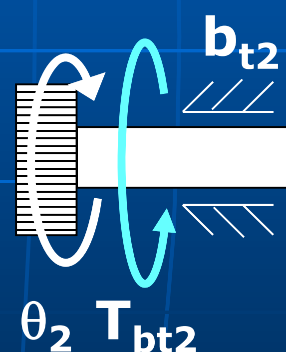
$$T_{J1} = J_1\ddot{\theta}_1$$

Viscous friction



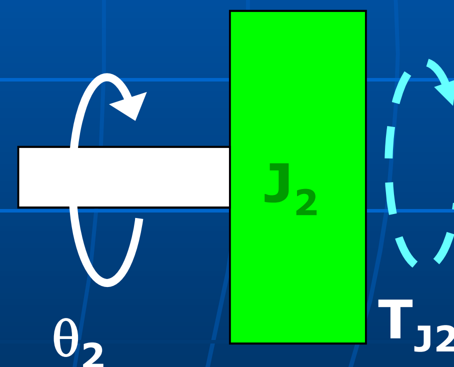
$$T_{bt1} = b_{t1}\dot{\theta}_1$$

Viscous friction



$$T_{bt2} = b_{t2}\dot{\theta}_2$$

Load Inertia



$$T_{J2} = J_2\ddot{\theta}_2$$

EXAMPLE – 2c

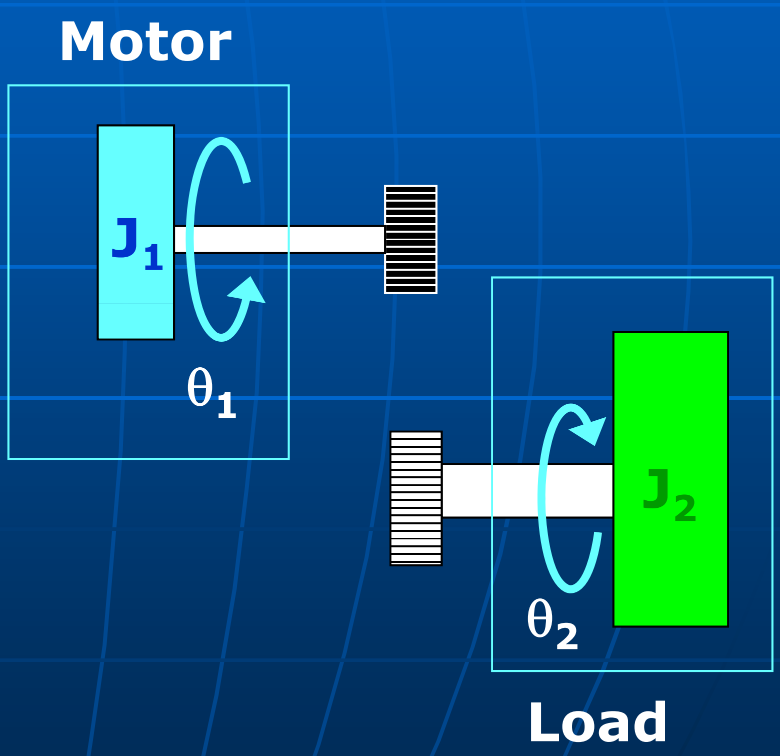
- To write the **structural equations**, the system is divided into two parts at the gear pair.

- **Compatibility equation (gear pair):**

$$\dot{\theta}_1 = n\dot{\theta}_2$$

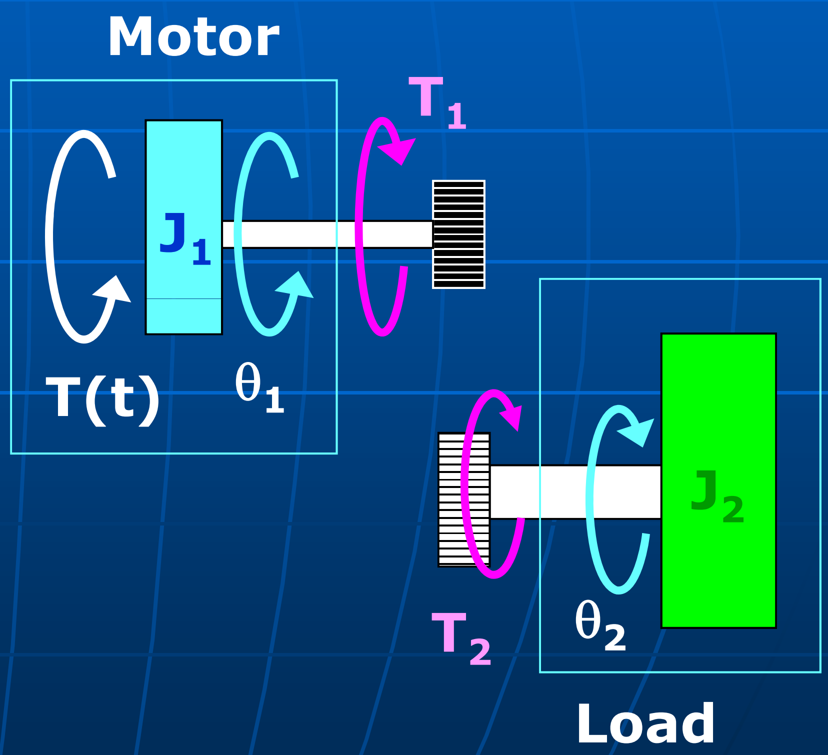
and/or

$$\ddot{\theta}_1 = n\ddot{\theta}_2$$



EXAMPLE – 2d

- To write the **continuity** equations, the torque reactions on both sides must be introduced.



- T_1 : Load (resistance) torque on the pinion.
- T_2 : Drive torque on the gear.

EXAMPLE – 2e

3 continuity equations:

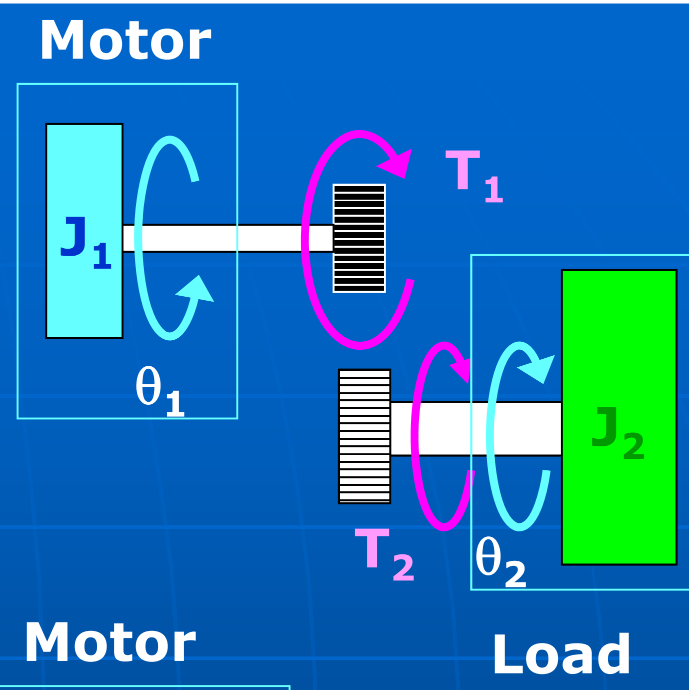
- **Torque balance on the gear pair :** $T_2 = nT_1$

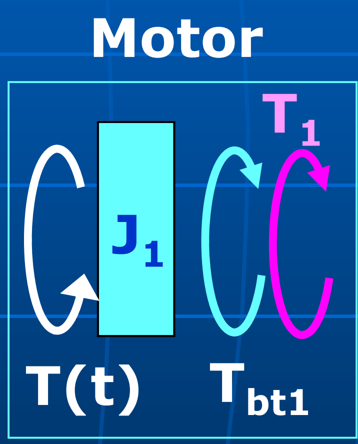
- **Net torque acting on motor inertia :**

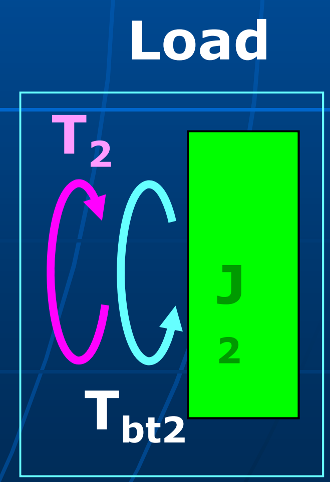
$$T_{J1} = T(t) - T_1 - T_{bt1}$$

- **Net torque acting on load inertia :**

$$T_{J2} = T_2 - T_{bt2}$$







EXAMPLE – 2f

- Note that T_1 and T_2 appearing in the continuity equations are internal reaction torques and are of no interest at this point. Thus eliminate them using the three continuity equations.

$$T_{J1} = T(t) - T_1 - T_{bt1} \quad \Rightarrow \quad T_1 = T(t) - T_{J1} - T_{bt1}$$

$$T_2 = nT_1$$

$$T_{J2} = T_2 - T_{bt2} \quad \Rightarrow \quad T_2 = T_{J2} + T_{bt2}$$

$$T_{J2} + T_{bt2} = n\left[T(t) - T_{J1} - T_{bt1}\right]$$

EXAMPLE – 2g

- Now, insert the elemental equations into the combined continuity equation,

$$T_{J2} = J_2\ddot{\theta}_2$$

$$T_{J1} = J_1\ddot{\theta}_1$$

$$T_{J2} + T_{bt2} = n\left[T(t) - T_{J1} - T_{bt1}\right]$$

$$T_{bt2} = b_{t2}\dot{\theta}_2$$

$$T_{bt1} = b_{t1}\dot{\theta}_1$$

$$J_2\ddot{\theta}_2 + b_{t2}\dot{\theta}_2 = n\left[T(t) - J_1\ddot{\theta}_1 - b_{t1}\dot{\theta}_1\right]$$

EXAMPLE – 2h

$$J_2\ddot{\theta}_2 + b_{t2}\dot{\theta}_2 = n\left[T(t) - J_1\ddot{\theta}_1 - b_{t1}\dot{\theta}_1\right]$$

$$\dot{\theta}_1 = n\dot{\theta}_2$$

$$\ddot{\theta}_1 = n\ddot{\theta}_2$$

- **Finally use the compatibility equations to eliminate θ_1 and its derivatives to obtain the input-output relation.**

$$J_2\ddot{\theta}_2 + b_{t2}\dot{\theta}_2 = n\left[T(t) - J_1\left(n\ddot{\theta}_2\right) - b_{t1}\left(n\dot{\theta}_2\right)\right]$$

$$\left(n^2 J_1 + J_2\right)\ddot{\theta}_2 + \left(n^2 b_{t1} + b_{t2}\right)\dot{\theta}_2 = nT(t)$$

Reading

- **Nise – Sections 2.5, 2.6, 2.7, 2.10 and 2.11 (Modeling only)**
- **Ogata – Section 3.1, Example problems A-3-14, 15, and 16**