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Unification of Stieltjes-Calogero type relations for the zeros of classical orthogonal polynomials

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The classical orthogonal polynomials (COPs) satisfy a second-order differential equation of the form $\sigma(x)y'' + \tau(x)y' + \lambda y = 0$, which is called the equation of hypergeometric type (EHT). It is shown that two numerical methods provide equivalent schemes for the discrete representation of the EHT. Thus, they lead to the same matrix eigenvalue problem. In both cases, explicit closed-form expressions for the matrix elements have been derived in terms only of the zeros of the COPs. On using the equality of the entries of the resulting matrices in the two discretizations, unified identities related to the zeros of the COPs are then introduced. Hence, most of the formulas in the literature known for the roots of Hermite, Laguerre and Jacobi polynomials are recovered as the particular cases of our more general and unified relationships. Furthermore, we present some novel results that were not reported previously.

Keywords: Stieltjes–Calogero relations; equation of hypergeometric type; classical orthogonal polynomials; pseudospectral methods; Galerkin with numerical integration scheme

1. Introduction

The first equation concerning the zeros of the classical orthogonal polynomials (COPs) dates back to more than a century ago. In 1885, Stieltjes [1, 2] established the relations

$$\sum_{\substack{i=1 \\ i\neq n}}^{N} \frac{1}{x_i - x_n} = -x_n, \qquad n = 1, 2, \ldots, N \tag{1}$$

where x_n denotes the roots of Hermite polynomial $H_N(x)$ of degree N. Stieltjes discovered the relations in (1) while dealing with the following electrostatic model: Place N movable unit charges at distinct points on the line, and determine the equilibrium position of these charges when the interaction forces arise from a logarithmic potential in the harmonic field [3, 4]. He showed that the equilibrium position is attained at the zeros of $H_N(x)$. Moreover, he obtained similar relations for the Laguerre and Jacobi polynomials by considering two other electrostatic models. These results are known as the Stieltjes relations. A short and light survey on the electrostatic interpretation of the zeros of some well-known families of orthogonal polynomials is given by Marcellán and co-workers [5]. Further results on electrostatic models can be found in [6–8].

Nearly a hundred years after Stieltjes, in [9–14], Calogero, Ahmed, Bruschi, Olshanetsky and Perelomov made a series of investigations and extended the results to the Bessel functions as well. Besides (1), the authors presented a number of different relations involving the zeros of the COPs, most of which were obtained during the study of certain integrable many-body problems in one dimension.

On the other hand, Ronveaux and Muldoon [15] revisited the Stieltjes relations in order to enlarge the class of differential equations to which the theory applies, and to find sum formulas not only for the zeros of the polynomial solutions but also for their derivatives. In [16, 17], Case generated sum rules for the powers of the zeros for polynomials satisfying higher order differential equations of particular forms having polynomial coefficients.

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Recently, Kudryashov and Demina [3] derived non-linear relations for some families of polynomials by comparing two power series. Two series come from a representation of the polynomial in terms of its zeros and from the Painlevé expansion of solution of the associated non-linear differential equation near a removable singular point. Then, in [18], Anghel extended the results of Kudryashov and Demina from polynomials to entire functions of finite order by using Gil's relations.

In this article, we deal with a unification of the relations of these kinds, which are satisfied by the zeros of the COPs. In this way, we can obtain the separate identities known for the Hermite, Laguerre and Jacobi polynomials as particular cases of our unified formulation by choosing the coefficient functions $\sigma(x)$ and $\tau(x)$ in the equation of hypergeometric type (EHT) appropriately. The idea behind our approach is numerical in character, which is quite different than those in the literature. To be specific, in section 2, we first construct the pseudospectral discretization of the EHT. Then, we use the Galerkin approach incorporated with the Gaussian quadrature as an alternative method. Finally, we prove that the two matrix representations of the EHT are identical in the resulting matrix eigenvalue problems. Hence, in section 3, we deduce several unified relations as required, on equating the corresponding entries of these matrices. To verify the accuracy of our general results valid for arbitrary values of the coefficients $\sigma(x)$ and $\tau(x)$, we specify them in section 4 for a recovery of the celebrated formulas related to the roots of the Hermite, Laguerre and Jacobi polynomials. New relations are also introduced in this section. Section 5 concludes the paper with further remarks as usual.

2. Pseudospectral and Galerkin with Gauss quadrature schemes for EHT

Consider the EHT

$$\sigma(x)y'' + \tau(x)y' + \lambda y = 0, \quad x \in (a,b) \subseteq \mathbb{R} \tag{2}$$

in which $\sigma(x)$ and $\tau(x)$ are polynomials of degree at most two and one, respectively, and λ is a real parameter. It can be written in the self-adjoint form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sigma(x)\rho(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \lambda\rho(x)y = 0 \tag{3}$$

where $\rho(x)$ is a weight function satisfying the Pearson equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sigma(x)\rho(x)\right] = \tau(x)\rho(x). \tag{4}$$

The EHT has polynomial solutions, say $y = p_n(x)$, of degree n for specific values of λ

$$\lambda := \lambda_n = -n\left[\tau' + \frac{1}{2}(n-1)\sigma''\right], \quad \tau' \neq 0, \tag{5}$$

standing for the simple and discrete eigenvalues of the problem, when n is a non-negative integer [19]. These polynomial solutions are characterized by the Rodrigues' formula

$$p_n(x) = \frac{K_n}{\rho(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\sigma^n(x)\rho(x)\right] = k_n x^n + \ldots \tag{6}$$

where k_n is the coefficient of the leading term and K_n denotes a renormalization constant. On the other hand, if the equation

$$\sigma(x)\rho(x)x^k = 0, \quad k = 0, 1, \ldots \tag{7}$$

is satisfied at the boundaries of (a,b) interval, then the polynomial solutions $p_n(x)$ are orthogonal with respect to the weight function $\rho(x)$ on (a,b) in the sense that

$$\int_a^b p_m(x)p_n(x)\rho(x)\mathrm{d}x = h_n^2\delta_{mn} \tag{8}$$

where h_n is a normalization constant and δ_{mn} denotes Kronecker's delta [19]. Moreover, the n-th degree polynomial solution $p_n(x)$ of the EHT has exactly n real and distinct zeros in the orthogonality interval (a,b) [19].

Now, consider a pseudospectral method that is based on N-th degree polynomial interpolation of a function $y(x)$ denoted by $I_N(x)$,

$$I_N(x) = \sum_{n=0}^{N}\ell_n(x)y_n, \tag{9}$$

where $y_n := y(x_n)$ are the actual values of the function $y(x)$ at the specified nodes $x = x_n$ for $n = 0, 1, \ldots, N$ [20]. The set of Lagrange interpolating polynomials $\{\ell_n(x)\}$ of degree N is defined by

$$\ell_n(x) = \frac{\psi_{N+1}(x)}{(x-x_n)\psi'_{N+1}(x_n)} \tag{10}$$

for each n, in which $\psi_{N+1}(x) = p_{N+1}(x)/h_{N+1}$ is the $(N+1)$-th degree normalized polynomial solution of the EHT, and x_n stands for the $N+1$ real and distinct roots of $p_{N+1}(x)$. The interpolant $I_N(x)$ and the function $y(x)$ agree, at least, at the nodes $y_n = I_N(x_n)$, because $\ell_n(x_m) = \delta_{mn}$. The derivatives of the function $y(x)$ can also be approximated by differentiating the interpolant $I_N(x)$. Indeed, making use of the so-called *differentiation matrix* defined by

$$\boldsymbol{D}^{(k)} := \left[d_{mn}^{(k)}\right] = \frac{\mathrm{d}^k}{\mathrm{d}x^k}[\ell_n(x)]\bigg|_{x=x_m}, \quad k = 1, 2, \ldots, N \tag{11}$$

for $m, n = 0, 1, \ldots, N$, we determine the approximate derivative values $\boldsymbol{y}^{(k)}$ by means of the algebraic system written in the matrix-vector form

$$\boldsymbol{y}^{(k)} = \boldsymbol{D}^{(k)}\boldsymbol{y}, \quad \boldsymbol{y}^{(k)} = \left[I_N^{(k)}(x_0), I_N^{(k)}(x_1), \ldots, I_N^{(k)}(x_N)\right]^{\mathrm{T}} \tag{12}$$

in terms of the function values y_n at the nodes, where $\boldsymbol{y} = [y_0, y_1, \ldots, y_N]^{\mathrm{T}}$.

In particular, the entries of the first-order and the second-order differentiation matrices are expressible, respectively, as

$$d_{mn}^{(1)} = \frac{1}{2}\begin{cases} \dfrac{2}{x_m - x_n}\dfrac{\psi'_{N+1}(x_m)}{\psi'_{N+1}(x_n)} & \text{if } m \neq n \\ -\dfrac{\tau(x_n)}{\sigma(x_n)} & \text{if } m = n \end{cases} \tag{13}$$

and

$$d_{mn}^{(2)} = \frac{1}{3}\begin{cases} -\dfrac{3}{x_m - x_n}\left[\dfrac{\tau(x_m)}{\sigma(x_m)} + \dfrac{2}{x_m - x_n}\right]\dfrac{\psi'_{N+1}(x_m)}{\psi'_{N+1}(x_n)} & \text{if } m \neq n \\ \dfrac{1}{\sigma(x_n)}\left\{\dfrac{\tau(x_n)}{\sigma(x_n)}\left[\sigma'(x_n) + \tau(x_n)\right] + N\left[\tau' + \frac{1}{2}(N+1)\sigma''\right]\right\} & \text{if } m = n \end{cases} \tag{14}$$

on differentiating (10), taking into account that $\psi_{N+1}(x_n) = 0$ and using (2) [21].

Now proposing the interpolant $I_N(x)$ in (9) to be an approximate solution of the EHT and requiring the satisfaction of the EHT at the nodal points x_m

$$-\sum_{n=0}^{N}\left[\sigma(x_m)\ell''_n(x_m) + \tau(x_m)\ell'_n(x_m)\right]y_n = \lambda\sum_{n=0}^{N}\ell_n(x_m)y_n = \lambda\sum_{n=0}^{N}\delta_{mn}y_n \tag{15}$$

for $m = 0, 1, \ldots, N$, we obtain a discrete representation

$$\widehat{\mathcal{K}}\boldsymbol{y} = \lambda\boldsymbol{y}, \quad \widehat{\mathcal{K}}_{mn} = -\left[\sigma(x_m)\ell''_n(x_m) + \tau(x_m)\ell'_n(x_m)\right] \tag{16}$$

of the EHT. Here, the vector $\boldsymbol{y} = [y_0, y_1, \ldots, y_N]^{\mathrm{T}}$ now contains the values of an eigenfunction $y(x)$ of the EHT at the nodal points, and the entries $\widehat{\mathcal{K}}_{mn}$ of the matrix $\widehat{\mathcal{K}}$ take the form

$$\widehat{\mathcal{K}}_{mn} = \begin{cases} \dfrac{2\sigma(x_m)}{(x_m - x_n)^2}\dfrac{\psi'_{N+1}(x_m)}{\psi'_{N+1}(x_n)} & \text{if } m \neq n \\ \dfrac{\tau(x_n)}{6\sigma(x_n)}\left[\tau(x_n) - 2\sigma'(x_n)\right] - \frac{1}{3}N\left[\tau' + \frac{1}{2}(N+1)\sigma''\right] & \text{if } m = n, \end{cases} \tag{17}$$

which is not symmetric [21]. However, it may be symmetrized by a similarity transformation of the form $\mathcal{K} = \boldsymbol{S}^{-1}\widehat{\mathcal{K}}\boldsymbol{S}$ in which $\boldsymbol{S} = \operatorname{diag}\{s_0, s_1, \ldots, s_m, \ldots, s_N\}$ is a diagonal matrix with entries

$$s_m = \sqrt{\sigma(x_m)}\,\psi'_{N+1}(x_m), \quad m = 0, 1 \ldots, N. \tag{18}$$

Then, we may replace the unsymmetric system (16) by the symmetric one $\mathcal{K}\boldsymbol{y} = \lambda\boldsymbol{y}$ where

$$\mathcal{K}_{mn} = \frac{1}{6}\begin{cases} \dfrac{12\sqrt{\sigma(x_m)\sigma(x_n)}}{(x_m - x_n)^2} & \text{if } m \neq n \\ \dfrac{\tau(x_n)}{\sigma(x_n)}\left[\tau(x_n) - 2\sigma'(x_n)\right] - 2N\left[\tau' + \dfrac{1}{2}(N+1)\sigma''\right] & \text{if } m = n \end{cases} \tag{19}$$

because the similar matrices share the same eigenvalues [21]. Hence, the eigenvalues (5) of the EHT for $n = 0, 1, \ldots, N$ are exactly determined by the symmetric matrix eigenvalue problem because the exact polynomial solution of the EHT is used in the construction of the approximate solution. That is, the EHT is satisfied exactly by $I_N(x)$ for $n \le N$. In other words, the n-th degree polynomial solution $y = p_n(x)$, for $n \le N$, and the interpolant $I_N(x)$ agree not only at the nodes x_n but also at any point x of the interval (a, b). This is, in fact, a direct consequence of the existence and uniqueness theorem for polynomial interpolation [22]. As a result, the interpolant $I_N(x)$ may be regarded as another rearrangement of each polynomial solution of the EHT.

The numerical treatment of a differential equation by using its actual solution has no meaning from a numerical point of view. Nevertheless, there is no inconvenience theoretically in doing so. As a matter of fact, our aim is not to solve the EHT numerically. The main objective is to obtain a discrete analogue of the EHT in which the matrix elements have closed-form analytical expressions in terms of the roots of its polynomial solutions. Actually, the construction of the matrix in (19) requires only the knowledge of zeros of an orthonormal polynomial solution $\psi_{N+1}(x)$ of the EHT for prescribed values of the coefficients $\sigma(x)$ and $\tau(x)$.

It should be noted here that the zeros of $p_{N+1}(x)$, and hence $\psi_{N+1}(x)$, can be computed as the eigenvalues of a tridiagonal matrix $\boldsymbol{R}$ of size $(N+1) \times (N+1)$ having the diagonal

$$R_{nn} = \eta_{n-1} - \eta_n, \quad \eta_{n-1} = n\left[\frac{\tau(0) + (n-1)\sigma'(0)}{\tau' + (n-1)\sigma''}\right], \quad n = 0, 1, \ldots, N \tag{20}$$

and the off-diagonal entries

$$R_{n,n+1} = R_{n+1,n} := A_n = \frac{k_n h_{n+1}}{h_n k_{n+1}}, \quad n = 0, 1, \ldots, N-1 \tag{21}$$

where k_n and h_n are defined in (6) and (8), respectively. In (20), the parameter η_{n-1} depends on $\sigma(x)$ and $\tau(x)$. It is also possible to express the A_n in terms of the coefficients of the EHT. For further information, we refer the readers to [23].

As an alternative numerical procedure, we multiply the self-adjoint form (3) of the EHT by $\ell_m(x)$, for $m = 0, 1, \ldots, N$, and integrate from a to b to obtain

$$-\sigma(x)y'(x)\ell_m(x)\rho(x)\,\Big|_a^b + \int_a^b \sigma(x)\rho(x)y'(x)\ell_m'(x)\mathrm{d}x = \lambda\int_a^b y(x)\rho(x)\ell_m(x)\mathrm{d}x. \tag{22}$$

Now inserting the N-th degree polynomial interpolant in (9) as an approximate solution for $y(x)$, we obtain

$$\sum_{n=0}^{N}\left[\int_a^b \ell_m'(x)\ell_n'(x)\sigma(x)\rho(x)\mathrm{d}x\right]y_n = \lambda\sum_{n=0}^{N}\left[\int_a^b \ell_n(x)\ell_m(x)\rho(x)\mathrm{d}x\right]y_n \tag{23}$$

where the leftmost term in (22) vanishes by the use of (7) because each $\ell_i(x)$ is a polynomial of degree N. This is the Galerkin method that results in another matrix representation of the EHT

$$\mathcal{A}\boldsymbol{y} = \lambda\mathcal{M}\boldsymbol{y} \tag{24}$$

in which the matrix $\mathcal{A}$ and the mass matrix $\mathcal{M}$ are defined by

$$\mathcal{A}_{mn} = \int_a^b \ell_m'(x)\ell_n'(x)\sigma(x)\rho(x)\mathrm{d}x \quad \text{and} \quad \mathcal{M}_{mn} = \int_a^b \ell_m(x)\ell_n(x)\rho(x)\mathrm{d}x, \tag{25}$$

respectively. Clearly, the integrands in (25) are polynomials of degrees at most $2N$, and therefore, the integrals may be evaluated exactly by the Gauss quadrature rule based on $(N+1)$ points. The degrees of the integrands can be rechecked by keeping in mind that $\sigma(x)$ is a polynomial of degree at most two, and $\ell_n(x)$ is a polynomial of exact degree N. Hence, we write

$$\mathcal{A}_{mn} = \sum_{i=0}^{N} \ell_m'(x_i)\ell_n'(x_i)\sigma(x_i)\omega_i \quad \text{and} \quad \mathcal{M}_{mn} = \sum_{i=0}^{N} \ell_m(x_i)\ell_n(x_i)\omega_i \tag{26}$$

where ω_i is referred to as the Christoffel numbers defined by [24]

$$\omega_i = \frac{1}{A_N\psi_{N+1}'(x_i)\psi_N(x_i)} = \frac{\lambda_{2N+2}}{2(N+1)\sigma(x_i)\left[\psi_{N+1}'(x_i)\right]^2}, \tag{27}$$

the determination of which depends on the COP used in the construction of the Gauss quadrature. Here, A_N is the parameter in (21), and λ_n the eigenvalues in (5). Then, it is not difficult to see that $\mathcal{M}$ reduces to a diagonal matrix with entries

$$\mathcal{M}_{mn} = \omega_n\delta_{mn} = \frac{\lambda_{2N+2}}{2(N+1)\sigma(x_n)\left[\psi_{N+1}'(x_n)\right]^2}\delta_{mn} \tag{28}$$

by virtue of $\ell_n(x_m) = \delta_{mn}$. On the other hand, the general entry $\mathcal{A}_{mn}$ of the matrix $\mathcal{A}$ is given by

$$\mathcal{A}_{mn} = \begin{cases} \dfrac{\lambda_{2N+2}}{2(N+1)\psi'_m\psi'_n}\left[\dfrac{\sigma_m\tau_n - \sigma_n\tau_m}{2\sigma_m\sigma_n(x_m - x_n)} + \displaystyle\sum_{\substack{i=0\\ i\neq m,n}}^{N} \dfrac{1}{(x_i - x_m)(x_i - x_n)}\right] & \text{if } m \neq n \\[2ex] \dfrac{\lambda_{2N+2}}{2(N+1)(\psi'_n)^2}\left[\left(\dfrac{\tau_n}{2\sigma_n}\right)^2 + \displaystyle\sum_{\substack{i=0\\ i\neq n}}^{N} \dfrac{1}{(x_i - x_n)^2}\right] & \text{if } m = n \end{cases} \tag{29}$$

where we have used (13) and the last equation in (27). In (29), we adopt the abbreviations $\psi'_k = \psi'_{N+1}(x_k)$, $\tau_k = \tau(x_k)$ and $\sigma_k = \sigma(x_k)$. In practice, when $m \neq n$, the three cases $(i \neq m, n)$, $(i = m,\ i \neq n)$ and $(i \neq m,\ i = n)$ should be handled separately. Likewise, when $m = n$, the two cases $(i \neq n)$ and $(i = n)$ require an individual treatment.

The generalized matrix eigenvalue problem in (24) can be reduced to a standard one $\widehat{\mathcal{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ immediately, where $\widehat{\mathcal{T}} = \mathcal{M}^{-1}\mathcal{A}$ is again unsymmetric. Fortunately, it is interesting to see that the matrix $\boldsymbol{S}$, whose entries are given by (18), also symmetrizes $\widehat{\mathcal{T}}$, so that $\mathcal{T} = \boldsymbol{S}^{-1}\widehat{\mathcal{T}}\boldsymbol{S}$ turns out to be a symmetric one with entries

$$\mathcal{T}_{mn} = \begin{cases} \sqrt{\sigma_m\sigma_n}\left[\dfrac{\sigma_m\tau_n - \sigma_n\tau_m}{2\sigma_m\sigma_n(x_m - x_n)} + \displaystyle\sum_{\substack{i=0\\ i\neq m,n}}^{N} \dfrac{1}{(x_i - x_m)(x_i - x_n)}\right] & \text{if } m \neq n \\[2ex] \sigma_n\left[\left(\dfrac{\tau_n}{2\sigma_n}\right)^2 + \displaystyle\sum_{\substack{i=0\\ i\neq n}}^{N} \dfrac{1}{(x_i - x_n)^2}\right] & \text{if } m = n. \end{cases} \tag{30}$$

Now we state the main theorem of this section.

Theorem 2.1

The two discrete representations $\mathcal{K}\boldsymbol{y} = \lambda\boldsymbol{y}$ and $\mathcal{T}\boldsymbol{y} = \lambda\boldsymbol{y}$ of the EHT suggested by the methods of pseudospectral and Galerkin incorporated with a numerical integration, respectively, are equivalent with $\mathcal{K} = \mathcal{T}$.

Proof

The eigenvalues of the matrices $\mathcal{K}$ in (19) and $\mathcal{T}$ in (30) coincide with the first $N+1$ eigenvalues $\lambda_n = -n\left[\tau' + \frac{1}{2}(n-1)\sigma''\right]$, for $n = 0, 1, \ldots, N$, of the EHT. Therefore, $\mathcal{K}$ and $\mathcal{T}$ are similar matrices. Furthermore, we can prove that they are equal. To this end, first note, from the definition $\widehat{\mathcal{T}} = \mathcal{M}^{-1}\mathcal{A}$ and from (28), that $\widehat{\mathcal{T}}_{mn} = \mathcal{A}_{mn}/w_m$, where $\mathcal{A}_{mn}$ may be rewritten as

$$\mathcal{A}_{mn} = \sigma(x)\rho(x)\ell_m(x)\ell'_n(x)\,\big|_a^b - \int_a^b \left[\ell'_n(x)\sigma(x)\rho(x)\right]'\ell_m(x)\mathrm{d}x \tag{31}$$

on integrating the first equation in (25) by parts. Now using (4) and (7), we have

$$\mathcal{A}_{mn} = -\int_a^b \left[\sigma(x)\ell''_n(x) + \tau(x)\ell'_n(x)\right]\ell_m(x)\rho(x)\mathrm{d}x, \tag{32}$$

which may be evaluated exactly by a Gauss quadrature based on $N+1$ points because the integrand is a polynomial of degree at most $2N$. So choosing the quadrature points to be the $N+1$ real and distinct zeros x_n of the polynomial solution $p_{N+1}(x)$ of the EHT, we find that

$$\mathcal{A}_{mn} = -\sum_{i=0}^{N}\left[\sigma(x_i)\ell''_n(x_i) + \tau(x_i)\ell'_n(x_i)\right]\ell_m(x_i)\omega_i \tag{33}$$

having only one non-zero term when $i = m$, that is,

$$\mathcal{A}_{mn} = -\omega_m\left[\sigma(x_m)\ell''_n(x_m) + \tau(x_m)\ell'_n(x_m)\right] \tag{34}$$

because $\ell_m(x_i) = \delta_{im}$. Comparing this and (16), we see that $\widehat{\mathcal{K}}_{mn} = \mathcal{A}_{mn}/w_m$ and that $\widehat{\mathcal{T}} = \widehat{\mathcal{K}}$. Hence, it follows from the definitions $\mathcal{T} = \boldsymbol{S}\widehat{\mathcal{T}}\boldsymbol{S}^{-1}$ and $\mathcal{K} = \boldsymbol{S}\widehat{\mathcal{K}}\boldsymbol{S}^{-1}$ that $\mathcal{T} = \mathcal{K}$, which completes the proof. □

3. Relations satisfied by the zeros of the COPs

Making use of the equality of the matrix elements $\mathcal{K}_{mn}$ and $\mathcal{T}_{mn}$ in (19) and (30), we derive here several algebraic equations satisfied by the real and distinct zeros of an arbitrary polynomial solution $p_n(x)$ of the EHT, $\sigma(x)y'' + \tau(x)y' + \lambda y = 0$ with specific λ values in (5).

Theorem 3.1
The zeros x_n of the $p_{N+1}(x)$ satisfy the relation

$$\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{x_i - x_n} = \frac{\tau(x_n)}{2\sigma(x_n)} \tag{35}$$

for each $n = 0, 1, \ldots, N$.

Proof
Equating the off-diagonal elements of (19) and (30), we immediately obtain

$$\sum_{\substack{i=0\\ i\neq m,n}}^{N} \frac{1}{(x_i - x_m)(x_i - x_n)} = \frac{2}{(x_m - x_n)^2} + \frac{1}{2(x_m - x_n)}\left[\frac{\tau(x_m)}{\sigma(x_m)} - \frac{\tau(x_n)}{\sigma(x_n)}\right], \tag{36}$$

which can be rewritten as

$$\sum_{\substack{i=0\\ i\neq m,n}}^{N} \left(\frac{1}{x_i - x_m} - \frac{1}{x_i - x_n}\right) - \frac{2}{x_m - x_n} - \frac{1}{2}\left[\frac{\tau(x_m)}{\sigma(x_m)} - \frac{\tau(x_n)}{\sigma(x_n)}\right] = 0 \tag{37}$$

where we have expanded the term under the summation symbol into its partial fractions. Rearranging the term $2(x_m - x_n)^{-1} = (x_m - x_n)^{-1} - (x_n - x_m)^{-1}$, we obtain

$$G_m(N) - G_n(N) = 0, \quad G_n(N) = \sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{x_i - x_n} - \frac{\tau(x_n)}{2\sigma(x_n)} \tag{38}$$

implying that $G_n(N)$ is independent of $n = 0, 1, \ldots, N$ for all N and that $G_n(N) = c$, where c is some constant. So it remains to show that $c = 0$. To this end, it is enough to consider the case of $N = 0$, so does $n = 0$, and $G_0(0) = -\tau(x_0)/[2\sigma(x_0)]$. In this case, x_0 is the only root of the first-degree polynomial solution $p_1(x)$ of the EHT, that is, $\sigma(x)p_1''(x) + \tau(x)p_1'(x) + \lambda p_1(x) = 0$. Now replacing x by x_0, we arrive at $\tau(x_0) = 0$ because p_1' is a non-zero constant, p_1'' is identically zero and $p_1(x_0) = 0$. That is, $G_0(0) = 0$, and hence, $G_n(N) = 0$ for all n and N, and the result follows. □

Corollary 3.2
The zeros x_n of the $p_{N+1}(x)$ satisfy an equation of the form

$$\sum_{n=0}^{N} \frac{\tau(x_n)}{\sigma(x_n)} = 0 \tag{39}$$

for each N.

Proof
Summing (35) over n, we obtain

$$\sum_{n=0}^{N} \frac{\tau(x_n)}{\sigma(x_n)} = 2\sum_{n=0}^{N}\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{x_i - x_n} \tag{40}$$

in which the right-hand side vanishes because the summand is unsymmetric in the dummy indices i and n. □

Theorem 3.3
The zeros x_n of the $p_{N+1}(x)$ satisfy the relation

$$\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i - x_n)^2} = -\frac{1}{3\sigma(x_n)}\left\{\left[\tfrac{1}{4}\tau(x_n) + \sigma'(x_n)\right]\frac{\tau(x_n)}{\sigma(x_n)} + N\left[\tau' + \tfrac{1}{2}(N+1)\sigma''\right]\right\} \tag{41}$$

for each $n = 0, 1, \ldots, N$.

Proof
This is a direct consequence of the equality of the matrices $\mathcal{K}$ and $\mathcal{T}$ in (19) and (30), respectively. □

Theorem 3.4
The zeros x_n of the $p_{N+1}(x)$ satisfy the relation

$$\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i-x_n)^3} = -\frac{1}{4}\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i-x_n)^2}\left[\frac{\tau(x_i)}{\sigma(x_i)} - \frac{\tau(x_n)}{\sigma(x_n)}\right] \tag{42}$$

for each $n = 0, 1, \ldots, N$.

Proof
By dividing (36) by $(x_n - x_m)$ and taking the sum over n, there follows

$$\begin{aligned}\sum_{\substack{n=0\\ n\neq m}}^{N} \frac{2}{(x_n-x_m)^3} &= -\frac{1}{2}\sum_{\substack{n=0\\ n\neq m}}^{N} \frac{1}{(x_n-x_m)^2}\left[\frac{\tau(x_n)}{\sigma(x_n)} - \frac{\tau(x_m)}{\sigma(x_m)}\right]\\ &\quad + \sum_{\substack{n=0\\ n\neq m}}^{N} \frac{1}{(x_n-x_m)} \sum_{\substack{i=0\\ i\neq m,n}}^{N} \frac{1}{(x_i-x_m)(x_i-x_n)}\end{aligned} \tag{43}$$

in which the rightmost term sums to zero because the summand is unsymmetric in the dummy variables n and i. Then, the replacement of the dummy indices n and m by i and n, respectively, yields the result. □

Theorem 3.5
The zeros x_n of the $p_{N+1}(x)$ satisfy an equation of the form

$$\begin{aligned}\sum_{n=0}^{N}\left[\frac{\tau(x_n)}{\sigma(x_n)} + 2\frac{\sigma'(x_n)}{\sigma(x_n)}\right]\frac{\tau(x_n)}{\sigma(x_n)} &= -2N\left[\tau' + \tfrac{1}{2}(N+1)\sigma''\right]\sum_{n=0}^{N}\frac{1}{\sigma(x_n)}\\ &\quad + \sum_{n=0}^{N}\sum_{\substack{i=0\\ i\neq n}}^{N}\frac{1}{x_i-x_n}\left[\frac{\tau(x_i)}{\sigma(x_i)} - \frac{\tau(x_n)}{\sigma(x_n)}\right]\end{aligned} \tag{44}$$

for each N.

Proof
Squaring equation (35) and summing over n, we write

$$\sum_{n=0}^{N}\left[\frac{\tau(x_n)}{2\sigma(x_n)}\right]^2 = \sum_{n=0}^{N}\sum_{\substack{i=0\\ i\neq n}}^{N}\frac{1}{(x_i-x_n)^2} + \sum_{n=0}^{N}\sum_{\substack{i=0\\ i\neq n}}^{N}\sum_{\substack{k=0\\ k\neq i,n}}^{N}\frac{1}{(x_k-x_i)(x_k-x_n)}. \tag{45}$$

Now, using (36) and theorem 3.3 and rearranging the result, we complete the proof. □

Theorem 3.6
The zeros x_n of the $p_{N+1}(x)$ satisfy an equation of the form

$$\sum_{n=0}^{N}\left[\tau(x_n) - 2\sigma'(x_n)\right]\frac{\tau(x_n)}{\sigma(x_n)} = (2\sigma'' - \tau')N(N+1) \tag{46}$$

for each N.

Proof
The proof is based on the well-known theorem of the basic linear algebra stating that the sum of eigenvalues of a matrix is equal to its trace. Thus, summing the diagonal elements $\mathcal{K}_{nn}$ in (19) of the matrix $\mathcal{K}$ and the eigenvalues λ_n in (5), we obtain the required result. □

Theorem 3.7
The zeros x_n of the $p_{N+1}(x)$ satisfy an equation of the form

$$\sum_{n=0}^{N}\sum_{\substack{i=0\\ i\neq n}}^{N}\frac{2\sigma(x_n)}{(x_i-x_n)^2} = -\left[\tau' + \tfrac{1}{3}(N-1)\sigma''\right]N(N+1) - \sum_{n=0}^{N}\frac{[\tau(x_n)]^2}{2\sigma(x_n)} \tag{47}$$

$$= -\left[\tfrac{1}{2}\tau' + \tfrac{1}{3}(N+2)\sigma''\right]N(N+1) - \sum_{n=0}^{N}\frac{\tau(x_n)\sigma'(x_n)}{\sigma(x_n)} \tag{48}$$

for each N.

Proof
The proof is similar to the previous one, where now the trace of the matrix $\mathcal{T}$ is determined by using $\mathcal{T}_{nn}$ in (30). Clearly, we employ theorem 3.6 to write (48). □

Theorem 3.8
The summation rule

$$\sum_{n=0}^{N} x_n = -(N+1)\left[\frac{\tau(0)+N\sigma'(0)}{\tau'+N\sigma''}\right] \tag{49}$$

holds for the zeros x_n of the $p_{N+1}(x)$ for each N.

Proof
The zeros of the COPs are the eigenvalues of the tridiagonal matrix $\boldsymbol{R}$ defined by (20) and (21). Thus, the trace of $\boldsymbol{R}$ may easily be calculated because the diagonal elements R_{nn} of $\boldsymbol{R}$ in (20) are telescoping. It is obvious that the result in (49) stands for a special case of the so-called Newton sums of forms

$$s_k = \sum_{n=0}^{N} x_n^k, \quad k = 1, 2, \ldots$$

when $\boldsymbol{k} = 1$. □

4. Application to the Hermite, Laguerre and Jacobi polynomials

4.1. *Relations satisfied by the zeros of the Hermite polynomials $H_n(x)$*

When $\sigma(x) = 1$ and $\tau(x) = -2x$ in the EHT, the suitably scaled polynomial solutions are the Hermite polynomials $H_n(x)$. In this case, we obtain from theorems 3.1, 3.3 and 3.4 the well-known Stieltjes–Calogero relations [1, 2, 13]

$$\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{x_i - x_n} = -x_n, \quad \sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i - x_n)^2} = \frac{1}{3}\left(2N - x_n^2\right), \quad \sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i - x_n)^3} = -\frac{1}{2}x_n, \tag{50}$$

respectively, for the zeros of $H_{N+1}(x)$. Equation (36) leads to another relation [13]

$$\frac{2}{(x_m - x_n)^2} = 1 + \sum_{\substack{i=0\\ i\neq m,n}}^{N} \frac{1}{(x_i - x_m)(x_i - x_n)}. \tag{51}$$

By corollary 3.2 or theorem 3.8, the trivial result $\sum_{n=0}^{N} x_n = 0$ is deduced, because the zeros of $H_n(x)$ are located symmetrically about the origin. This tells us that the sum of all the odd powers of the roots is also zero. From theorem 3.5 (or 3.6) and theorem 3.7, we have, respectively,

$$\sum_{n=0}^{N} x_n^2 = \frac{1}{2}N(N+1) = \sum_{n=0}^{N}\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i - x_n)^2}. \tag{52}$$

4.2. *Relations satisfied by the zeros of the Laguerre polynomials $L_n^{\gamma}(x)$*

When $\sigma(x) = x$ and $\tau(x) = \gamma + 1 - x$, the EHT in (2) reduces to the Laguerre differential equation. Therefore, with these coefficient functions, the unified formulas give rise several relations for the roots $x_n \in (0, \infty)$ of the Laguerre polynomials $L_{N+1}^{\gamma}(x)$. First, from theorem 3.1, we obtain the Stieltjes relation [11, 12, 25]

$$\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{x_i - x_n} = -\frac{1}{2}\left(1 - \frac{\gamma+1}{x_n}\right). \tag{53}$$

Then, corollary 3.2 gives

$$\sum_{n=0}^{N}\left(1 - \frac{\gamma+1}{x_n}\right) = 0 \quad \Rightarrow \quad \sum_{n=0}^{N} \frac{1}{x_n} = \frac{N+1}{\gamma+1} \tag{54}$$

the sum of reciprocals of the roots x_n [11]. We have from theorem 3.3

$$\sum_{\substack{i=0\\i\neq n}}^{N}\frac{1}{(x_i-x_n)^2}=-\frac{1}{3x_n}\left[\frac{(\gamma+1-x_n)(\gamma+5-x_n)}{4x_n}-N\right]$$
$$=-\frac{(\gamma+1)(\gamma+5)}{12x_n^2}+\frac{2(N+1)+\gamma+1}{6x_n}-\frac{1}{12}, \tag{55}$$

and theorem 3.4 in conjunction with (53) and (54) gives the result

$$\sum_{\substack{i=0\\i\neq n}}^{N}\frac{1}{(x_i-x_n)^3}=\frac{\gamma+1}{4x_n^2}\sum_{\substack{i=0\\i\neq n}}^{N}\left(\frac{1}{x_i-x_n}-\frac{1}{x_i}\right)=\frac{\gamma+1}{4x_n^2}\left[\frac{\gamma+1-x_n}{2x_n}-\left(\frac{N+1}{\gamma+1}-\frac{1}{x_n}\right)\right]$$
$$=\frac{(\gamma+1)(\gamma+3)}{8x_n^3}-\frac{2(N+1)+\gamma+1}{8x_n^2}, \tag{56}$$

which may be checked with those of [3,11,12]. On the other hand, theorem 3.5 gives

$$\sum_{n=0}^{N}\frac{(\gamma+1-x_n)(\gamma+3-x_n)}{x_n^2}=2N\sum_{n=0}^{N}\frac{1}{x_n}-(\gamma+1)\sum_{n=0}^{N}\frac{1}{x_n}\sum_{\substack{i=0\\i\neq n}}^{N}\frac{1}{x_i} \tag{57}$$

from which the sum [11]

$$\sum_{n=0}^{N}\frac{1}{x_n^2}=\frac{(N+1)(N+\gamma+2)}{(\gamma+1)^2(\gamma+2)} \tag{58}$$

is obtained by excluding the term $i=n$ in the last sum and using (54). Now the sum of (55) over n implies that [10]

$$\sum_{n=0}^{N}\sum_{\substack{i=0\\i\neq n}}^{N}\frac{1}{(x_i-x_n)^2}=\frac{N(N+1)}{4(\gamma+2)} \tag{59}$$

where we have used (54) and (58). The summation formula [14]

$$\sum_{n=0}^{N}\frac{(\gamma+1-x_n)(\gamma-1-x_n)}{x_n}=N(N+1)\quad\Rightarrow\quad\sum_{n=0}^{N}x_n=(N+1)(N+\gamma+1) \tag{60}$$

may be derived from theorem 3.6. The last formula may also be deduced by theorem 3.8 directly. Finally, equation (48) in theorem 3.7 gives a new formula

$$\sum_{n=0}^{N}\sum_{\substack{i=0\\i\neq n}}^{N}\frac{x_n}{(x_i-x_n)^2}=\frac{1}{4}N(N+1) \tag{61}$$

for the roots of the Laguerre polynomial $L_{N+1}^{\gamma}(x)$ that is free of the parameter γ. Furthermore, two interesting formulas

$$\sum_{n=0}^{N}\sum_{\substack{i=0\\i\neq n}}^{N}\frac{\gamma+2-x_n}{(x_i-x_n)^2}=0,\qquad\sum_{n=0}^{N}\sum_{\substack{i=0\\i\neq n}}^{N}\frac{\gamma+2+x_n}{(x_i-x_n)^2}=\frac{1}{2}N(N+1) \tag{62}$$

are introduced from (59) and (61), which we could not find them elsewhere. By the way, equation (36) leads to a relation

$$\sum_{\substack{i=0\\i\neq m,n}}^{N}\frac{1}{(x_i-x_m)(x_i-x_n)}=\frac{2}{(x_m-x_n)^2}-\frac{\gamma+1}{2x_mx_n}, \tag{63}$$

which was most likely not reported previously.

4.3. *Relations satisfied by the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$*

The Jacobi polynomials are solutions of the EHT with $\tau(x) = \beta - \alpha - (\alpha + \beta + 2)x$ and $\sigma(x) = 1 - x^2$. Now the unified general formulas of section 3 suggest numerous relations for the zeros $x_n \in (-1, 1)$ of $P_{N+1}^{(\alpha,\beta)}(x)$. The famous Stieltjes relation [2, 12]

$$\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{x_i - x_n} = -\frac{1}{2}\left(\frac{\alpha+1}{1-x_n} - \frac{\beta+1}{1+x_n}\right) \tag{64}$$

is written down from theorem 3.1. Corollary 3.2 implies that

$$(\alpha+1)\sum_{n=0}^{N} \frac{1}{1-x_n} - (\beta+1)\sum_{n=0}^{N}\frac{1}{1+x_n} = 0. \tag{65}$$

Using theorem 3.6 with some manipulations, we have

$$(\alpha^2-1)\sum_{n=0}^{N} \frac{1}{1-x_n} + (\beta^2-1)\sum_{n=0}^{N}\frac{1}{1+x_n} = \frac{1}{2}(N+1)(N+\alpha+\beta+2)(\alpha+\beta-2) \tag{66}$$

leading to the relations

$$\sum_{n=0}^{N} \frac{1}{1-x_n} = \frac{(N+1)(N+\alpha+\beta+2)}{2(\alpha+1)}, \quad \sum_{n=0}^{N} \frac{1}{1+x_n} = \frac{(N+1)(N+\alpha+\beta+2)}{2(\beta+1)}, \tag{67}$$

which are most probably new. These relations suggest some other new formulas. For instance, the addition and subtraction of the equations in (67) imply that

$$\sum_{n=0}^{N} \frac{1}{1-x_n^2} = \frac{(N+1)(N+\alpha+\beta+2)(\alpha+\beta+2)}{4(\alpha+1)(\beta+1)}, \quad \sum_{n=0}^{N} \frac{x_n}{1-x_n^2} = \frac{(N+1)(N+\alpha+\beta+2)(\beta-\alpha)}{4(\alpha+1)(\beta+1)}, \tag{68}$$

respectively. We also have

$$\sum_{n=0}^{N} \frac{x_n}{1-x_n} = \frac{(N+1)(N-\alpha+\beta)}{2(\alpha+1)}, \quad \sum_{n=0}^{N} \frac{x_n}{1+x_n} = -\frac{(N+1)(N+\alpha-\beta)}{2(\beta+1)} \tag{69}$$

because $x_n/(1-x_n) = 1/(1-x_n) - 1$ and $x_n/(1+x_n) = 1 - 1/(1+x_n)$. Furthermore, by simple algebraic manipulations, there follows

$$\sum_{n=0}^{N} \frac{1+x_n}{1-x_n} = \frac{(N+1)(N+\beta+1)}{\alpha+1}, \quad \sum_{n=0}^{N} \frac{1-x_n}{1+x_n} = \frac{(N+1)(N+\alpha+1)}{\beta+1}. \tag{70}$$

Similar relations may be derived by suitably combining the relations in (67)–(70). Observe that the equations from (67) to (70) are in accordance with the property $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ of the Jacobi polynomials.

Theorems 3.3 and 3.4, on the other hand, reduce to [9, 12]

$$\begin{aligned}\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i-x_n)^2} = \frac{1}{12}\Bigg[&\frac{2N(N+\alpha+\beta+3)+(\alpha+3)(\beta+3)-4}{1-x_n} - \frac{(\alpha+1)(\alpha+5)}{(1-x_n)^2}\\ &+ \frac{2N(N+\alpha+\beta+3)+(\alpha+3)(\beta+3)-4}{1+x_n} - \frac{(\beta+1)(\beta+5)}{(1+x_n)^2}\Bigg]\end{aligned} \tag{71}$$

and

$$\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i-x_n)^3} = -\frac{1}{4}\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{1}{(x_i-x_n)^2}\left(\frac{\beta+1}{1+x_i} - \frac{\alpha+1}{1-x_i} - \frac{\beta+1}{1+x_n} + \frac{\alpha+1}{1-x_n}\right), \tag{72}$$

respectively. After a little algebra and using (64) and the relations in (67), the last one gives the known relation [12]

$$\sum_{\substack{i=0\\ i\neq n}}^{N}\frac{1}{(x_i-x_n)^3}=\frac{1}{8}\left[\frac{(\beta+1)(\beta+3)}{(1+x_n)^3}-\frac{(\alpha+1)(\alpha+3)}{(1-x_n)^3}+\frac{(\alpha+1)(\beta+1)}{(1+x_n)(1-x_n)^2}-\frac{(\alpha+1)(\beta+1)}{(1+x_n)^2(1-x_n)}\right.$$
$$\left.+\frac{(N+1)(N+\alpha+\beta+2)}{(1-x_n)^2}-\frac{(N+1)(N+\alpha+\beta+2)}{(1+x_n)^2}\right], \tag{73}$$

which is written here in a more neatly form.

It is worth mentioning that theorem 3.7 yields

$$\sum_{n=0}^{N}\sum_{\substack{i=0\\ i\neq n}}^{N}\frac{1-x_n^2}{(x_i-x_n)^2}=\frac{1}{12}N(N+1)(4N+3\alpha+3\beta+2), \tag{74}$$

which seems to be new. Finally, the summation rule [14]

$$\sum_{n=0}^{N}x_n=\frac{(N+1)(\beta-\alpha)}{2N+\alpha+\beta+2} \tag{75}$$

may easily be found from theorem 3.8. If this is combined with an appropriate one in (67)–(70), further relations may be derived. However, we do not present them here in order not to overfill the content with similar formulas.

5. Conclusion

In this paper, we have elucidated the general and unified structures of the formulas generating specific algebraic equations for the roots of the polynomials that satisfy the EHT in (2). Actually, the statements of the theorems in section 3, except those of 3.1 and 3.3 [9, 26], are reported for the first time within this generality. Therefore, well-known and certain novel results for the zeros of the COPs associated with the names Hermite, Laguerre and Jacobi are presented in section 4 as special cases. Note that similar equations for the zeros of derivatives $p_n^{(k)}(x)$ may be generated by using the differential–difference relations

$$\frac{d^k}{dx^k}H_n(x)=\frac{2^k n!}{(n-k)!}H_{n-k}(x),\qquad \frac{d^k}{dx^k}L_n^{\gamma}(x)=(-1)^k L_{n-k}^{\gamma+k}(x)$$

and

$$\frac{d^k}{dx^k}P_n^{(\alpha,\beta)}(x)=\frac{1}{2^k}(n+\alpha+\beta+1)_k P_{n-k}^{(\alpha+k,\beta+k)}(x)$$

for the COPs, where $(a)_m=a(a+1)\dots(a+m-1)$ is the Pochhammer symbol.

As another remark, it may be of some interest to recall that the pseudospectral differentiation matrices $\boldsymbol{D}^{(k)}$ of order k have zero row sums [27, 28]. This suggests several Stieltjes–Calogero relations of new kinds. To be specific, the first-order differentiation matrix in (13) provides an identity of the form

$$\sum_{\substack{n=0\\ n\neq m}}^{N}\frac{1}{(x_m-x_n)\psi'_{N+1}(x_n)}=\frac{\tau(x_m)}{2\sigma(x_m)\psi'_{N+1}(x_m)}. \tag{76}$$

Keeping in mind that $\psi_{N+1}(x)$ stands for a normalized polynomial solution of the EHT, we have in the Hermite case, for example,

$$\sum_{\substack{n=0\\ n\neq m}}^{N}\frac{1}{(x_m-x_n)H_N(x_n)}=-\frac{x_m}{H_N(x_m)} \tag{77}$$

where the recursion $H'_{N+1}(x)=2(N+1)H_N(x)$ or $\psi'_{N+1}(x_n)=\sqrt{2(N+1)}\psi_N(x)$ has been used. In principle, it is also possible to evaluate the sums of forms

$$\sum_{\substack{i=0\\ i\neq n}}^{N}\frac{\tau(x_i)}{(x_i-x_n)^M}\quad\text{or}\quad\sum_{\substack{i=0\\ i\neq n}}^{N}\frac{\sigma(x_i)}{(x_i-x_n)^M}\quad\text{for}\quad M=1,2,\dots$$

by means of the idea employed in section 3. As a simple and typical example, multiplying (35) throughout by $\tau(x_n)$, and adding and subtracting the term $\tau(x_i)$ to the numerator on the left-hand side, we have

$$\sum_{\substack{i=0\\ i\neq n}}^{N} \frac{\tau(x_n)-\tau(x_i)+\tau(x_i)}{x_i-x_n} = \frac{[\tau(x_n)]^2}{2\sigma(x_n)} \quad \Rightarrow \quad \sum_{\substack{i=0\\ i\neq n}}^{N} \frac{\tau(x_i)}{x_i-x_n} = N\tau' + \frac{[\tau(x_n)]^2}{2\sigma(x_n)}$$

where $\tau(x_n) - \tau(x_i) = (x_n - x_i)\tau'$ if the Taylor polynomial expansion $\tau(x) = \tau(0) + \tau' x$ of $\tau(x)$ about zero is considered. Unfortunately, however, when $M > 3$, the labour involved in finding and equating the powers of the matrix elements $\mathcal{K}_{mn}$ and $\mathcal{T}_{mn}$ increases dramatically, making the procedure useless from a practical viewpoint.

The present results have been obtained under the orthogonality assumption in (8). Nevertheless, most of the results are valid much more generally. For example, the results for the Laguerre polynomials $L^{\gamma}_{N+1}(x)$ are derived for a real parameter $\gamma > -1$. However, they may be extended to be valid for an arbitrary complex constant $\gamma \neq -1, -2, \ldots, -(N+1)$ by means of analytic continuation. The same idea can be used in the case of the Jacobi polynomials as well, to extend the validity of the results from real parameters $\alpha, \beta > -1$ to arbitrary complex numbers α and β where $\alpha + \beta \neq -2, -3, \ldots, -2(N+1)$.

More general second-order ordinary differential equations than equation (2) with polynomial coefficients have sometimes polynomial solutions as well. To be specific, the exceptional Laguerre and Jacobi or the Heun-like equations are the typical examples for equations of this type [29,30]. It seems that the present technique with appropriate modifications may be applied to determine similar relations between the zeros of polynomial solutions. Furthermore, it may be generalized to deal with the roots of discrete orthogonal polynomials like Hahn, Charlier, Krawchouk and Meixner, which is presently an ongoing research.

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