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On the Orthogonality of q-Classical Polynomials of the Hahn Class

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Abstract. The central idea behind this review article is to discuss in a unified sense the orthogonality of all possible polynomial solutions of the q-hypergeometric difference equation on a q-linear lattice by means of a qualitative analysis of the q-Pearson equation. To be more specific, a geometrical approach has been used by taking into account every possible rational form of the polynomial coefficients in the q-Pearson equation, together with various relative positions of their zeros, to describe a desired q-weight function supported on a suitable set of points. Therefore, our method differs from the standard ones which are based on the Favard theorem, the three-term recurrence relation and the difference equation of hypergeometric type. Our approach enables us to extend the orthogonality relations for some well-known q-polynomials of the Hahn class to a larger set of their parameters.

Key words: q-polynomials; orthogonal polynomials on q-linear lattices; q-Hahn class

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1 Introduction

The so-called q-polynomials are of great interest inside the class of special functions since they play an important role in the treatment of several problems such as Eulerian series and continued fractions [8, 15], q-algebras and quantum groups [23, 24, 33] and q-oscillators [4, 10, 18], and references therein, among others.

A q-analog of the Chebychev's discrete orthogonal polynomials is due to Markov in 1884 [7, p. 43], which can be regarded as the first example of a q-polynomial family. In 1949, Hahn introduced the q-Hahn class [19] including the big q-Jacobi polynomials, on the exponential lattice although he did not use this terminology. In fact, he did not give the orthogonality relations of the big q-Jacobi polynomials in [19] which was done by Andrew and Askey [7]. During the last decades the q-polynomials have been studied by many authors from different points of view. There are two most recognized approaches. The first approach, initiated by the work of Askey and Wilson [9] (see also Andrews and Askey [7]) is based on the basic hypergeometric series [8, 17]. The second approach is due to Nikiforov and Uvarov [29, 31] and uses the analysis of difference equations on non-uniform lattices. The readers are also referred to the surveys [11, 28, 30, 32]. These approaches are associated with the so-called q-Askey scheme [21] and the Nikiforov–Uvarov scheme [30], respectively. Another approach was published in [27] where the authors proved several characterizations of the q-polynomials starting from the so-called distributional q-Pearson equation (for the non q-case see, e.g., [16, 26] and references therein).

In particular, in [27] a classification of all possible families of orthogonal polynomials on the exponential lattice was established, and latter on in [5] the comparison with the q-Askey and Nikiforov–Uvarov schemes was done, resulting in two new families of orthogonal polynomials. Furthermore, an important contribution to the theory of (orthogonal) q-polynomials, and in particular, to the theory of orthogonal q-polynomials on the linear exponential lattice, appeared in the recent book [21]. The corresponding table is generally called the q-Hahn tableau (see, e.g., Koornwinder [23]). The q-polynomials belonging to this class are the solutions of the q-difference equation of hypergeometric type (q-EHT) [19]

$$\sigma_1(x;q)D_{q^{-1}}D_q y(x,q)+\tau(x,q)D_q y(x,q)+\lambda(q)y(x,q)=0. \tag{1.1}$$

One way of deriving the q-EHT (1.1) whose bounded solutions are the q-polynomials of the Hahn class, is to discretize the classical differential equation of hypergeometric type (EHT)

$$\sigma(x)y''+\tau(x)y'+\lambda y=0, \tag{1.2}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree, respectively, and λ is a constant [2, 11, 26, 28, 31]. To this end, we can use the approximations (see e.g. [31, § 13, p. 142])

$$y'(x)\sim\frac{1}{1+q}[D_q y(x)+qD_{q^{-1}}y(x)] \qquad \text{and} \qquad y''(x)\sim\frac{2q}{1+q}D_qD_{q^{-1}}y(x) \qquad \text{as} \quad q\to 1$$

for the derivatives in (1.2), where we use the standard notation for the q and q^{-1}-Jackson derivatives of $y(x)$ [17, 20], i.e.,

$$D_\zeta y(x)=\frac{y(x)-y(\zeta x)}{(1-\zeta)x}, \qquad \zeta\in\mathbb{C}\setminus\{0,\pm1\}$$

for $x\neq 0$ and $D_\zeta y(0)=y'(0)$, provided that $y'(0)$ exists. This leads to the q-EHT (1.1) where

$$\sigma_1(x;q):=\frac{2}{1+q}\left[\sigma(x)-\frac{1}{2}(q-1)x\tau(x)\right],$$
$$\tau(x,q):=\tau(x), \qquad \lambda(q):=\lambda, \qquad y(x,q):=y(x).$$

Notice here the relations $D_q=D_{q^{-1}}+(q-1)xD_qD_{q^{-1}}$ and $D_qD_{q^{-1}}=q^{-1}D_{q^{-1}}D_q$ so that (1.1) can be rewritten in the equivalent form

$$\sigma_2(x;q)D_qD_{q^{-1}}y(x,q)+\tau(x,q)D_{q^{-1}}y(x,q)+\lambda(q)y(x,q)=0, \tag{1.3}$$

where

$$\sigma_2(x,q):=q\left[\sigma_1(x,q)+\left(1-q^{-1}\right)x\tau(x,q)\right]. \tag{1.4}$$

It should be noted that the q-EHT (1.1) and (1.3) correspond to the second order linear difference equations of hypergeometric type on the linear exponential lattices $x(s)=c_1q^s+c_2$ and $x(s)=c_1q^{-s}+c_2$, respectively [2, 11, 28].

Notice also that (1.1) (or (1.3)) can be written in a very convenient form [5, 21, 22]

$$\sigma_2(x,q)D_q y(x,q)-q\sigma_1(x,q)D_{q^{-1}}y(x,q)+(q-1)x\lambda(q)y(x,q)=0,$$

where the coefficients $\sigma_1(x;q)$ and $\sigma_2(x;q)$ are polynomials of at most 2nd degree and $\tau(x,q)$ is a 1st degree polynomial in x.

Notice that the q-EHT (1.1) can be written in the self-adjoint form

$$D_q\left[\sigma_1(x,q)\rho(x,q)D_{q^{-1}}y(x)\right] + q^{-1}\lambda(q)\rho(x,q)y(x) = 0,$$

where ρ is a function satisfying the so-called q-Pearson equation $D_q\left[\sigma_1(x,q)\rho\right] = q^{-1}\tau(x,q)\rho$ that can be written as

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(qx,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}, \tag{1.5}$$

or, equivalently,

$$\sigma_2(x,q)\rho(x,q) = q\sigma_1(qx,q)\rho(qx,q). \tag{1.6}$$

In this paper we study, without loss of generality, the q-EHT (1.1), assume $0 < q < 1$ and take $\lambda(q)$ as

$$\lambda(q) := \lambda_n(q) = -[n]_q\left[\tau'(0,q) + \frac{1}{2}[n-1]_{q^{-1}}\sigma_1''(0,q)\right], \qquad n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$$

since we are interested only in the polynomial solutions [2, 11, 28]. For more details on the q-polynomials of the q-Hahn tableau we refer the readers to the works [1, 2, 3, 5, 11, 13, 21, 23, 25, 28, 29, 30, 31, 32], and references therein.

In this paper, we deal with the orthogonality properties of the q-polynomials of the q-Hahn tableau from a different viewpoint than the one used in [21]. In [21], the authors presented a unified study of the orthogonality of q-polynomials based on the Favard theorem. Here, the main idea is to provide a relatively simple geometrical analysis of the q-Pearson equation by taking into account every possible rational form of the polynomial coefficients of the q-difference equation. Roughly, our qualitative analysis is concerned with the examination of the behavior of the graphs of the ratio $\rho(qx,q)/\rho(x,q)$ by means of the definite right hand side (r.h.s.) of (1.5) in order to find out a suitable q-weight function. Such a qualitative analysis implies all possible orthogonality relations among the polynomial solutions of the q-difference equation in question. Moreover, it allows us to extend the orthogonality relations for some well-known q-polynomials of the Hahn class to a larger set of their parameters (see Sections 4.1 and 4.2). A first attempt of using a geometrical approach for studying the orthogonality of q-polynomials of the q-Hahn class was presented in [13]. However, the study is far from being complete and only some partial results were obtained. We will fill this gap in this review paper.

Our main goal is to study each orthogonal polynomial system or sequence (OPS), which is orthogonal with respect to (w.r.t.) a q-weight function $\rho(x,q) > 0$ satisfying the q-Pearson equation as well as certain boundary conditions (BCs) to be introduced in Section 2. For each family of polynomial solutions of (1.1) we search for the ones that are orthogonal in a suitable intervals depending on the range of the parameters coming from the coefficients of (1.1) and the corresponding q-Pearson equation. Hence, in Section 2, we present the candidate intervals by inspecting the BCs as well as some preliminary results. Theorems which help to calculate q-weight functions are given in Section 3. Section 4 deals with the qualitative analysis including the theorems stating the main results of our article. The last section concludes the paper with some final remarks.

2 The orthogonality and preliminary results

We first introduce the so-called q-Jackson integrals and afterward a well known theorem for the orthogonality of polynomial solutions of (1.1) in order to make the article self-contained [2, 12, 28].

The q-Jackson integrals for $q \in (0,1)$ [17, 20] are defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_{j=0}^{\infty} q^j f(q^j a) \qquad \text{and}$$
$$\int_a^0 f(x)d_qx = (1-q)(-a)\sum_{j=0}^{\infty} q^j f(q^j a) \tag{2.1}$$

if $a > 0$ and $a < 0$, respectively. Therefore, we have

$$\int_a^b f(x)d_qx := \int_0^b f(x)d_qx - \int_0^a f(x)d_qx \qquad \text{and}$$
$$\int_a^b f(x)d_qx := \int_a^0 f(x)d_qx + \int_0^b f(x)d_qx, \tag{2.2}$$

when $0 < a < b$ and $a < 0 < b$, respectively. Furthermore, we make use of the *improper* q-Jackson integrals

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j f(q^j) \qquad \text{and}$$
$$\int_{-\infty}^{\infty} f(x)d_qx = (1-q)\sum_{j=-\infty}^{\infty} q^j[f(q^j)+f(-q^j)], \tag{2.3}$$

where the second one is sometimes called the *bilateral* q-integral. The q^{-1}-*Jackson integrals* are defined similarly. For instance, the improper q^{-1}-*Jackson integral* on (a,∞) is given by

$$\int_a^{\infty} f(x)d_{q^{-1}}x = \big(q^{-1}-1\big)a\sum_{j=0}^{\infty} q^{-j} f\big(q^{-j}a\big), \qquad a > 0 \tag{2.4}$$

provided that $\lim_{j\to\infty} q^{-j}f(q^{-j}a) = 0$ and the series is convergent.

Theorem 2.1. *Let ρ be a function satisfying the q-Pearson equation (1.5) in such a way that the BCs*

$$\sigma_1(x,q)\rho(x,q)x^k\big|_{x=a,b} = \sigma_2\big(q^{-1}x,q\big)\rho\big(q^{-1}x,q\big)x^k\big|_{x=a,b} = 0, \qquad k \in \mathbb{N}_0 \tag{2.5}$$

also hold. Then the sequence $\{P_n(x,q)\}$ of polynomial solutions of (1.1) are orthogonal on (a,b) w.r.t. $\rho(x,q)$ in the sense that

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.6}$$

where $d_n(q)$ and δ_{mn} denote the norm of the polynomials P_n and the Kronecker delta, respectively. Analogously, if the conditions

$$\sigma_2(x,q)\rho(x,q)x^k\big|_{x=a,b} = \sigma_1(qx,q)\rho(qx,q)x^k\big|_{x=a,b} = 0, \qquad k \in \mathbb{N}_0 \tag{2.7}$$

are fulfilled, the q-polynomials then satisfy the relation

$$\int_a^b P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}. \tag{2.8}$$

Remark 2.2. The relation (2.6) means that the polynomials $P_n(x, q)$ are orthogonal with respect to a measure supported on the set of points $\{q^k a\}_{k\in\mathbb{N}_0}$ and $\{q^k b\}_{k\in\mathbb{N}_0}$. Since we are interested in the positive definite cases, i.e., when $\rho(x, q) > 0$, then,

- when $a = 0$, the measure is supported on the set of points $\{q^k b\}_{k\in\mathbb{N}_0}$ in $(0, b]$;
- when $a > 0$, the measure should be supported on the finite set of points $\{q^k b\}_{k=0}^{N}$ being $a = q^{N+1}b$;
- when $a < 0$, the measure is supported on the set $\{q^k a\}_{k\in\mathbb{N}_0} \bigcup \{q^k b\}_{k\in\mathbb{N}_0}$ in $[a, 0) \bigcup (0, b]$.

A similar analysis can be done for the relation (2.8).

According to Theorem 2.1, we have to determine an interval (a, b) in which ρ is q-integrable and $\rho > 0$ on the lattice points of the types $\alpha q^{\pm k}$ and $\beta q^{\pm k}$ for $k \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$. Such a weight function will be a solution of the q-Pearson equation (1.5). To this end, a qualitative analysis of the q-Pearson equation is presented by a detailed inspection of the r.h.s. of (1.5). Note that the r.h.s. of (1.5) consists of the polynomial coefficients σ_1 and σ_2 of the q-EHT which can be made definite for possible forms of the coefficients. As a result, the possible behavior of ρ on the left hand side (l.h.s.) of (1.5) and the candidate intervals can be obtained accordingly.

OPSs on finite (a, b) intervals

First assume that (a, b) denotes a finite interval. We list the following possibilities for finding ρ which obeys the BCs in (2.5) or in (2.7).

PI. This is the simplest case where σ_1 vanishes at both $x = a$ and b, i.e., $\sigma_1(a, q) = \sigma_1(b, q) = 0$. Using (1.5) rewritten of the form

$$\rho\big(q^{-1}x, q\big) = \frac{q\sigma_1(x, q)}{\sigma_2(q^{-1}x, q)}\rho(x, q) \tag{2.9}$$

we see that the function $\rho(x, q)$ becomes zero at the points $q^{-k}a$ and $q^{-k}b$ for $k \in \mathbb{N}$. However, we have to take into consideration three different situations.

(i) Let $a < 0 < b$. Since the points $q^{-k}a$ and $q^{-k}b$ lie outside the interval (a, b) and BCs are fulfilled at $x = a$ and b, there could be an OPS w.r.t. a measure supported on the union of the set of points $\{aq^k\}_{k\in\mathbb{N}_0}$ and $\{bq^k\}_{k\in\mathbb{N}_0}$ in $[a, 0) \cup (0, b]$, if ρ is positive.

(ii) Let $0 < a < b$. In this case $\rho(x, q)$ vanishes at the points $q^{-k}a$ in (a, b) and $q^{-k}b$ out of (a, b). Then, the only possibility to have an OPS on $(a, b]$ depends on the existence of N such that $q^{N+1}b = a$. This condition, however, implies that $bq^k = aq^{-(N-k)}$ and that ρ vanishes at bq^k for $k = 0, 1, \ldots, N$, and, therefore, it must be rejected. The similar statement is true when $a < b < 0$, which can be obtained by a simple linear scaling transformation so that it does not represent an independent case.

(iii) Let $a = 0 < b$ (or, $a < b = 0$). This case is much more involved. First of all, if $a = 0$ is a zero of $\sigma_1(x, q)$ then it is a zero of $\sigma_2(x, q)$ as well, both containing a factor x. Therefore, the r.h.s. of q-Pearson equation (1.5) can be simplified and **PI**(i)-(ii) are not valid anymore. In fact, in this case an OPS w.r.t. a measure supported on the set of points $\{bq^k\}_{k\in\mathbb{N}_0}$ in $(0, b]$ can be defined.

PII. The relation in (1.6) suggest an alternative possibility to define an OPS on (a, b). Namely, if $q^{-1}a$ and $q^{-1}b$ are both zeros of $\sigma_2(x, q)$, by using (1.5) rewritten of the form

$$\rho(qx, q) = \frac{q^{-1}\sigma_2(x, q)}{\sigma_1(qx, q)}\rho(x, q), \tag{2.10}$$

it follows that $\rho(x,q)$ vanishes at the points $q^k a$ and $q^k b$ for $k \in \mathbb{N}_0$. Then two different situations appear depending on whether $a < 0 < b$ or $0 < a < b$. In the first case, $\rho(x,q) = 0$ at the points $q^k a$ and $q^k b$ for $k \in \mathbb{N}_0$ in $[a,b]$, which is not interesting. In the second case, the $q^k b$ are in $[a,b]$ whereas the $q^k a$ remain out of $[a,b]$, so that we could have an OPS if there exists N such that $q^{-N-1}a = q^{-1}b$. However, since $q^{-k}a = q^{N-k}b$, ρ vanishes at the $q^{-k}a$ which are in $[a,b]$ as well.

PIII. Let $q^{-1}a$ and b be the roots of σ_2 and σ_1, respectively. Then we see, from (2.9) and (2.10), that $\rho = 0$ at $x = q^{-k}b$ for $k \in \mathbb{N}$ and at $x = q^k a$ for $k \in \mathbb{N}_0$. That is, if $a < 0 < b$, $\rho = 0$ on $x \in (a, 0)$ and, therefore, an OPS can not be constructed on (a,b) unless $a \to 0^-$. In this limiting case of $x \in (0, b]$, it can be possible to introduce a desired weight function supported on the set $\{bq^k\}_{k\in\mathbb{N}_0}$. If $0 < a < b$, on the other hand, ρ vanishes for $x < a$ and $x > b$. Thus there could be an OPS w.r.t. a measure supported on the finite set of points $\{q^k b\}_{k=0}^N$ provided that $q^{N+1}b = a$ for some finite N integer. Alternatively, we can define an equivalent OPS w.r.t. a measure supported on the equivalent finite set of points $\{q^{-k}a\}_{k=0}^N$ provided now that $q^{-N-1}a = b$, where N is a finite integer. Note that in the limiting case of $a \to 0^+$ the set of points $\{q^k b\}_{k\in\mathbb{N}_0}$ becomes infinity.

PIV. Assume that a and $q^{-1}b$ are the roots of $\sigma_1(x,q)$ and $\sigma_2(x,q)$, respectively. Then, from (2.9) and (2.10), it follows that $\rho(x,q)$ vanishes at the points $q^{-k}a$, $k \in \mathbb{N}$ and $q^k b$, $k \in \mathbb{N}_0$. So, if $a < 0 < b$, it is not possible to find a weight function satisfying the BCs. Nevertheless, as in **PIII**, in the limiting case of $b \to 0^+$ an OPS w.r.t. a measure supported at the points $q^k a$, $k \in \mathbb{N}_0$ in $[a, 0)$ can be constructed. If $0 < a < b$, there is no possibility to introduce an OPS. Note that when $a = 0 < b$, an OPS also does not exist.

OPSs on infinite intervals

Assume now that (a,b) is an infinite interval. Without any loss of generality, let a be a finite number and $b \to \infty$. In fact, the system on the infinite interval $(-\infty, b)$ is not independent which may be transformed into (a,∞) on replacing x by $-x$. Obviously one BC in (2.5) reads as

$$\lim_{b\to\infty} \sigma_1(b,q)\rho(b,q)b^k = 0 \qquad \text{or} \qquad \lim_{b\to\infty} \sigma_2(b,q)\rho(b,q)b^k = 0, \qquad k \in \mathbb{N}_0,$$

and there are additional cases for $x = a$.

PV. If $x = a \neq 0$ is root of $\sigma_1(x,q)$ then, from (2.9), $\rho(x,q)$ vanishes at the points $q^{-k}a$ for $k \in \mathbb{N}$ which are interior points of (a,∞) when $a > 0$. Therefore there is no OPS on (a,∞) for $a > 0$. If $a < 0$ we can find a q-weight function in $[a, 0) \cup (0,\infty)$ supported on the union of the sets $\{q^k a\}_{k\in\mathbb{N}_0}$ and $\{q^{\pm k}\alpha\}_{k\in\mathbb{N}_0}$ for arbitrary $\alpha > 0$ where α can be taken as unity. If $a = 0$, on the other hand, then a weight function in $(0,\infty)$ can be defined at the points $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$.

PVI. If $x = q^{-1}a$ is a root of $\sigma_2(x,q)$, as we have already discussed, ρ is zero at $q^k a$ for $k \in \mathbb{N}_0$. Therefore, for $a > 0$ a q-weight function can exist in $(q^{-1}a, \infty)$ supported on the set of points $\{q^{-k}a\}_{k\in\mathbb{N}}$. An OPS does not exist if $a < 0$. Finally, if $a = 0$ it is possible to find a ρ on $(0,\infty)$ supported at the points $q^{\pm k}\alpha$ for arbitrary $\alpha > 0$ and $k \in \mathbb{N}_0$.

PVII. Finally, we consider the possibility of satisfying the BC

$$\lim_{a\to-\infty} \sigma_1(a,q)\rho(a,q)a^k = 0$$

in the limiting case as $a \to -\infty$. If this condition holds a weight function and, hence, an OPS w.r.t. a measure supported on the set of points $\{\pm q^{\pm k}\alpha\}_{k\in\mathbb{N}_0}$, for arbitrary $\alpha > 0$, can be defined.

The aforementioned considerations are expressible as a theorem.

Theorem 2.3. *Let $a_1(q)$, $b_1(q)$ and $a_2(q)$, $b_2(q)$ denote the zeros of $\sigma_1(x,q)$ and $\sigma_2(x,q)$, respectively. Let ρ be a bounded and non-negative function satisfying the q-Pearson equation (1.5) as well as the BCs (2.5) or (2.7). Then ρ is a desired weight function for the polynomial solutions $P_n(x,q)$ of (1.1) only in the following cases:*

1. *Let $a < 0 < b$, where $a = a_1(q)$ and $b = b_1(q)$. Then ρ is supported on $\{q^k a\}_{k\in\mathbb{N}_0} \bigcup \{q^k b\}_{k\in\mathbb{N}_0}$ and*

$$\int_{a_1(q)}^{b_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.11}$$

where the q-Jackson integral is of type (2.2).

2. *Let $a = 0 < b$, where $b = a_1(q)$. Then ρ is supported on the set of points $\{q^k b\}_{k\in\mathbb{N}_0}$ in $(0, b]$ and*

$$\int_{0}^{a_1(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.12}$$

where the q-Jackson integral is of type (2.1).

3. *Let $0 < a < b$, where $a = a_2(q)$ and $b = q^{-1}a_1(q)$. Then ρ is supported on the finite set of points $\{q^{-k}a\}_{k=0}^{N}$ when $q^{-N-1}a = b$ and*

$$\int_{a_2(q)}^{q^{-1}a_1(q)=q^{-N-1}a_2(q)} P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}, \tag{2.13}$$

which is the finite sum of the form

$$\int_{a_2(q)}^{q^{-N-1}a_2(q)} [\cdot]d_{q^{-1}}x$$
$$= \left(1-q^{-1}\right)a_2(q)\sum_{k=0}^{N} P_n(q^{-k}a_2(q),q)P_m(q^{-k}a_2(q),q)\rho(q^{-k}a_2(q),q).$$

4. *Let $a = a_1(q) < 0$ and $b \to \infty$. Then ρ is supported on the set $\{q^k a\}_{k\in\mathbb{N}_0} \bigcup \{q^{\mp k}\alpha\}_{k\in\mathbb{N}_0}$ for arbitrary $\alpha > 0$ and*

$$\int_{a_1(q)}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx := \int_{a_1(q)}^{0}[\cdot]d_qx + \int_0^\infty [\cdot]d_qx = d_n^2(q)\delta_{mn}, \tag{2.14}$$

where the first q-Jackson integral is of type (2.1) and the second one is of type (2.3), respectively.

5. *Let $a = a_2(q) > 0$ and $b \to \infty$. Then ρ is supported on the set of points $\{q^{-k}a\}_{k\in\mathbb{N}_0}$ in $[a, \infty)$ and*

$$\int_{a_2(q)}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_{q^{-1}}x = d_n^2(q)\delta_{mn}, \tag{2.15}$$

where the q^{-1}-Jackson integral is of type (2.4).

6. *Let $a = 0$ and $b \to \infty$. Then ρ is supported on the set of points $\{q^{\pm k}\alpha\}_{k\in\mathbb{N}_0}$ for arbitrary $\alpha > 0$ and*

$$\int_{0}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.16}$$

where the q-Jackson integral is of type (2.3).

7. *Let $a \to -\infty$ and $b \to \infty$. Then ρ is supported on the set of points $\{\mp q^{\pm k}\alpha\}_{k\in\mathbb{N}_0}$ for arbitrary $\alpha > 0$ and*

$$\int_{-\infty}^{\infty} P_n(x,q)P_m(x,q)\rho(x,q)d_qx = d_n^2(q)\delta_{mn}, \tag{2.17}$$

where the bilateral q-Jackson integral is of type (2.3).

Before starting our analysis, let us mention that in accordance with [5, 27, 30] the q-polynomials can be classified by means of the degrees of the polynomial coefficients σ_1 and σ_2 and the fact that either $\sigma_1(0,q)\sigma_2(0,q) \neq 0$ or $\sigma_1(0,q) = \sigma_2(0,q) = 0$. Therefore, we can define two classes, namely, the non-zero ($\emptyset$) and zero (0) classes corresponding to the cases where $\sigma_1(0,q)\sigma_2(0,q) \neq 0$ and $\sigma_1(0,q) = \sigma_2(0,q) = 0$, respectively (this is a consequence of the fact that $\sigma_2(0,q) = q\sigma_1(0,q)$, i.e., σ_1 and σ_2 both have the same constant terms). In each class we consider all possible degrees of the polynomial coefficients $\sigma_1(x,q)$ and $\sigma_2(x,q)$ as shown in [27, p. 182]. We follow the notation introduced in [5, 27], i.e., the statement $\emptyset$-Laguerre/Jacobi implies that $\deg\sigma_2 = 1$, $\deg\sigma_1 = 2$, and $\sigma_1(0,q)\sigma_2(0,q) \neq 0$ and the statement 0-Jacobi/Laguerre indicates that $\deg\sigma_2 = 2$, $\deg\sigma_1 = 1$ and $\sigma_1(0,q) = \sigma_2(0,q) = 0$.

In the following we use the Taylor polynomial expansion for the coefficients

$$\begin{aligned}
&\tau(x,q) = \tau'(0,q)x + \tau(0,q), \qquad \tau'(0,q) \neq 0,\\
&\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x^2 + \sigma_1'(0,q)x + \sigma_1(0,q) = \frac{1}{2}\sigma_1''(0,q)[x - a_1(q)][x - b_1(q)],\\
&\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x + \sigma_2(0,q) = \frac{1}{2}\sigma_2''(0,q)[x - a_2(q)][x - b_2(q)].
\end{aligned} \tag{2.18}$$

Theorem 2.4 (classification of the OPS of the q-Hahn class [5, 27])**.** *All orthogonal polynomial solutions of the q-difference equations (1.1) and (1.3) can be classified as follows:*

1. $\emptyset$-***Jacobi/Jacobi*** *polynomials where $\sigma_2''(0,q) \neq 0$ and $\sigma_1''(0,q) \neq 0$ with $\sigma_1(0,q)\sigma_2(0,q) \neq 0$.*
2. $\emptyset$-***Jacobi/Laguerre*** *polynomials where $\sigma_2''(0,q) \neq 0$ and $\sigma_1''(0,q) = 0$, $\sigma_1'(0,q) \neq 0$ with $\sigma_1(0,q)\sigma_2(0,q) \neq 0$.*
3. $\emptyset$-***Jacobi/Hermite*** *polynomials where $\sigma_2''(0,q) \neq 0$, $\sigma_1''(0,q) = 0$, $\sigma_1'(0,q) = 0$ and $\sigma_1(0,q) \neq 0$ with $\sigma_1(0,q)\sigma_2(0,q) \neq 0$.*
4. $\emptyset$-***Laguerre/Jacobi*** *polynomials where $\sigma_2''(0,q) = 0$, $\sigma_2'(0,q) \neq 0$ and $\sigma_1''(0,q) \neq 0$ with $\sigma_1(0,q)\sigma_2(0,q) \neq 0$.*
5. $\emptyset$-***Hermite/Jacobi*** *polynomials where $\sigma_2''(0,q) = 0$, $\sigma_2'(0,q) = 0$, $\sigma_2(0,q) \neq 0$ and $\sigma_1''(0,q) \neq 0$ with $\sigma_1(0,q)\sigma_2(0,q) \neq 0$.*
6. 0-***Jacobi/Jacobi*** *polynomials where $\sigma_2''(0,q) \neq 0$, $\sigma_2'(0,q) \neq 0$ and $\sigma_1''(0,q) \neq 0$, $\sigma_1'(0,q) \neq 0$ with $\sigma_2(0,q) = \sigma_1(0,q) = 0$.*
7. 0-***Jacobi/Laguerre*** *polynomials where $\sigma_2''(0,q) \neq 0$, $\sigma_2'(0,q) \neq 0$ and $\sigma_1''(0,q) = 0$, $\sigma_1'(0,q) \neq 0$ with $\sigma_2(0,q) = \sigma_1(0,q) = 0$.*
8. 0-***Bessel/Jacobi*** *polynomials where $\sigma_2''(0,q) \neq 0$, $\sigma_2'(0,q) = 0$ and $\sigma_1''(0,q) \neq 0$, $\sigma_1'(0,q) \neq 0$ with $\sigma_2(0,q) = \sigma_1(0,q) = 0$.*
9. 0-***Bessel/Laguerre*** *polynomials where $\sigma_2''(0,q) \neq 0$, $\sigma_2'(0,q) = 0$ and $\sigma_1''(0,q) = 0$, $\sigma_1'(0,q) \neq 0$ with $\sigma_2(0,q) = \sigma_1(0,q) = 0$.*
10. 0-***Laguerre/Jacobi*** *polynomials where $\sigma_2''(0,q) = 0$, $\sigma_2'(0,q) \neq 0$ and $\sigma_1''(0,q) \neq 0$, $\sigma_1'(0,q) \neq 0$ with $\sigma_2(0,q) = \sigma_1(0,q) = 0$.*

3 The q-weight function

In the following sections we will discuss the solutions of the q-Pearson equation (1.5) defined on the q-linear lattices enumerated in Remark 2.2. Since it is a linear difference equation on a given lattice, its solution can be uniquely determined by the equation (1.5) and the boundary conditions (for more details on the general theory of linear difference equations, see e.g. [14, § 1.2]). In fact, the explicit form of a q-weight function can be deduced by means of Theorem 3.1.

Theorem 3.1. *Let f satisfy the difference equation*

$$\frac{f(qx;q)}{f(x;q)} = \frac{a(x;q)}{b(x;q)}, \tag{3.1}$$

such that the limits $\lim_{x\to 0} f(x;q) = f(0,q)$ and $\lim_{x\to\infty} f(x;q) = f(\infty,q)$ exist, where $a(x;q)$ and $b(x;q)$ are definite functions. Then $f(x;q)$ admits the two q-integral representations

$$f(x,q) = f(0,q)\exp\left[\int_0^x \frac{1}{(q-1)t}\ln\left[\frac{a(t,q)}{b(t,q)}\right] d_q t\right] \tag{3.2}$$

and

$$f(x,q) = f(\infty,q)\exp\left[\int_x^\infty \frac{1}{(1-q^{-1})t}\ln\left[\frac{a(t,q)}{b(t,q)}\right] d_{q^{-1}} t\right] \tag{3.3}$$

provided that the integrals are convergent.

Proof. Taking the logarithms of both sides of (3.1), multiplying by $1/(q-1)t$ and then integrating from 0 to x, we have

$$\int_0^x \frac{1}{(q-1)t}\ln\left[\frac{f(qt,q)}{f(t,q)}\right] d_q t = \int_0^x \frac{1}{(q-1)t}\ln\left[\frac{a(t,q)}{b(t,q)}\right] d_q t.$$

The l.h.s. is expressible as

$$\begin{aligned}\int_0^x \frac{1}{(q-1)t}\ln\left[\frac{f(qt,q)}{f(t,q)}\right] d_q t &= \lim_{n\to\infty}\sum_{j=0}^{n}\left[\ln\left(f(q^j x,q)\right) - \ln\left(f(q^{j+1}x,q)\right)\right]\\ &= \ln\left[f(x,q)\right] - \ln\left[f(0,q)\right],\end{aligned}$$

which completes the proof, on using the fact that $f(q^{n+1}x,q) \to f(0,q)$ as $n\to\infty$ for $0<q<1$. The second representation (3.3) can be proven in a similar way. ■

This theorem can be used to derive the q-weight functions for every σ_1 and σ_2. However, here we take into account the quadratic coefficients leading to $\emptyset$-Jacobi/Jacobi and 0-Jacobi/Jacobi cases. The results, some of which can be found in [5], are stated by the next theorem.

Theorem 3.2. *In the $\emptyset$-Jacobi/Jacobi case, let $\sigma_1(x,q)$ and $\sigma_2(x,q)$ be of forms (2.18) in which $\sigma_1''(0,q)a_1(q)b_1(q) \neq 0$ and $\sigma_2''(0,q)a_2(q)b_2(q) \neq 0$. And let, in 0-Jacobi/Jacobi case, $b_1(q) = b_2(q) = 0$, $\sigma_1''(0,q) \neq 0$ and $\sigma_2''(0,q) \neq 0$. Then a solution $\rho(x,q)$ of q-Pearson equation (1.5) is expressible in the equivalent forms shown in Table 1 where $H^{(1)}(x)$ is given by [5]*

$$H^{(1)}(x) = \sqrt{x^{\log_q x - 1}} = |x|^{-\frac{1}{2}} q^{\frac{1}{2}(\log_q x)^2}. \tag{3.4}$$

Table 1. Expressions for the q-weight function $\rho(x,q)$ in the Jacobi/Jacobi cases.

$\emptyset$-Jacobi/Jacobi case	
1.	$\dfrac{(a_1^{-1}qx,\, b_1^{-1}qx;q)_\infty}{(a_2^{-1}x,\, b_2^{-1}x;q)_\infty}$
2.	$\lvert x\rvert^\alpha \dfrac{(b_1^{-1}qx, a_2q/x;q)_\infty}{(a_1/x, b_2^{-1}x;q)_\infty}$ where $q^\alpha = \dfrac{q^{-2}\sigma_2''(0,q)b_2}{\sigma_1''(0,q)b_1}$
0-Jacobi/Jacobi case	
3.	$\lvert x\rvert^\alpha \dfrac{(a_1^{-1}qx;q)_\infty}{(a_2^{-1}x;q)_\infty}$ where $q^\alpha = \dfrac{q^{-2}\sigma_2''(0,q)a_2}{\sigma_1''(0,q)a_1}$
4.	$\lvert x\rvert^\alpha\, H^{(1)}(x)(qa_1^{-1}x, qa_2/x;q)_\infty$ where $q^\alpha = \dfrac{q^{-2}\sigma_2''(0,q)}{\sigma_1''(0,q)a_1}$

Proof. We start proving the first expression in Table 1. Keeping in mind that $q^{-1}\sigma_2(0,q) = \sigma_1(0,q)$ and that $q^{-1}\sigma_2''(0,q)a_2(q)b_2(q) = \sigma_1''(0,q)a_1(q)b_1(q)$ we have from (1.5)

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}\sigma_2''(0,q)[x-a_2(q)][x-b_2(q)]}{\sigma_1''(0,q)[qx-a_1(q)][qx-b_1(q)]} = \frac{[1-a_2^{-1}(q)x][1-b_2^{-1}(q)x]}{[1-a_1^{-1}(q)qx][1-b_1^{-1}(q)qx]} \tag{3.5}$$

which gives

$$\rho(x,q) = \rho(0,q)\exp\left\{\int_0^x \frac{1}{(q-1)t}\left[\ln\left(1-a_2^{-1}t\right)\left(1-b_2^{-1}t\right) - \ln\left(1-a_1^{-1}qt\right)\left(1-b_1^{-1}qt\right)\right]d_qt\right\}$$

on using (3.2). By definition (2.1) of the q-integral, we first obtain

$$\begin{aligned}\rho(x,q) = \rho(0,q)\exp\Bigg\{&\ln\prod_{k=0}^\infty\left(1-a_1^{-1}q^{k+1}x\right)\left(1-b_1^{-1}q^{k+1}x\right)\\ &-\ln\prod_{k=0}^\infty\left(1-a_2^{-1}q^kx\right)\left(1-b_2^{-1}q^kx\right)\Bigg\}\end{aligned}$$

and, therefore,

$$\rho(x,q) = \rho(0,q)\frac{(a_1^{-1}qx, b_1^{-1}qx;q)_\infty}{(a_2^{-1}x, b_2^{-1}x;q)_\infty}, \qquad \rho(0,q)\neq 0. \tag{3.6}$$

This implies that $a_1(q)q^{-1-k}$ and $b_1(q)q^{-1-k}$ for $k\geq 0$ are zeros of ρ. Furthermore, $a_2(q)q^{-j}$ and $b_2(q)q^{-j}$ for $j\geq 0$ stand for the simple poles of ρ. Note here that $\rho(0,q)$ can be made unity, and $a_1(q)$, $b_1(q)$, $a_2(q)$ and $b_2(q)$ are non-zero constants. Therefore the solution in (3.6) is continuous everywhere except at the simple poles.

To show 4, we rewrite the q-Pearson equation in the form

$$\frac{\rho(qx,q)}{\rho(x,q)} = \frac{ax[1-a_2(q)/x]}{[1-a_1^{-1}(q)qx]}, \qquad a = \frac{q^{-2}\sigma_2''(0,q)}{\sigma_1''(0,q)a_1(q)} \tag{3.7}$$

and assume that ρ is a product of three functions $\rho(x,q) = f(x,q)g(x,q)h(x,q)$. Hence, if f, g and h are solutions of

$$\frac{f(qx,q)}{f(x,q)} = ax, \qquad \frac{g(qx,q)}{g(x,q)} = \frac{1}{[1-a_1^{-1}(q)qx]} \qquad\text{and}\qquad \frac{h(qx,q)}{h(x,q)} = \left[1-\frac{a_2(q)}{x}\right] \tag{3.8}$$

respectively, then $\rho = fgh$ is a solution of (3.7). A solution of (3.8) for $f(x,q)$ is of the form $f(x,q) = |x|^{\alpha} H^{(1)}(x)$, where $H^{(1)}(x)$ is given by (3.4) and $\alpha \neq 0$ is such that $q^{\alpha} = a$, which can be verified by direct substitution. The equation in (3.8) for $g(x,q)$ can be solved in a way similar to that of (3.5). So we find that $g(x,q) = g(0,q)(a_1^{-1}qx;q)_{\infty}$, where $g(0,q) = 1$. The expression (3.2) is not suitable in finding $h(x,q)$ which gives a divergent infinite product. Instead, we employ (3.3) so that the equation for $h(x,q)$ becomes

$$\frac{h(q^{-1}x,q)}{h(x,q)} = \frac{1}{[1 - qa_2(q)/x]}$$

whose solution is of the form $h(x,q) = h(\infty,q)(qa_2/x;q)_{\infty}$, where $h(\infty,q)$ can be taken again as unity without loss of generality. Clearly $h(x,q)$ is uniformly convergent in any compact subset of the complex plane that does not contain the point at the origin. Moreover, the product converges to an arbitrary constant c, which has been set to unity, as $x \to \infty$. Thus,

$$\rho(x,q) = f(x,q)g(x,q)h(x,q) = |x|^{\alpha} H^{(1)}(x)\big(qa_1^{-1}x, qa_2/x;q\big)_{\infty}.$$

In order to obtain the expressions 2 and 3 in Table 1 for the weight function we use the same procedure as before, but starting from the q-Pearson equation written in the forms

$$\frac{\rho(qx,q)}{\rho(x,q)} = a\frac{[1 - a_2(q)/x][1 - b_2^{-1}(q)x]}{[1 - a_1(q)q^{-1}/x][1 - b_1^{-1}(q)qx]}, \qquad a = \frac{q^{-2}\sigma_2''(0,q)b_2(q)}{\sigma_1''(0,q)b_1(q)} \tag{3.9}$$

and

$$\frac{\rho(qx,q)}{\rho(x,q)} = a\frac{[1 - a_2^{-1}(q)x]}{[1 - a_1^{-1}(q)qx]}, \qquad a = \frac{q^{-2}\sigma_2''(0,q)a_2(q)}{\sigma_1''(0,q)a_1(q)}, \tag{3.10}$$

respectively. This completes the proof. ■

Remark 3.3. Notice that for getting the expressions of the weight function we have used the q-Pearson equation rewritten in different forms, namely (3.5), (3.7), (3.9) and (3.10), and different solution procedure in each case, therefore, it is not surprising that ρ has several equivalent representations displayed in Table 1. However, they all satisfy the same linear equation and, therefore, they differ only by a multiplicative constant.

For the sake of the completeness, let us obtain the analytic representations of q-weight functions satisfying (1.5) for the other cases.

Theorem 3.4. *Let σ_1 and σ_2 be polynomials of at most 2nd degree in x as the form (2.18). Then a solution $\rho(x,q)$ of q-Pearson equation (1.5) for each $\emptyset$-Jacobi/Laguerre, $\emptyset$-Jacobi/Hermite, $\emptyset$-Laguerre/Jacobi, $\emptyset$-Hermite/Jacobi, 0-Jacobi/Laguerre, 0-Bessel/Jacobi, 0-Bessel/Laguerre and 0-Laguerre/Jacobi case is expressible in the equivalent forms shown in Table 2 where $H^{(1)}(x)$ is given by (3.4).*

Proof. The proof is similar to the previous one. That is, to obtain the second formula for the $\emptyset$-Jacobi/Laguerre family we rewrite the q-Pearson equation (1.5) in the form

$$\frac{\rho(qx,q)}{\rho(x,q)} = ax\frac{[1 - a_2/x][1 - b_2/x]}{[1 - a_1q^{-1}/x]}, \qquad a = \frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)}$$

and then apply the same procedure described in the proof of the previous theorem. ■

Table 2. Expressions for the q-weight function $\rho(x,q)$ for the other cases.

$\emptyset$-Jacobi/Laguerre	1. $\dfrac{(a_1^{-1}qx;q)_\infty}{(a_2^{-1}x,\, b_2^{-1}x;q)_\infty}$ 2. $\lvert x\rvert^\alpha\, H^{(1)}(x)\dfrac{(a_2q/x,b_2q/x;q)_\infty}{(a_1/x;q)_\infty}$, $q^\alpha=\dfrac{\frac12\sigma_2''(0,q)q^{-2}}{\sigma_1'(0,q)}$ 3. $\lvert x\rvert^{\alpha+1}\,[H^{(1)}(x)]^2(qa_1^{-1}x,qa_2/x,qb_2/x;q)_\infty$, $q^\alpha=-\dfrac{q^{-2}\frac12\sigma_2''(0,q)}{\sigma_1'(0,q)a_1}$
$\emptyset$-Jacobi/Hermite	1. $\dfrac{1}{(a_2^{-1}x,b_2^{-1}x;q)_\infty}$ 2. $\lvert x\rvert^\alpha\,[H^{(1)}(x)]^2(a_2q/x,b_2q/x;q)_\infty$, $q^\alpha=\dfrac{\frac12\sigma_2''(0,q)q^{-1}}{\sigma_1(0,q)}$
$\emptyset$-Laguerre/Jacobi	1. $\dfrac{(a_1^{-1}qx,b_1^{-1}qx;q)_\infty}{(a_2^{-1}x;q)_\infty}$ 2. $\lvert x\rvert^\alpha\,\dfrac{(qa_2/x,qb_1^{-1}x;q)_\infty}{(a_1/x;q)_\infty}$, $q^\alpha=-\dfrac{q^{-2}\sigma_2'(0,q)}{\frac12\sigma_1''(0,q)b_1}$
$\emptyset$-Hermite/Jacobi	$(a_1^{-1}qx,b_1^{-1}qx;q)_\infty$
0-Jacobi/Laguerre	1. $\lvert x\rvert^\alpha\,\dfrac{1}{(a_2^{-1}x;q)_\infty}$, $q^\alpha=-\dfrac{q^{-2}\frac12\sigma_2''(0,q)a_2}{\sigma_1'(0,q)}$ 2. $\lvert x\rvert^\alpha\, H^{(1)}(x)(qa_2/x;q)_\infty$, $q^\alpha=\dfrac{q^{-2}\frac12\sigma_2''(0,q)}{\sigma_1'(0,q)}$
0-Bessel/Jacobi	$\lvert x\rvert^\alpha\, H^{(1)}(x)(a_1^{-1}qx;q)_\infty$, $q^\alpha=-\dfrac{q^{-2}\frac12\sigma_2''(0,q)}{\frac12\sigma_1''(0,q)a_1}$
0-Bessel/Laguerre	$\lvert x\rvert^\alpha\, H^{(1)}(x)$, $q^\alpha=\dfrac{q^{-2}\frac12\sigma_2''(0,q)}{\sigma_1'(0,q)}$
0-Laguerre/Jacobi	$\lvert x\rvert^\alpha\,(a_1^{-1}qx;q)_\infty$, $q^\alpha=-\dfrac{q^{-2}\sigma_2'(0,q)}{\frac12\sigma_1''(0,q)a_1}$

4 The orthogonality of q-polynomials

In this section we study the orthogonality of all families of q-polynomials of the Hahn class by taking into account the rational function on the r.h.s. of the q-Pearson equation (1.5). Since it is the ratio of two polynomials σ_1 and σ_2 of at most second degree, we deal with a definite rational function having at most two zeros and two poles. In the analysis of the unknown quantity $\rho(qx,q)/\rho(x,q)$ on the l.h.s. of (1.5), we sketch roughly its graph by using every possible form of the definite rational function in question. In particular, we split the x-interval into subintervals according to whether $\rho(qx,q)/\rho(x,q) < 1$ or $\rho(qx,q)/\rho(x,q) > 1$, which yields valuable information about the monotonicity of $\rho(x,q)$. Other significant properties of ρ are provided by the asymptotes, if there exist any, of $\rho(qx,q)/\rho(x,q)$. A full analysis along these lines is sufficient for a complete characterization of the orthogonal q-polynomials.

Since $\emptyset$-Jacobi/Jacobi and $\emptyset$-Jacobi/Laguerre families include new results, we present the detailed analysis of these two cases in Sections 4.1 and 4.2, respectively. For the other families we only include a list leading to a positive definite OPS and compare them with those obtained in [21]. For a complete study see [6].

4.1 q-classical $\emptyset$-Jacobi/Jacobi polynomials

Let the coefficients σ_2 and σ_1 be quadratic polynomials in x such that $\sigma_1(0,q)\sigma_2(0,q)\neq 0$. If σ_1 is written in terms of its roots, i.e., $\sigma_1(x,q)=\frac12\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$ then, from (1.4),

$$\begin{aligned}\sigma_2(x,q)=&\left[\frac{\sigma_1''(0,q)}{2}+\left(1-q^{-1}\right)\tau'(0,q)\right]qx^2\\&-\left[\frac{\sigma_1''(0,q)}{2}(a_1+b_1)-\left(1-q^{-1}\right)\tau(0,q)\right]qx+\frac{\sigma_1''(0,q)}{2}a_1b_1,\end{aligned}$$

where $\frac{1}{2}\sigma_1''(0,q)+(1-q^{-1})\tau'(0,q)\neq 0$ by hypothesis. Then q-Pearson equation (1.5) takes the form

$$f(x,q):=\frac{\rho(qx,q)}{\rho(x,q)}=\frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)}=\left[1+\frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\frac{[x-a_2(q)][x-b_2(q)]}{[qx-a_1(q)][qx-b_1(q)]} \quad (4.1)$$

provided that the discriminant, denoted by Δ_q,

$$\Delta_q:=\left[a_1(q)+b_1(q)-\frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]^2-4a_1(q)b_1(q)\left[1+\frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right],$$

of the quadratic polynomial in the nominator of $f(x,q)$ in (4.1) is non-zero. Here $x=a_2$ and $x=b_2$ denote the zeros of f, and they are constant multiples of the roots of $\sigma_2(x,q)$.

We see that the lines $x=q^{-1}a_1$ and $x=q^{-1}b_1$ stand for the vertical asymptotes of $f(x,q)$ and the point $y=1$ is always its y-intercept since $\sigma_2(0,q)=q\sigma_1(0,q)$. Moreover, the locations of the zeros of f are determined by the straightforward lemma.

Lemma 4.1. *Define the parameter*

$$\Lambda_q=\frac{1}{q^2}\left[1+\frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\neq 0 \quad (4.2)$$

so that the line $y=\Lambda_q$ denotes the horizontal asymptote of $f(x,q)$. Then we encounter the following cases for the roots of the equation $f(x,q)=0$.

Case 1. *If $\Lambda_q>0$ and $a_1(q)<0<b_1(q)$, f has two real and distinct roots with opposite signs.*

Case 2. *If $\Lambda_q>0$ and $0<a_1(q)<b_1(q)$, there exist three possibilities*

- **(a)** *if $\Delta_q>0$, f has two real roots with the same signs,*
- **(b)** *if $\Delta_q=0$, f has a double root,*
- **(c)** *if $\Delta_q<0$, f has a pair of complex conjugate roots.*

Case 3. *If $\Lambda_q<0$ and $a_1(q)<0<b_1(q)$, there exist three possibilities*

- **(a)** *if $\Delta_q>0$, f has two real roots with the same signs,*
- **(b)** *if $\Delta_q=0$, f has a double root,*
- **(c)** *if $\Delta_q<0$, f has a pair of complex conjugate roots.*

Case 4. *If $\Lambda_q<0$ and $0<a_1(q)<b_1(q)$, f has two real distinct roots with opposite signs.*

From (4.1) it is clear that we need to consider the cases $\Lambda_q>1$ and $0<\Lambda_q<1$ separately. Now, our strategy consists of sketching first the graphs of $f(x,q)$ depending on all possible relative positions of the zeros of σ_1 and σ_2. To obtain the behaviors of q-weight functions ρ from the graphs of $f(x,q)=\rho(qx,q)/\rho(x,q)$, we divide the real line into subintervals in which ρ is either monotonic decreasing or increasing. We take into consideration only the subintervals where $\rho>0$. Note that if ρ is initially positive then we have $\rho>0$ everywhere in an interval where $\rho(qx,q)/\rho(x,q)>0$. Then we find suitable intervals in cooperation with Theorem 2.3.

In Fig. 1A, the intervals $(q^{-1}a_1,a_2)$ and $(b_2,q^{-1}b_1)$ are rejected immediately since f is negative. The subinterval (a_2,b_2) should also be rejected in which $\rho=0$ by **PII**. For the same reason $(q^{-1}b_1,\infty)$ and $(-\infty,q^{-1}a_1)$, by symmetry, are not suitable by **PV**. Therefore, an OPS fails to exist.

Let us analyze the problem presented in Fig. 1B. The positivity of ρ implies that the intervals $(q^{-1}a_1,a_2)$ and $(q^{-1}b_1,b_2)$ should be eliminated. With the transformation $x=-t$, we eliminate

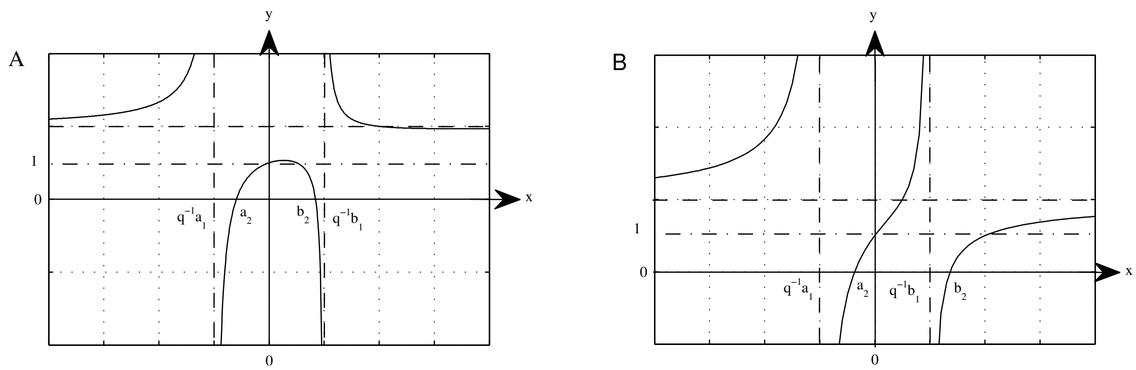

Figure 1. The graph of $f(x,q)$ in **Case 1** with $\Lambda_q > 1$. In A, the zeros are in order $q^{-1}a_1 < a_2 < 0 < b_2 < q^{-1}b_1$, and in B, $q^{-1}a_1 < a_2 < 0 < q^{-1}b_1 < b_2$.

also $(-\infty, q^{-1}a_1)$ by **PV**. The interval $(a_2, q^{-1}b_1)$ is not suitable too, by **PIII**. So it remains only (b_2, ∞) to examine which coincides with the $5th$ case in Theorem 2.3. Since $\rho(qx,q)/\rho(x,q) = 1$ at $x_0 = -\tau(0,q)/\tau'(0,q) > b_2(q)$, then ρ is increasing on (b_2, x_0) and decreasing on (x_0, ∞). As is shown from the figure f has a finite limit as $x \to +\infty$ so that we could have the case $\rho \to 0$ as $x \to \infty$. Even if $\rho \to 0$ as $x \to \infty$, we must show also that $\sigma_1(x,q)\rho(x,q)x^k \to 0$ as $x \to \infty$ to satisfy the BC. In fact, instead of the usual q-Pearson equation we have to consider the equation

$$g(x,q) := \frac{\sigma_1(qx,q)\rho(qx,q)(qx)^k}{\sigma_1(x,q)\rho(x,q)x^k} = q^k \frac{\sigma_1(x,q) + (1-q^{-1})x\tau(x,q)}{\sigma_1(x,q)} = q^k \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(x,q)} \qquad (4.3)$$

in case of an infinite interval, what we call it here the *extended* q-Pearson equation to determine the behavior of the quantity $\sigma_1(x,q)\rho(x,q)x^k$ as $x \to \infty$, which has been easily derived from (1.5). It is obvious that the extended q-Pearson equation is the difference equation not for the weight function $\rho(x,q)$ but for $\sigma_1(x,q)\rho(x,q)x^k$.

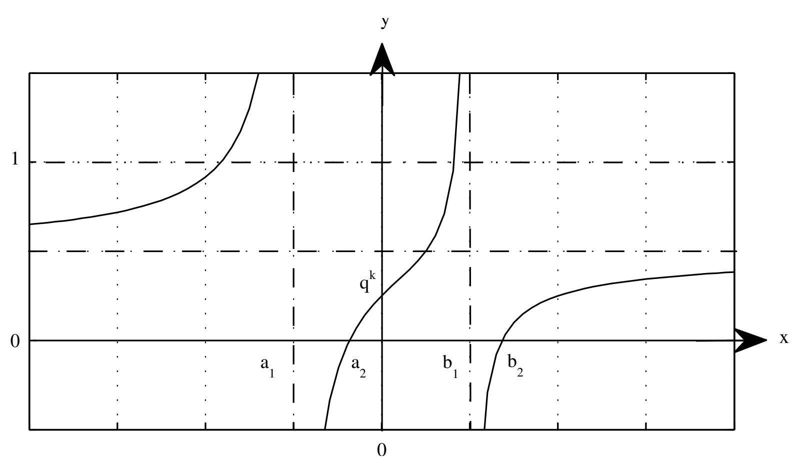

Figure 2. The graph of $g(x,q)$ corresponding to Fig. 1B.

In Fig. 2 we draw the graph of a typical g for some $0 < q < 1$, where k is large enough. From this figure we see that $g < 1$ for $x > b_2$ so that $\sigma_1(x,q)\rho(x,q)x^k$ does not vanish at ∞ since it is increasing as x increases. Thus we cannot find a weight function ρ on (b_2, ∞).

From Fig. 3A, we first eliminate the intervals $(a_2, q^{-1}a_1)$ and $(q^{-1}b_1, b_2)$ because of the positivity of ρ. The interval (b_2, ∞) coincides again with the $5th$ case in Theorem 2.3. However, $f(x,q) < 1$ on this interval so that ρ is increasing on (b_2, ∞) which implies that ρ can not vanish as $x \to \infty$. Thus $\sigma_1(x,q)\rho(x,q)x^k$ is never zero as $x \to \infty$ for some $k \in \mathbb{N}_0$. The same is

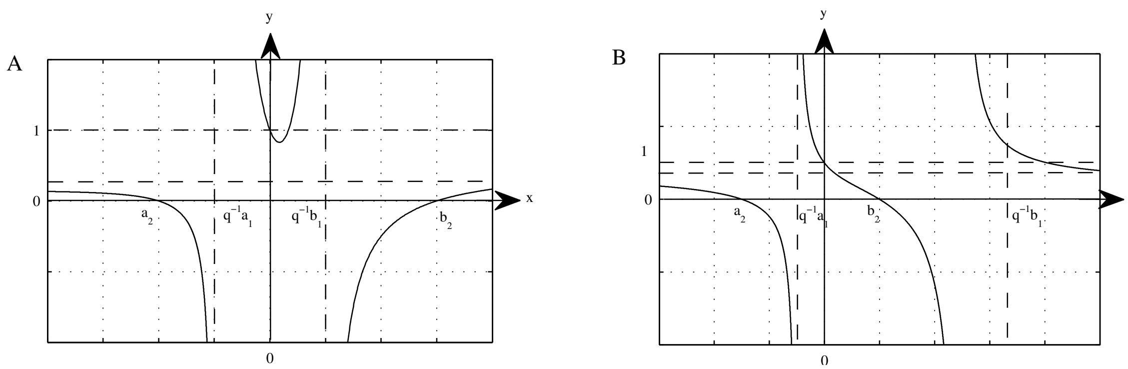

Figure 3. The graph of $f(x,q)$ in **Case 1** with $0 < \Lambda_q < 1$. In A, the zeros are in order $a_2 < q^{-1}a_1 < 0 < q^{-1}b_1 < b_2$ and in B, $a_2 < q^{-1}a_1 < 0 < b_2 < q^{-1}b_1$.

true for $(-\infty, a_2)$ by symmetry. For the last subinterval $(q^{-1}a_1, q^{-1}b_1)$, we face the $1th$ case in Theorem 2.3. Since $\rho(qx,q)/\rho(x,q) = 1$ at $q^{-1}a_1 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$, then ρ is increasing on $(q^{-1}a_1, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Furthermore, $\rho(qx,q)/\rho(x,q) \to \infty$, and hence $\rho \to 0$, as $x \to q^{-1}a_1^+$ and $x \to q^{-1}b_1^-$. As a result, the typical shape of ρ is shown in Fig. 4 assuming a positive initial value of ρ in each subinterval. Then, an OPS with such a weight function in Fig. 4 supported on the union of set of points $\{q^k a_1(q)\}_{k\in\mathbb{N}_0}$ and $\{q^k b_1(q)\}_{k\in\mathbb{N}_0}$ exists (see Theorem 2.3-**1**). This OPS can be stated in the Theorem 4.2.

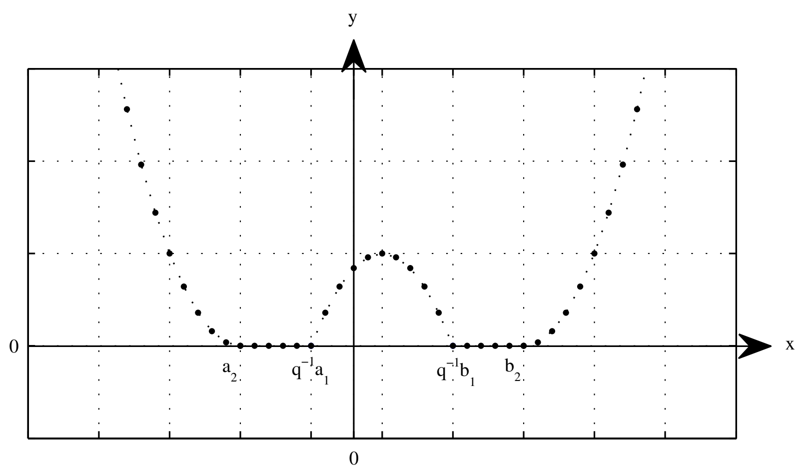

Figure 4. The graph of $\rho(x,q)$ associated with the case in Fig. 3A.

Theorem 4.2. *Consider the case where $a_2 < a_1 < 0 < b_1 < b_2$ and $0 < q^2\Lambda_q < 1$. Let $a = a_1(q)$ and $b = b_1(q)$ be zeros of $\sigma_1(x,q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (see equation 1 in Table 1)*

$$\rho(x,q) = \frac{(qa^{-1}x, qb^{-1}x; q)_\infty}{(a_2^{-1}x, b_2^{-1}x; q)_\infty}, \tag{4.4}$$

*supported on $\{q^k a\}_{k\in\mathbb{N}_0} \bigcup \{q^k b\}_{k\in\mathbb{N}_0}$, (see (2.11) of Theorem 2.3-**1**).*

The OPS in Theorem 4.2 coincides with the case VIIa1 in Chapter 10 of [21, p. 292 and p. 318]. In fact, a typical example of this family is the big q-Jacobi polynomials $P_n(x; a, b, c; q)$ satisfying the q-EHT with the coefficients

$$\sigma_1(x,q) = q^{-2}(x-a_1)(x-b_1), \qquad \sigma_2(x,q) = abq(x-a_2)(x-b_2),$$

$$\tau(x,q)=\frac{1-abq^2}{(1-q)q}x+\frac{a(bq-1)+c(aq-1)}{1-q} \qquad \text{and} \qquad \lambda_n(q)=q^{-n}[n]_q\frac{1-abq^{n+1}}{q-1}, \quad (4.5)$$

where $a_1=cq$, $b_1=aq$, $a_2=b^{-1}c$ and $b_2=1$. The conditions $0<q^2\Lambda_q<1$ and $a_2<a_1<0<b_1<b_2$ give the known constrains $c<0$, $0<b<q^{-1}$ and $0<a<q^{-1}$ on the parameters of $P_n(x;a,b,c;q)$ with orthogonality on $\{cq,cq^2,cq^3,\dots\}\bigcup\{\dots,aq^3,aq^2,aq\}$ in the sense (2.11) where

$$d_n^2=(a-c)q(1-q)\frac{(q,abq^2,a^{-1}cq,ac^{-1}q;q)_\infty}{(aq,bq,cq,abc^{-1}q;q)_\infty}\frac{(q,abq;q)_n}{(abq,abq^2;q)_{2n}}$$
$$\times\left(aq,bq,cq,abc^{-1}q;q\right)_n(-ac)^n q^{\frac{n(n+3)}{2}}.$$

It should be noted that the difference between these conditions and those of Fig. 3 comes from the fact that we have considered not only the conditions on ρ but also on $\sigma_1\rho$ in Theorem 4.2. Finally, the analysis of the case in Fig. 3B does not yield an OPS.

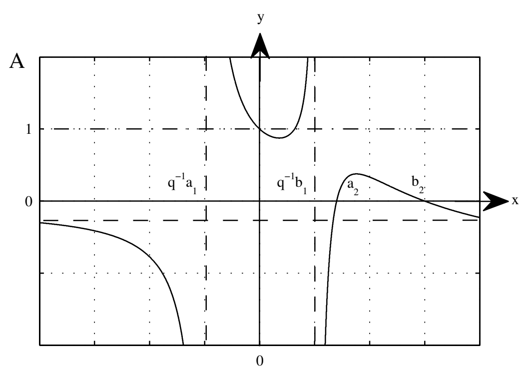

Figure 5. The graph of $f(x,q)$ in **Case 3** with $\Lambda_q<0$. In A, we have **Case 3(a)** with $q^{-1}a_1<0<q^{-1}b_1<a_2<b_2$, and In B, we have **Case 3(c)** with $q^{-1}a_1<0<q^{-1}b_1$ and $a_2,b_2\in\mathbb{C}$.

In Fig. 5A, the only suitable interval is $(q^{-1}a_1,q^{-1}b_1)$ which coincides with the $1st$ case in Theorem 2.3. In fact, $\rho(qx,q)/\rho(x,q)=1$ at $q^{-1}a_1<x_0=-\tau(0,q)/\tau'(0,q)<q^{-1}b_1$, then it follows that ρ is increasing on $(q^{-1}a_1,x_0)$ and decreasing on $(x_0,q^{-1}b_1)$ with $\rho\to0$ as $x\to q^{-1}a_1^+$ and $x\to q^{-1}b_1^-$ since $\rho(qx,q)/\rho(x,q)\to\infty$. Notice that the BCs (2.5) hold at $x=a_1$ and $x=b_1$. Then, there exists an OPS w.r.t. a ρ supported on the set of points $\{q^ka_1\}_{k\in\mathbb{N}_0}\bigcup\{q^kb_1\}_{k\in\mathbb{N}_0}$. Thus, we have the following result.

Theorem 4.3. *Consider the case $a_1<0<b_1<a_2\le b_2$, and $q^2\Lambda_q<0$. Let $a=a_1(q)$ and $b=b_1(q)$ be zeros of $\sigma_1(x,q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n\in\mathbb{N}_0$ orthogonal w.r.t. the weight function* (4.4) *supported on $\{q^ka\}_{k\in\mathbb{N}_0}\bigcup\{q^kb\}_{k\in\mathbb{N}_0}$ (see* (2.11) *of Theorem* 2.3-**1**).

An example of this family is again the big q-Jacobi polynomials which are orthogonal on the set $\{cq,cq^2,cq^3,\dots\}\bigcup\{\dots,aq^3,aq^2,aq\}$. They satisfy the q-EHT with the coefficients in (4.5) where $a_1=cq$, $b_1=aq$, $a_2=b^{-1}c$ and $b_2=1$. This case corresponds to the case VIIa1 in Chapter 10 of [21, p. 292 and p. 318]. However, notice that the conditions, $a_1<0<b_1<a_2\le b_2$ and $q^2\Lambda_q<0$, lead to the new constrains $c<0$, $b<0$, $abc^{-1}q\le1$ and $0<a<q^{-1}$, which give a larger set of parameters for the orthogonality of the big q-Jacobi polynomials than the one reported in [21, p. 319].

In Fig. 5B, the interval $(q^{-1}a_1,q^{-1}b_1)$ coincides with the $1st$ case in Theorem 2.3. Notice that $\rho(qx,q)/\rho(x,q)=1$ at $q^{-1}a_1<x_0=-\tau(0,q)/\tau'(0,q)<q^{-1}b_1$, then ρ is increasing on $(q^{-1}a_1,x_0)$ and decreasing on $(x_0,q^{-1}b_1)$ with $\rho(x,q)\to0$ as $x\to q^{-1}a_1^+$ and $x\to q^{-1}b_1^-$ since

$\rho(qx,q)/\rho(x,q) \to \infty$. Then there is a suitable ρ supported on the set $\{q^k a_1\}_{k\in\mathbb{N}_0} \bigcup \{q^k b_1\}_{k\in\mathbb{N}_0}$. Therefore, the following theorem holds:

Theorem 4.4. *Consider the case $a_1 < 0 < b_1$, a_2, $b_2 \in \mathbb{C}$ and $q^2\Lambda_q < 0$. Let $a = a_1(q)$ and $b = b_1(q)$ be zeros of $\sigma_1(x,q)$. Then there exists a sequence of polynomials $\{P_n\}$ for $n \in \mathbb{N}_0$ orthogonal w.r.t. the weight function (4.4), supported on $\{q^k a\}_{k\in\mathbb{N}_0} \bigcup \{q^k b\}_{k\in\mathbb{N}_0}$, (see (2.11) of Theorem 2.3-**1**) with*

$$d_n^2 = (b_1 - a_1)(1-q)q^{n(n-1)/2}(-a_1 b_1)^n \frac{(q, q^{-1}a_2^{-1}b_2^{-1}a_1 b_1; q)_n}{(q^{-1}a_2^{-1}b_2^{-1}a_1 b_1, a_2^{-1}b_2^{-1}a_1 b_1; q)_{2n}}$$
$$\times \left(a_2^{-1}a_1, a_2^{-1}b_1, b_2^{-1}a_1, b_2^{-1}b_1; q\right)_n \frac{(q, qb_1 a_1^{-1}, qa_1 b_1^{-1}, a_2^{-1}b_2^{-1}a_1 b_1; q)_\infty}{(a_2^{-1}a_1, a_2^{-1}b_1, b_2^{-1}a_1, b_2^{-1}b_1; q)_\infty},$$

where $q^2\Lambda_q = a_1 b_1 a_2^{-1} b_2^{-1}$, $a_2 = i\alpha$, $b_2 = \overline{a_2} = -i\alpha$, $\alpha \in \mathbb{R}$.

This case is included in the case VIIa1 in Chapter 10 of [21, p. 292 and p. 318] (with $\gamma_2 = \overline{\gamma_1}$) but it is not mentioned there. In fact, this case is similar to the big q-Jacobi polynomials studied in (**Cases 1** in Fig. 3A and **Case 3(a)** in Fig. 5A). The difference is that the roots $a_2(q)$ and $b_2(q)$ are complex numbers.

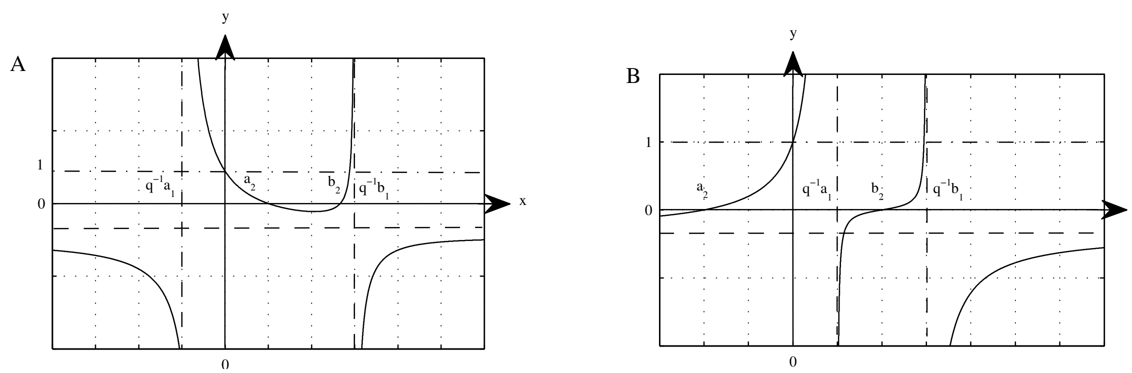

Figure 6. The graph of $f(x,q)$ with $\Lambda_q < 0$. In A, we have **Case 3(a)** with $q^{-1}a_1 < 0 < a_2 < b_2 < q^{-1}b_1$, and in B, we have **Case 4** with $a_2 < 0 < q^{-1}a_1 < b_2 < q^{-1}b_1$.

In Fig. 6A, the only possible interval is $[b_2, q^{-1}b_1)$ which corresponds to the $3th$ case in Theorem 2.3. In this case $\rho(qx,q)/\rho(x,q) = 1$ at $b_2 < x_0 = -\tau(0,q)/\tau'(0,q) < q^{-1}b_1$, then it follows that ρ is increasing on $[b_2, x_0)$ and decreasing on $(x_0, q^{-1}b_1)$. Furthermore, $\rho(qb_2,q) = 0$ and $\rho(x,q) \to 0$ as $x \to q^{-1}b_1^-$ since $\rho(qb_2,q)/\rho(b_2,q) = 0$ and $\rho(qx,q)/\rho(x,q) \to \infty$ as $x \to q^{-1}b_1^-$. Thus there is a suitable ρ supported on the set of points $\{q^{-k}b_2\}_{k=0}^N$ where $q^{-N-1}b_2 = q^{-1}b_1$ (see Theorem 2.3-**3**). Hence we state the following theorem.

Theorem 4.5. *Consider the case where $a_1 < 0 < a_2 \le b_2 < b_1$ and $q^2\Lambda_q < 0$. Let $a = b_2(q)$ and $b = q^{-1}b_1(q)$ be zeros of $\sigma_2(x,q)$ and $\sigma_1(qx,q)$, respectively. Then there exists a finite family of polynomials $\{P_n\}$ orthogonal w.r.t. the weight function (see equation 2 in Table 1)*

$$\rho(x,q) = |x|^\iota \frac{(\frac{qa}{x}, b^{-1}x; q)_\infty}{(\frac{a_1}{x}, a_2^{-1}x; q)_\infty}, \qquad q^\iota = \frac{q^{-3}\sigma_2''(0,q)a_2}{\sigma_1''(0,q)b} \tag{4.6}$$

*supported on the set of points $\{q^{-k}a\}_{k=0}^N$ where $q^{-N-1}a = b$ (see (2.13) of Theorem 2.3-**3**).*

A typical example of this family is the q-Hahn polynomials satisfying (1.1) and (1.3) with

$$\sigma_1(x,q) = q^{-2}(x-a_1)(x-b_1), \qquad \sigma_2(x,q) = \alpha\beta q(x-a_2)(x-b_2),$$

$$\tau(x,q)=\frac{1-\alpha\beta q^2}{(1-q)q}x+\frac{\alpha q^{-N}+\alpha\beta q-\alpha-q^{-N-1}}{1-q} \qquad \text{and}$$
$$\lambda_n(q)=-q^{-n}[n]_q\frac{1-\alpha\beta q^{n+1}}{1-q}, \tag{4.7}$$

where $a_1=\alpha q$, $b_1=q^{-N}$, $a_2=\beta^{-1}q^{-N-1}$ and $b_2=1$. They are orthogonal on the set of points $\{1,q^{-1},q^{-2},\dots,q^{-N}\}$ in the sense (2.13) where

$$d_n^2=\frac{(q,q^{N+1},\beta^{-1},\alpha^{-1}\beta^{-1}q^{-N-1};q)_\infty}{(\alpha q,\beta q^{N+1},\beta^{-1}q^{-N},\alpha^{-1}\beta^{-1}q^{-1};q)_\infty} \times\frac{(q,\alpha\beta q,\alpha q,q^{-N},\beta q,\alpha\beta q^{N+2};q)_n}{(\alpha\beta q,\alpha\beta q^2;q)_{2n}(q^{-1}-1)^{-1}}(-\alpha q^{-N})^n q^{\frac{n(n+1)}{2}}.$$

The conditions $a_1<0<a_2\le b_2<b_1$ and $q^2\Lambda_q<0$ lead to the orthogonality relation for the q-Hahn polynomials that is valid in a larger set of the parameters, $\alpha<0$ and $\beta\ge q^{-N-1}$. This new parameter set is not mentioned in [21].

In Fig. 6B, the only possible interval is $[b_2,q^{-1}b_1)$ which corresponds to the one described in Theorem 2.3-**3**. A similar analysis shows that there exists a q-weight function defined on the interval $[b_2,q^{-1}b_1)$ supported at the points $q^{-k}b_2$ for $k=0,1,\dots,N$ where $q^{-N-1}b_2=q^{-1}b_1$ which lead to the following theorem:

Theorem 4.6. *Consider the case where $a_2<0<a_1<b_2<b_1$ and $q^2\Lambda_q<0$. Let $a=b_2(q)$ and $b=q^{-1}b_1(q)$ be zeros of $\sigma_2(x,q)$ and $\sigma_1(qx,q)$, respectively. Then there exists a finite family of polynomials $\{P_n\}$ orthogonal w.r.t. the weight function (4.6) supported on the set of points $\{q^{-k}a\}_{k=0}^N$ where $q^{-N-1}a=b$ (see (2.13) of Theorem 2.3-***3***).*

An example of this family is again the q-Hahn polynomials orthogonal on the finite set of points $\{1,q^{-1},q^{-2},\dots,q^{-N}\}$. They satisfy (1.1) and (1.3) with the coefficients (4.7) where $a_1=\alpha q$, $b_1=q^{-N}$, $a_2=\beta^{-1}q^{-N-1}$ and $b_2=1$. The conditions $a_2<0<a_1<b_2<b_1$ and $q^2\Lambda_q<0$ lead to another new constrains $0<\alpha<q^{-1}$ and $\beta<0$ on the parameters of the q-Hahn polynomials which extend the orthogonality relation for the q-Hahn polynomials and it has been not reported in [21].

For the sake of the shortness, in the following theorem, we summarize the other cases coming from the analysis of the $\emptyset$-Jacobi/Jacobi family.

Theorem 4.7. *The other positive definite orthogonal polynomial solutions of the q-EHT (1.1) and (1.3) for the $\emptyset$-Jacobi/Jacobi case can be divided into two cases.*

Case 2(a). *$0<a_1<a_2<b_1<b_2$ and $0<q^2\Lambda_q<1$. The q-Hahn polynomials satisfying the q-EHT with the coefficients (4.7) where $a_1=\alpha q$, $b_1=q^{-N}$, $a_2=1$ and $b_2=\beta^{-1}q^{-N-1}$. They are orthogonal w.r.t. the weight function (see equation* 2 *in Table 1) on the set of points $\{1,q^{-1},q^{-2},\dots,q^{-N}\}$ with $0<\alpha<q^{-1}$ and $0<\beta<q^{-1}$ (see (2.13)). In the literature, this relation is usually written as a finite sum* [21, p. 367]. *This case corresponds to the case* IIIb9 *in Chapter* 11 *of* [21, p. 366].

Case 2(a). *$0<a_2\le b_2<a_1\le b_1$ and $q^2\Lambda_q>1$. The q-Hahn polynomials satisfying the q-EHT with the coefficients (4.7) where $a_1=q^{-N}$, $b_1=\alpha q$, $a_2=\beta^{-1}q^{-N-1}$ and $b_2=1$. They are orthogonal w.r.t. the weight function (see equation* 2 *in Table 1) on the set of points $\{1,q^{-1},q^{-2},\dots,q^{-N}\}$ with $\alpha\ge q^{-N-1}$ and $\beta\ge q^{-N-1}$. The authors did not mention this different set of the parameters for the q-Hahn polynomials in* [21]. *However it is given in* [22, p. 76].

4.2 q-classical $\emptyset$-Jacobi/Laguerre polynomials

Let the coefficients σ_2 and σ_1 be quadratic and linear polynomials in x, respectively, such that $\sigma_1(0,q)\sigma_2(0,q) \neq 0$. If σ_1 is written in terms of its root, i.e., $\sigma_1(x,q) = \sigma_1'(0,q)[x - a_1(q)]$, $a_1(q) = -\frac{\sigma_1(0,q)}{\sigma_1'(0,q)}$ then from (1.4)

$$\sigma_2(x,q) = (q-1)\tau'(0,q)x^2 + \left[q\sigma_1'(0,q) + (q-1)\tau(0,q)\right]x - q\sigma_1'(0,q)a_1(q),$$

where $\tau'(0,q) \neq 0$ by hypothesis. Then the q-Pearson equation (1.5) takes the form

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} = \frac{(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}[x-a_2(q)][x-b_2(q)]}{qx - a_1(q)} \tag{4.8}$$

provided that the discriminant denoted by Δ_q,

$$\Delta_q := \left[1 + \frac{(1-q^{-1})\tau(0,q)}{\sigma_1'(0,q)}\right]^2 + 4a_1(q)\big(1-q^{-1}\big)\frac{\tau'(0,q)}{\sigma_1'(0,q)}$$

of the quadratic polynomial in the nominator of f in (4.8) is non-zero. Note that here $x = a_2$ and $x = b_2$ are roots of f which are constant multiples of the roots of σ_2. Moreover, $x = q^{-1}a_1$ is the vertical asymptote of f and $y = 1$ is its y-intercept since $\sigma_2(0,q) = q\sigma_1(0,q)$. On the other hand, the locations of the zeros of f are introduced by the following straightforward lemma.

Lemma 4.8. *Let $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1'(0,q)} \neq 0$. Then, we have the following cases for the roots of the equation $f(x,q) = 0$.*

Case 1. *If Λ_q and $a_1(q)$ have opposite signs, then there are two real distinct roots with opposite signs.*

Case 2. *If Λ_q and $a_1(q)$ have same signs, then there exist three possibilities*

(a) *if $\Delta_q > 0$, f has two real roots with same signs,*

(b) *if $\Delta_q = 0$, f has a double root,*

(c) *if $\Delta_q < 0$, f has a pair of complex conjugate roots.*

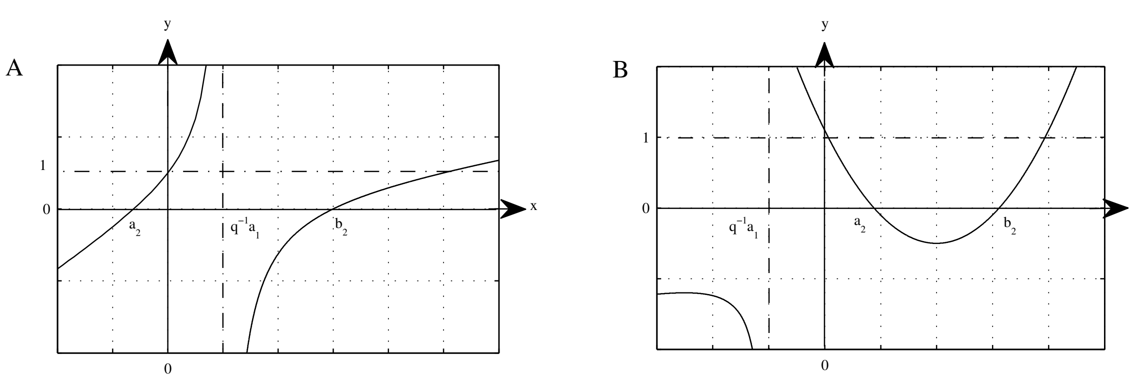

Figure 7. The graph of $f(x,q)$ with $\Lambda_q < 0$. In A, we have **Case 1** with $a_2 < 0 < q^{-1}a_1 < b_2$, and in B, we have **Case 2(a)** $q^{-1}a_1 < 0 < a_2 < b_2$.

In Fig. 7A, we first start with the positivity condition of q-weight function which allows us to exclude the intervals $(-\infty, a_2)$ and $(q^{-1}a_1, b_2)$. Moreover, due to **PIII** $(a_2, q^{-1}a_1)$ can not be used. On the other hand, the interval (b_2, ∞) coincides with the $5th$ case of Theorem 2.3.

Notice that since $\rho(qx,q)/\rho(x,q)=1$ at $x_0=-\tau(0,q)/\tau'(0,q)>b_2$, ρ is decreasing on (x_0,∞). Moreover, since $\rho(qx,q)/\rho(x,q)$ has an infinite limit as $x\to+\infty$, we have $\rho\to 0$ as $x\to\infty$. However, since it is infinite interval, we should check that $\sigma_1(x,q)\rho(x,q)x^k\to 0$ as $x\to\infty$ by using extended q-Pearson equation (4.3). The graph of the function g defined in (4.3) looks like the one for f. Then the analysis of the extended q-Pearson equation leads to $\sigma_1(x,q)\rho(x,q)x^k\to 0$ as $x\to\infty$. Therefore, there exists a suitable ρ supported on the set of points $\{q^{-k}b_2\}_{k\in\mathbb{N}_0}$. Thus, we have the following theorem.

Theorem 4.9. *Let $a_2<0<a_1<b_2$ and $\Lambda_q<0$. Let $a=b_2(q)$ be the zero of $\sigma_2(x,q)$ and $b\to\infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n\in\mathbb{N}_0$ orthogonal w.r.t. the weight function (see the 2nd expression of the $\emptyset$-Jacobi/Laguerre case in Table 2)*

$$\rho(x,q)=|x|^{\alpha}\,H^{(1)}(x)\frac{(qa_2/x,qa/x;q)_\infty}{(a_1/x;q)_\infty},\qquad q^{\alpha}=\frac{q^{-2}\frac{1}{2}\sigma_2''(0,q)}{\sigma_1'(0,q)}\tag{4.9}$$

*supported on $\{q^{-k}a\}_{k\in\mathbb{N}_0}$ (see (2.15) of Theorem 2.3-**5**). Here $H^{(1)}(x)$ is defined by (3.4).*

The OPS in Theorem 4.9 coincides with the case IIa2 in Chapter 11 of [21, p. 337 and p. 358]. An example of this family is the q-Meixner polynomials $M_n(x;b,c;q)$ satisfying the q-EHT with

$$\begin{aligned}&\sigma_1(x,q)=cq^{-2}(x-a_1),\qquad \sigma_2(x,q)=(x-a_2)(x-b_2),\\&\tau(x,q)=-\frac{1}{1-q}x+\frac{cq^{-1}-bc+1}{1-q}\qquad\text{and}\qquad \lambda_n(q)=\frac{[n]_q}{1-q},\end{aligned}\tag{4.10}$$

where $a_1=bq$, $a_2=-bc$ and $b_2=1$. The conditions $\Lambda_q<0$ and $a_2<0<a_1<b_2$ give us the known constrains $c>0$ and $0<b<q^{-1}$ on the parameters of $M_n(x;b,c;q)$ with orthogonality on $\{1,q^{-1},q^{-2},\dots\}$ in the sense (2.15) where

$$d_n^2=(q^{-1}-1)c^{2n}q^{-n(2n+1)}\big(q,-c^{-1}q,bq;q\big)_n\frac{(q,-c;q)_\infty}{(bq;q)_\infty}.$$

In the literature, this relation can be found as an infinite sum [21, p. 360].

In Fig. 7B, the only possible interval is (b_2,∞). An analogous analysis as the one that has been done for the case in Fig. 7A yields $\rho\to 0$ as $x\to\infty$. Moreover, $\sigma_1(x,q)\rho(x,q)x^k\to 0$ as $x\to\infty$ for $k\in\mathbb{N}_0$ by (4.3), then there exists a q-weight function on $[b_2,\infty)$ supported at the points $q^{-k}b_2$ for $k\in\mathbb{N}_0$. Thus we have the following result.

Theorem 4.10. *Let $a_1<0<a_2\le b_2$ and $\Lambda_q<0$. Let $a=b_2$ be the zero of $\sigma_2(x,q)$ and $b\to\infty$. Then, there exists a sequence of polynomials $(P_n)_n$ for $n\in\mathbb{N}_0$ orthogonal w.r.t. the weight function (4.9) (see Theorem 4.9) supported on $\{q^{-k}a\}_{k\in\mathbb{N}_0}$ (see (2.15) of Theorem 2.3-**5**).*

A typical example of this family is again the q-Meixner polynomials orthogonal on the set of points $\{1,q^{-1},q^{-2},\dots\}$. They satisfy the q-EHT with the coefficients (4.10) where $a_1=bq$, $a_2=-bc$ and $b_2=1$. This set of q-Meixner polynomials corresponds to the case IIa2 in Chapter 11 of [21, p. 337 and p. 358] and their orthogonality relation is valid in a larger set of parameters. In fact, the conditions $a_1<0<a_2\le b_2$ and $\Lambda_q<0$ yield $c>0$, $b<0$ and $0<-bc\le 1$. This was not reported in [21].

To conclude this subsection we summarize the other cases in the following theorem.

Theorem 4.11. *The other positive definite orthogonal polynomial solutions of the q-EHT (1.1) and (1.3) for the $\emptyset$-Jacobi/Laguerre case can be divided into three cases.*

Case 2(a). *$a_2\le b_2<a_1<0$ and $\Lambda_q<0$. The polynomials with orthogonality on the union of the sets $\{a_1,a_1q,a_1q^2,\dots\}$ and $\{q^{\pm k}\alpha\}_{k\in\mathbb{N}_0}$ for arbitrary $\alpha>0$ w.r.t. the weight function (see the 1st expression of the $\emptyset$-Jacobi/Laguerre case in Table 2) in the sense (2.14) with*

$$d_n^2=(1-q)q^{-n(2n-1)}\left(a_2b_2a_1^{-1}\right)^{2n}\left(q,a_2^{-1}a_1,b_2^{-1}a_1;q\right)_n$$

$$\times \frac{(q, a_1, qa_1^{-1}, a_2^{-1}b_2^{-1}a_1, qa_2b_2a_1^{-1}; q)_\infty}{(a_2^{-1}a_1, b_2^{-1}a_1, a_2^{-1}, b_2^{-1}, qa_2, qb_2; q)_\infty}. \tag{4.11}$$

This case coincides with the case VIa2 *in Chapter* 10 *of* [21, p. 285 and p. 315].

Case 2(c). *$a_1 < 0$, $a_2, b_2 \in \mathbb{C}$ and $\Lambda_q < 0$. The polynomials with orthogonality on the union of the sets $\{a_1, a_1q, a_1q^2, \dots\}$ and $\{q^{\pm k}\alpha\}_{k\in\mathbb{N}_0}$ for arbitrary $\alpha > 0$ w.r.t. the weight function (see the 1st expression of the $\emptyset$-Jacobi/Laguerre case in Table 2) in the sense (2.14) where the norm is given by (4.11). This case coincides with the case* VIa1 *in Chapter* 10 *of* [21, p. 285 and p. 315]. *The orthogonality relation of this OPS has the same form as in the previous* **Case 2(a)** *but now the zeros of σ_2 are complex numbers.*

Case 2(a). *$0 < a_2 \le b_2 < a_1$ and $\Lambda_q > 0$. The quantum q-Kravchuk polynomials satisfying the q-EHT with*

$$\sigma_1(x,q) = -q^{-2}(x-a_1), \qquad \sigma_2(x,q) = p(x-a_2)(x-b_2),$$
$$\tau(x,q) = -\frac{p}{1-q}x + \frac{p-q^{-1}+q^{-N-1}}{1-q}q \qquad \textit{and} \qquad \lambda_n(q) = \frac{p}{1-q}[n]_q,$$

where $a_1 = q^{-N}$, $a_2 = p^{-1}q^{-N-1}$ and $b_2 = 1$. They are orthogonal w.r.t. the weight function (see the 3th expression of $\emptyset$-Jacobi/Laguerre in Table 2) on the set of points $\{1, q^{-1}, q^{-2}, \dots q^{-N}\}$ with $p \ge q^{-N-1}$ in the sense (2.13) where

$$d_n^2 = \left(q^{-1}-1\right)\frac{1}{(p^{-1}q^{-N};q)_N}p^{-2n}q^{-n(2n+1)}\left(q, pq, q^{-N}; q\right)_n\left(q, p^{-1}q^{-N}, q^{N+1}; q\right)_\infty.$$

In the literature, this relation is usually written as a finite sum [21, p. 362]. *This case coincides with the case* IIb1 *in Chapter* 11 *of* [21, p. 337 and p. 361].

4.3 q-classical $\emptyset$-Jacobi/Hermite polynomials

Let the coefficients σ_2 and σ_1 be quadratic and constant polynomials in x, respectively, such that $\sigma_1(0,q)\sigma_2(0,q) \ne 0$. If $\sigma_1(x,q) = \sigma_1(0,q) \ne 0$ then, from (1.4),

$$\sigma_2(x,q) = q\left[\sigma_1(x,q) + \left(1-q^{-1}\right)x\tau(x,q)\right] = (q-1)\tau'(0,q)x^2 + (q-1)\tau(0,q)x + q\sigma_1(0,q),$$

where $\tau'(0,q) \ne 0$ by hypothesis. Then the q-Pearson equation (1.5) takes the form

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{q^{-1}\sigma_2(x,q)}{\sigma_1(qx,q)} = \left(1-q^{-1}\right)\frac{\tau'(0,q)}{\sigma_1(0,q)}[x-a_2(q)][x-b_2(q)] \tag{4.12}$$

provided that the discriminant denoted by Δ_q,

$$\Delta_q := \left[(1-q^{-1})\frac{\tau(0,q)}{\sigma_1(0,q)}\right]^2 - 4(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1(0,q)}$$

of f in (4.12) is non-zero. Notice that y-intercept of f is $y = 1$ since $\sigma_2(0,q) = q\sigma_1(0,q)$. Moreover, $x = a_2$ and $x = b_2$ indicate its zeros which are constant multiples of the roots of σ_2. The following straightforward lemma allows us to determine the locations of the zeros of f.

Lemma 4.12. *Let $\Lambda_q = \frac{\tau'(0,q)}{\sigma_1(0,q)} \ne 0$. Then we encounter the following cases for the roots of the equation $f(x,q) = 0$.*

Case 1. *If $\Lambda_q > 0$, f has two real distinct roots with opposite signs.*

Case 2. *If $\Lambda_q < 0$, there exist three possibilities*

(a) *if $\Delta_q > 0$, f has two real roots with same signs,*

(b) *if $\Delta_q = 0$, f has a double root,*

(c) *if $\Delta_q < 0$, f has a pair of complex conjugate roots.*

Theorem 4.13. *The positive definite orthogonal polynomial solutions of the q-EHT* (1.1) *and* (1.3) *for the $\emptyset$-Jacobi/Hermite case can be divided into two cases.*

Case 2(a). $0 < a_2 \le b_2$ *and* $\Lambda_q < 0$*. The Al-Salam–Carlitz II polynomials satisfying the q-EHT with*

$$\sigma_1(x,q) = aq^{-1}, \qquad \sigma_2(x,q) = (x-a_2)(x-b_2),$$
$$\tau(x,q) = \frac{1}{q-1}x - \frac{1+a}{q-1} \qquad \text{and} \qquad \lambda_n(q) = \frac{1}{1-q}[n]_q,$$

where $a_2 = a$, $b_2 = 1$. They are orthogonal w.r.t. the weight function (see expression 2 for the $\emptyset$-Jacobi/Hermite case in Table 2*) on the set of points $\{1, q^{-1}, q^{-2}, \dots\}$ with $0 < a \le 1$ in the sense* (2.15) *where*

$$d_n^2 = \big(q^{-1}-1\big)a^n q^{-n^2}(q;q)_n(q;q)_\infty.$$

In the literature, this relation is usually written as an infinite sum [21, p. 357]*. This case coincides with the case* Ia1 *in Chapter* 11 *of* [21, p. 335 and pp. 355–357].

Case 2(c). $a_2(q), b_2(q) \in \mathbb{C}$ *and* $\Lambda_q < 0$*. The discrete q-Hermite II polynomials satisfying the q-EHT with*

$$\sigma_1(x,q) = q^{-1}, \qquad \sigma_2(x,q) = (x-a_2)(x-b_2),$$
$$\tau(x,q) = \frac{1}{q-1}x \qquad \text{and} \qquad \lambda_n(q) = \frac{1}{1-q}[n]_q,$$

where $a_2 = -i, b_2 = i \in \mathbb{C}$. They are orthogonal w.r.t. the weight function (see expression 1 *for the $\emptyset$-Jacobi/Hermite in Table* 2*) on the set of points $\{\pm q^{\mp k}\}_{k\in\mathbb{N}_0}$ in the sense* (2.17) *where*

$$d_n^2 = (1-q)q^{-n^2}(q;q)_n \frac{(q,-q,-1,-1,-q;q)_\infty}{(i,-i,-iq,iq,-i,i,iq,-iq;q)_\infty}.$$

This case corresponds to the case Ia1 *in Chapter* 11 *and case* Va2 *in Chapter* 10 *of* [21, p. 335, pp. 355–356, p. 283 and pp. 314–315].

4.4 q-classical $\emptyset$-Laguerre/Jacobi polynomials

Let the coefficients σ_2 and σ_1 be linear and quadratic polynomials in x, respectively, such that $\sigma_1(0,q)\sigma_2(0,q) \neq 0$. If σ_1 is written in terms of its roots, i.e., $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$, then from (1.4) $\sigma_2(x,q) = \sigma_2'(0,q)x + \sigma_2(0,q)$ where

$$\sigma_2'(0,q) = -q\left[\frac{1}{2}\sigma_1''(0,q)[a_1(q)+b_1(q)] - (1-q^{-1})\tau(0,q)\right] \neq 0 \qquad \text{and}$$
$$\sigma_2(0,q) = \frac{1}{2}\sigma_1''(0,q)qa_1(q)b_1(q) \neq 0$$

provided that $\tau'(0,q) = -\frac{\frac{1}{2}\sigma_1''(0,q)}{(1-q^{-1})}$. Therefore, the q-Pearson equation (1.5) takes the form

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{-\left[a_1(q)+b_1(q) - \frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right][x-a_2(q)]}{[qx-a_1(q)][qx-b_1(q)]},$$

where $\left[a_1(q)+b_1(q)-\frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]a_2(q)=a_1(q)b_1(q)$. Let us point out that $f(x,q)$ intersects the y-axis at the point $y=1$ since $\sigma_2(0,q)=q\sigma_1(0,q)$. On the other hand, we consider the cases depending on the signs of zeros of σ_1 and Λ_q defined by

$$\Lambda_q := \left[a_1(q)+b_1(q)-\frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right].$$

Case 1. $\Lambda_q<0$ with $a_1<0<b_1$, **Case 2.** $\Lambda_q>0$ with $0<a_1<b_1$, **Case 3.** $\Lambda_q<0$ with $0<a_1<b_1$.

Theorem 4.14. *The positive definite orthogonal polynomial solutions of the q-EHT* (1.1) *and* (1.3) *for the $\emptyset$-Laguerre/Jacobi case can be divided into two cases.*

Case 1. $a_1<0<b_1<a_2$ *and* $\Lambda_q<0$. *The big q-Laguerre polynomials satisfying the q-EHT with*

$$\sigma_1(x,q)=q^{-2}(x-a_1)(x-b_1),\qquad \sigma_2(x,q)=-abq(x-a_2),$$
$$\tau(x,q)=-\frac{q^{-1}}{q-1}x+\frac{a+b-abq}{q-1}\qquad\text{and}\qquad \lambda_n(q)=\frac{q^{-n}}{q-1}[n]_q,$$

where $a_1=bq$, $b_1=aq$ *and* $a_2=1$. *They are orthogonal w.r.t. the weight function* (*see the* 1*st expression of the $\emptyset$-Laguerre/Jacobi case in Table* 2) *on* $\{bq,bq^2,bq^3,\dots\}\bigcup\{\dots,aq^3,aq^2,aq\}$ *with* $b<0$ *and* $0<a<q^{-1}$ *in the sense* (2.11) *where*

$$d_n^2=(a-b)q(1-q)(-ab)^nq^{n(n+3)/2}(q;q)_n(aq,bq;q)_n\frac{(q,a^{-1}bq,ab^{-1}q;q)_\infty}{(aq,bq;q)_\infty}.$$

This case coincides with the case VIIa1 *in Chapter* 10 *of* [21, p. 292 and p. 318].

Case 2. $0<a_1<a_2<b_1$ *and* $\Lambda_q>0$. *The affine q-Kravchuk polynomials satisfying the q-EHT with*

$$\sigma_1(x,q)=q^{-1}(x-a_1)(x-b_1),\qquad \sigma_2(x,q)=-pq^{1-N}(x-a_2),$$
$$\tau(x,q)=\frac{1}{1-q}x-\frac{pq+q^{-N}-pq^{1-N}}{1-q}\qquad\text{and}\qquad \lambda_n(q)=\frac{1}{q-1}[n]_{q^{-1}},$$

where $a_1=pq$, $b_1=q^{-N}$ *and* $a_2=1$. *They are orthogonal w.r.t. the weight function* (*see the* 2*nd expression of the $\emptyset$-Laguerre/Jacobi case in Table* 2) *on* $\{1,q^{-1},q^{-2},\dots,q^{-N}\}$ *with* $0<p<q^{-1}$ *in the sense* (2.13) *where*

$$d_n^2=(-1)^np^{n-N}\left(q^{-1}-1\right)q^{-N(n+1)}q^{n(n+1)/2}\left(q,pq,q^{-N};q\right)_n\frac{(q,q^{N+1};q)_\infty}{(pq;q)_\infty}.$$

In the literature, this relation is usually written as a finite sum [21, p. 364]. *This case coincides with the case* IIIb3 *in Chapter* 11 *of* [21, p. 343 and p. 363].

4.5 q-classical $\emptyset$-Hermite/Jacobi polynomials

Let the coefficients σ_2 and σ_1 be constant and quadratic polynomials in x, respectively, such that $\sigma_1(0,q)\sigma_2(0,q)\neq 0$. If σ_1 can be written in terms of its roots, i.e., $\sigma_1(x,q)=\frac{1}{2}\sigma_1''(0,q)[x-a_1(q)][x-b_1(q)]$, then, from (1.4)

$$\sigma_2(x,q)=\sigma_2(0,q)=q\frac{1}{2}\sigma_1''(0,q)a_1(q)b_1(q)$$

provided that $(1-q^{-1})\tau'(0,q) = -\frac{1}{2}\sigma_1''(0,q)$ and $(1-q^{-1})\tau(0,q) = \frac{1}{2}\sigma_1''(0,q)[a_1(q)+b_1(q)]$. Therefore, the q-Pearson equation (1.5) becomes

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{a_1(q)b_1(q)}{[qx-a_1(q)][qx-b_1(q)]}.$$

Notice that the point $y=1$ is y-intercept of f.

Theorem 4.15. *The positive definite orthogonal polynomial solutions of the q-EHT* (1.1) *and* (1.3) *for the $\emptyset$-Hermite/Jacobi case appear only when $a_1 < 0 < b_1$. It corresponds to the Al-Salam–Carlitz I polynomials which satisfy the q-EHT with*

$$\sigma_1(x,q) = q^{-1}(x-a_1)(x-b_1), \qquad \sigma_2(x,q) = a,$$
$$\tau(x,q) = \frac{1}{1-q}x - \frac{1+a}{1-q} \qquad \textit{and} \qquad \lambda_n(q) = \frac{q^{1-n}}{q-1}[n]_q,$$

where $a_1 = a$ and $b_1 = 1$. They are orthogonal w.r.t. the weight function (see the $\emptyset$-Hermite/Jacobi case in Table 2*) on the set $\{a, qa, q^2a, \dots\} \bigcup \{\dots, q^2, q, 1\}$ with $a < 0$ in the sense* (2.11) *where*

$$d_n^2 = (1-a)(-a)^n q^{\binom{n}{2}}(1-q)(q;q)_n \big(q, aq, a^{-1}q; q\big)_\infty.$$

This case coincides with the case VIIa1 *in Chapter* 10 *of* [21, p. 292 and pp. 318–320]*. We note that the discrete q-Hermite I polynomials are special case of Al-Salam–Carlitz I polynomials (see* [21, p. 320] *for further details).*

4.6 q-classical 0-Jacobi/Jacobi polynomials

Let the coefficients σ_2 and σ_1 be quadratic polynomials in x such that $\sigma_1(0,q) = \sigma_2(0,q) = 0$. If σ_1 is written as $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x-a_1(q)]$ then, from (1.4) $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x$ where

$$\frac{1}{2}\sigma_2''(0,q) = q\left[\frac{1}{2}\sigma_1''(0,q) + \big(1-q^{-1}\big)\tau'(0,q)\right] \neq 0,$$
$$\sigma_2'(0,q) = q\big(1-q^{-1}\big)\tau(0,q) - \frac{1}{2}\sigma_1''(0,q)a_1(q).$$

Then it follows from (1.5) that

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \left[1 + \frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]\frac{[x-a_2(q)]}{q[qx-a_1(q)]}, \qquad x \neq 0 \tag{4.13}$$

provided that $\left[1 + \frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]a_2(q) = \left[a_1(q) - \frac{(1-q^{-1})\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]$. Let us point out that Λ_q defined in (4.2) is also horizontal asymptote of $f(x,q)$ in (4.13). Moreover, f intersects the y-axis at the point

$$y := y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right].$$

In the zero cases notice that one of the boundary of (a,b) interval could be zero. Therefore for such a case we need to know the behavior of ρ at the origin.

Lemma 4.16. *If $0 < y_0 < 1$, then $\rho(z,q) \to 0$ as $z \to 0$. Otherwise it diverges to $\mp\infty$.*

Proof. From (2.10), we write $\rho(q^k x, q) = \rho(x, q)\prod_{i=0}^{k-1} \frac{q^{-1}\sigma_2(q^i x, q)}{\sigma_1(q^{i+1}x, q)}$ from which

$$\rho(q^k x, q) = q^{-k}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]^k \frac{(x/a_2(q);q)_k}{(qx/a_1(q);q)_k}\rho(x,q)$$

is obtained. Taking $k \to \infty$ the result follows. ■

Here we introduce the following two cases which include all possibilities.

Case 1. $\Lambda_q > 0$ with **(a)** $0 < y_0 < 1$ or **(b)** $y_0 > 1$ or **(c)** $y_0 < 0$.

Case 2. $\Lambda_q < 0$ with **(a)** $0 < y_0 < 1$ or **(b)** $y_0 > 1$ or **(c)** $y_0 < 0$.

Theorem 4.17. *The positive definite orthogonal polynomial solutions of the q-EHT* (1.1) *and* (1.3) *for the* 0*-Jacobi/Jacobi case can be divided into three cases.*

Case 1(a). $0 < a_1 < a_2$*,* $0 < q^2\Lambda_q < 1$ *and* $0 < qy_0 < 1$*. The little q-Jacobi polynomials satisfying the q-EHT with*

$$\sigma_1(x,q) = q^{-2}x(x-a_1), \qquad \sigma_2(x,q) = abqx(x-a_2),$$
$$\tau(x,q) = \frac{1-abq^2}{(1-q)q}x + \frac{aq-1}{(1-q)q} \qquad \text{and} \qquad \lambda_n(q) = -q^{-n}[n]_q\frac{1-abq^{n+1}}{1-q}, \tag{4.14}$$

where $a_1 = 1$ *and* $a_2 = b^{-1}q^{-1}$*. They are orthogonal w.r.t. the weight function (see equation* 3 *in Table* 1*) on* $\{\dots, q^2, q, 1\}$ *with* $0 < a < q^{-1}$ *and* $0 < b < q^{-1}$ *the sense* (2.12) *where*

$$d_n^2 = a^n q^{n^2}(1-q)\frac{(q, abq;q)_n}{(abq, abq^2;q)_{2n}}(aq, bq;q)_n\frac{(q, abq^2;q)_\infty}{(aq, bq;q)_\infty}.$$

In the literature, this relation can be found as an infinite sum [21, p. 312]*. This case corresponds to the case* IVa3 *in Chapter* 10 *of* [21, p. 312].

Case 2(a). $a_2 < 0 < a_1$*,* $q^2\Lambda_q < 0$ *and* $0 < qy_0 < 1$*. The little q-Jacobi polynomials with orthogonality on the set of points* $\{\dots, q^2, q, 1\}$ *w.r.t. the weight function (see equation* 3 *in Table* 1*) with* $0 < a < q^{-1}$ *and* $b < 0$*. They satisfy the q-EHT with the coefficients in* (4.14) *where* $a_1 = 1$ *and* $a_2 = b^{-1}q^{-1}$*. This case coincides with the case* IVa4 *in Chapter* 10 *of* [21, p. 278 and p. 312]*. This extends the orthogonality relation of the little q-Jacobi polynomials for* $0 < a < q^{-1}$ *and* $0 < b < q^{-1}$ *to a larger set of the parameters* $0 < a < q^{-1}$ *and* $b < 0$*. Notice that combining this with the previous Case 1(a) one can obtain the orthogonality relation of the little q-Jacobi polynomials for* $0 < aq < 1$*,* $bq < 1$*.*

Case 2(c). $0 < a_2 < a_1$*,* $q^2\Lambda_q < 0$ *and* $qy_0 < 0$*. The q-Kravchuk polynomials satisfying the q-EHT with*

$$\sigma_1(x,q) = q^{-2}x(x-a_1), \qquad \sigma_2(x,q) = -px(x-a_2),$$
$$\tau(x,q) = \frac{1+pq}{(1-q)q}x - \frac{p+q^{-N-1}}{1-q} \qquad \text{and} \qquad \lambda_n(q) = -q^{-n}[n]_q\frac{1+pq^n}{1-q},$$

where $a_1 = q^{-N}$ *and* $a_2 = 1$ *They are orthogonal w.r.t. the weight function (see equation* 4 *in Table* 1*) on* $\{1, q^{-1}, q^{-2}, \dots, q^{-N}\}$ *with* $p > 0$ *in the sense* (2.13) *where*

$$d_n^2 = (q^{-1}-1)p^{-N}q^{-\binom{N+1}{2}}\left(-q^{-N}p\right)^n q^{n^2}\frac{1+p}{1+pq^{2n}}(-pq;q)_N\left(q, q^{N+1};q\right)_\infty\frac{(q, -pq^{N+1};q)_n}{(-p, q^{-N};q)_n}.$$

In the literature, this relation is usually written as a finite sum [22, p. 98]*. This case is not mentioned in* [21] *for* $0 < q < 1$*. However the q-Kravchuk polynomials with this set of parameters are described in* [22, p. 98].

4.7 q-classical 0-Jacobi/Laguerre polynomials

Let σ_2 and σ_1 be quadratic and linear polynomials in x, respectively, such that $\sigma_2(0,q) = \sigma_1(0,q) = 0$. If $\sigma_1(x,q) = \sigma_1'(0,q)x$, then from (1.4), $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 + \sigma_2'(0,q)x$ where

$$\frac{1}{2}\sigma_2''(0,q) = q\big(1-q^{-1}\big)\tau'(0,q) \neq 0 \qquad \text{and}$$
$$\sigma_2'(0,q) = q\big[\sigma_1'(0,q) + \big(1-q^{-1}\big)\tau(0,q)\big] \neq 0$$

provided that $(1-q^{-1})\tau(0,q) \neq -\sigma_1'(0,q)$. For this case the q-Pearson equation reads

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = q^{-1}\big(1-q^{-1}\big)\frac{\tau'(0,q)}{\sigma_1'(0,q)}[x - a_2(q)], \tag{4.15}$$

where $-(1-q^{-1})\frac{\tau'(0,q)}{\sigma_1'(0,q)}a_2(q) = 1 + \frac{(1-q^{-1})\tau(0,q)}{\sigma_1'(0,q)}$. Let us point out that f intersects the y-axis at the point

$$y := y_0 = q^{-1}\left[1 + \frac{(1-q^{-1})\tau(0,q)}{\sigma_1'(0,q)}\right].$$

Notice that for the zero cases one of the boundary of (a,b) interval could be zero. This requires to find the behavior of ρ at the origin.

Lemma 4.18. *If $0 < y_0 < 1$, then $\rho(z,q) \to 0$ as $z \to 0$. Otherwise it diverges to $\mp\infty$.*

Proof. From (4.15) it follows that

$$\rho(q^k x,q) = q^{-k}\left[1 + \frac{(1-q^{-1})\tau(0,q)}{\sigma_1'(0,q)}\right]^k (x/a_2(q);q)_k\rho(x,q)$$

from where the result follows. ∎

Again we identify the cases depending on σ_2, $\Lambda_q := \frac{\tau'(0,q)}{\sigma_1'(0,q)}$ and y_0. **Case 1.** $\Lambda_q > 0$, $a_2 > 0$ and $y_0 > 1$, **Case 2.** $\Lambda_q < 0$, $a_2 < 0$ and $0 < y_0 < 1$, **Case 3.** $\Lambda_q < 0$, $a_2 > 0$ and $y_0 < 0$.

Theorem 4.19. *The positive definite orthogonal polynomial solutions of the q-EHT (1.1) and (1.3) for the 0-Jacobi/Laguerre case can be divided into two cases.*

Case 2. *$a_2 < 0$, $\Lambda_q < 0$ and $0 < qy_0 < 1$. The q-Laguerre polynomials satisfying the q-EHT with*

$$\sigma_1(x,q) = q^{-2}x, \qquad \sigma_2(x,q) = q^{\alpha}x(x-a_2),$$
$$\tau(x,q) = -\frac{q^{\alpha}}{1-q}x + \frac{q^{-1}-q^{\alpha}}{1-q} \qquad \textit{and} \qquad \lambda_n(q) = [n]_q\frac{q^{\alpha}}{1-q},$$

where $a_2 = -1$. They are orthogonal w.r.t. the weight function (see the 1st expression of the 0-Jacobi/Laguerre case in Table 2) on $\{q^{\pm k}\}_{k\in\mathbb{N}_0}$ with $\alpha > -1$ in the sense (2.16) where

$$d_n^2 = q^{-n}(1-q)\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\frac{(q,-q^{\alpha+1},-q^{-\alpha};q)_\infty}{(q^{\alpha+1},-q,-q;q)_\infty}.$$

This case coincides with the case IIIa2 *in Chapter* 10 *of* [21, p. 272 and pp. 309–311].

Case 3. *$a_2 > 0$, $\Lambda_q < 0$ and $qy_0 < 0$. The q-Charlier polynomials satisfying the q-EHT with*

$$\sigma_1(x,q) = aq^{-2}x, \qquad \sigma_2(x,q) = x(x-a_2),$$

$$\tau(x,q) = -\frac{1}{1-q}x + \frac{a+q}{(1-q)q} \qquad \textit{and} \qquad \lambda_n(q) = [n]_q\frac{1}{1-q},$$

where $a_2 = 1$. They are orthogonal w.r.t. the weight function (see the 2nd expression of the 0-Jacobi/Laguerre case in Table 2) on $\{1, q^{-1}, q^{-2}, \dots\}$ with $a > 0$ in the sense (2.15) where

$$d_n^2 = a^{2n}q^{-n(2n+1)}\left(-a^{-1}q, q; q\right)_n(-a, q; q)_\infty.$$

In the literature, this relation is usually written as an infinite sum [21, p. 360]*. This case coincides with the case* IIa2 *in Chapter* 11 *of* [21, p. 337 and pp. 358–360]*.*

4.8 q-classical 0-Bessel/Jacobi polynomials

Let σ_2 and σ_1 be quadratic polynomials in x, respectively, such that $\sigma_2'(0,q) = 0$ and $\sigma_2(0,q) = \sigma_1(0,q) = 0$. If $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x - a_1(q)]$, $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} \neq -\frac{1}{(1-q^{-1})}$ and $\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = \frac{a_1(q)}{(1-q^{-1})}$, then from (1.4) we have $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 = q\left[\frac{1}{2}\sigma_1''(0,q) + (1-q^{-1})\tau'(0,q)\right]x^2$. As a result, the q-Pearson equation (1.5) becomes

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{\left[1 + \frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right]x}{q[qx - a_1(q)]}.$$

Let us point out that $f(x,q)$ passes through the origin and the line

$$y = \Lambda_q := q^{-2}\left[1 + \frac{(1-q^{-1})\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right] \neq 0$$

is its horizontal asymptote.

Theorem 4.20. *The positive definite orthogonal polynomial solutions of the q-EHT (1.1) and (1.3) for the 0-Bessel/Jacobi case appear only when $a_1 > 0$ and $q^2\Lambda_q < 0$. This corresponds to the alternative q-Charlier (q-Bessel) polynomials which satisfy the q-EHT with*

$$\sigma_1(x,q) = -q^{-2}x(x - a_1), \qquad \sigma_2(x,q) = ax^2,$$
$$\tau(x,q) = -\frac{1+aq}{(1-q)q}x + \frac{1}{(1-q)q} \qquad \textit{and} \qquad \lambda_n(q) = q^{-n}[n]_q\frac{1+aq^n}{1-q},$$

where $a_1 = 1$. They are orthogonal w.r.t. the weight function (see the 0-Bessel/Jacobi case in Table 2) on $\{\dots, q^2, q, 1\}$ with $a > 0$ in the sense (2.12) where

$$d_n^2 = a^n q^{n(3n-1)/2}(-aq, q; q)_\infty \frac{(q, -a; q)_n}{(-a, -aq; q)_{2n}}.$$

In the literature, this relation can be found as an infinite sum [21, p. 314]*. This case coincides with the case* IVa5 *in Chapter* 10 *of* [21, p. 278 and p. 313]*.*

4.9 q-classical 0-Bessel/Laguerre polynomials

Let σ_2 and σ_1 be quadratic and linear polynomials in x, respectively, such that $\sigma_2'(0,q) = 0$ and $\sigma_2(0,q) = \sigma_1(0,q) = 0$. If $\sigma_1(x,q) = \sigma_1'(0,q)x$, then, from (1.4) $\sigma_2(x,q) = \frac{1}{2}\sigma_2''(0,q)x^2 = q(1-q^{-1})\tau'(0,q)x^2$ provided that $(1-q^{-1})\tau(0,q) = -\sigma_1'(0,q)$. So the q-Pearson equation is now

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = q^{-1}\left(1 - q^{-1}\right)\frac{\tau'(0,q)}{\sigma_1'(0,q)}x.$$

Clearly, f passes through the origin. According to the sign of $\Lambda_q := \frac{\tau'(0,q)}{\sigma_1'(0,q)}$ we have only one possible case.

Theorem 4.21. *The positive definite orthogonal polynomial solutions of the q-EHT* (1.1) *and* (1.3) *for the* 0*-Bessel/Laguerre case appear only when $a_2 = 0$, $\Lambda_q < 0$ and $qy_0 = 0$. This corresponds to the Stieltjes–Wigert polynomials which satisfy the q-EHT with*

$$\sigma_1(x,q) = q^{-2}x, \qquad \sigma_2(x,q) = x^2,$$
$$\tau(x,q) = -\frac{1}{1-q}x + \frac{1}{(1-q)q} \qquad \text{and} \qquad \lambda_n(q) = [n]_q \frac{1}{1-q}.$$

They are orthogonal w.r.t. the weight function (*see the* 0*-Bessel/Laguerre case in Table* 2) *on $\{q^{\mp k}\}_{k\in\mathbb{N}_0}$ in the sense* (2.16) *where*

$$d_n^2 = q^{-n}(1-q)\frac{(-tq, -1/t, q; q)_\infty}{(q^2;q)_n}.$$

This case coincides with the case IIIa2 *in Chapter* 10 *of* [21, p. 272 and p. 309].

4.10 q-classical 0-Laguerre/Jacobi polynomials

Let σ_2 and σ_1 be linear and quadratic polynomials in x, respectively, such that $\sigma_2(0,q) = \sigma_1(0,q) = 0$. If $\sigma_1(x,q) = \frac{1}{2}\sigma_1''(0,q)x[x - a_1(q)]$ and $\frac{\tau'(0,q)}{\frac{1}{2}\sigma_1''(0,q)} = -\frac{1}{(1-q^{-1})}$, then from (1.4) we get $\sigma_2(x,q) = \sigma_2'(0,q)x = q\left[(1-q^{-1})\tau(0,q) - \frac{1}{2}\sigma_1''(0,q)a_1(q)\right]x$. Therefore, the q-Pearson equation has the form

$$f(x,q) := \frac{\rho(qx,q)}{\rho(x,q)} = \frac{(1-q^{-1})\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)} - a_1(q)}{q[qx - a_1(q)]}.$$

Notice that $y = 0$ is the horizontal asymptote of $f(x,q)$, and its y-intercept is

$$y := y_0 = q^{-1}\left[1 - \frac{(1-q^{-1})}{a_1(q)}\frac{\tau(0,q)}{\frac{1}{2}\sigma_1''(0,q)}\right].$$

Theorem 4.22. *The positive definite orthogonal polynomial solutions of the q-EHT* (1.1) *and* (1.3) *for the* 0*-Laguerre/Jacobi case appear only when $a_1 > 0$ and $0 < qy_0 < 1$. This corresponds to the little q-Laguerre* (*Wall*) *polynomials which satisfy the q-EHT with*

$$\sigma_1(x,q) = q^{-2}x(a_1 - x), \qquad \sigma_2(x,q) = ax,$$
$$\tau(x,q) = -\frac{1}{(1-q)q}x + \frac{1-aq}{(1-q)q} \qquad \text{and} \qquad \lambda_n(q) = \frac{q^{-n}}{1-q}[n]_q,$$

where $a_1 = 1$. They are orthogonal w.r.t. the weight function (*see the* 0*-Laguerre/Jacobi case in Table* 2) *on the set of points $\{\dots, q^2, q, 1\}$ with $0 < a < q^{-1}$ in the sense* (2.12) *where*

$$d_n^2 = a^n q^{n^2}\frac{(q;q)_\infty}{(aq;q)_\infty}(q, aq; q)_n.$$

In the literature, this relation can be found as an infinite sum [21, p. 312]. *This case coincides with the case* IVa4 *in Chapter* 10 *of* [21, p. 278 and p. 312].

5 Concluding remarks

The q-polynomials of the Hahn class have been revisited by use of a direct and very simple geometrical approach based on the qualitative analysis of solutions of the q-Pearson (1.5) and the *extended* q-Pearson (4.3) equations. By this way, it is shown that it is possible to introduce in a unified manner all orthogonal polynomial solutions of the q-EHT, which are orthogonal w.r.t. a measure supported on some set of points in certain intervals. In this review article we are able to extend the well known orthogonality relations for the big q-Jacobi polynomials (see Theorems 4.3 and 4.4), q-Hahn polynomials (see Theorems 4.5 and 4.6), and for the q-Meixner polynomials (see Theorem 4.10) to a larger set of their parameters.

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