# An alternative series solution to the isotropic quartic oscillator in $N$ dimensions

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The series solution of the $N$-dimensional isotropic quartic oscillator weighted by an appropriate function which exhibits the correct asymptotic behavior of the wave function is presented. The numerical performance of the solution in Hill's determinant picture is excellent, and yields the energy spectrum of the system to any desired accuracy for the full range of the coupling constant. Furthermore, it converges to the well-known exact solution of the unperturbed harmonic oscillator wave function, when the anharmonic interaction vanishes.

## 1. Introduction

The computation of the energy eigenvalues for spherically symmetric states of the $N$-dimensional Schrödinger equation

$$\left[-\frac{d^2}{dr^2}-\frac{N-1}{r}\frac{d}{dr}+\frac{l(l+N-2)}{r^2}+r^2+\beta r^4\right]\Psi(r)=E\Psi(r),$$
$$r\in[0,\infty)\,, \tag{1.1}$$

has been subjected to intensive study, especially, in one dimension both in quantum field theory and in chemical physics for a long time. Several methods have been introduced and applied usually to the quartic oscillator in (1.1) as it describes the most natural and the first nontrivial perturbation problem over the classical harmonic oscillator solution. Actually, Bell et al. [1] calculated the eigenvalues of (1.1) using the complete basis of eigenfunctions of the $N$-dimensional isotropic harmonic oscillator in the Rayleigh–Ritz variational method. A few years ago, Witwit [2] performed those calculations by perturbative and power series techniques employing again the renormalized exact solution of the harmonic oscillator to characterize the wave function $\Psi(r)$. A renormalized Rayleigh–Schrödinger perturbation theory was also developed by Vrscay in [3]. More recently, two- and three-dimensional quartic oscillators were treated by means of the phase-integral approach [4]. Some other different methods for the quartic oscillator, as well as for more general anharmonic oscillators, may be found in refs. [5–8].

In a preceding paper [9] by the present author, an alternative series solution to the two-dimensional equation corresponding to (1.1) was presented. More specifically, an asymptotically correct wave function was constructed which yields the exact solution of the harmonic oscillator as a special case. Therefore, the main purpose of this article is to extend such a construction to $N$-mode oscillators. In order to make the present article self-contained, we sketch here some of the findings and arguments of [9] adapted to the $N$-dimensional case.

It is clear that the usual boundary conditions of the problem are the regularity and the appropriately vanishing behavior of the wave function at the origin and infinity, respectively. The first condition requires that $\Psi(r)$ behaves like $r^l$ as $r \to 0$. Therefore, it is convenient to transform the dependent variable form $\Psi(r)$ to $\Phi(r)$, where

$$\Psi(r) = r^l \Phi(r) . \tag{1.2}$$

Now the equation satisfied by $\Phi(r)$ is

$$\left( -\frac{d^2}{dr^2} - \frac{2l+N-1}{r}\frac{d}{dr} + r^2 + \beta r^4 \right)\Phi(r) = E\Phi(r) \tag{1.3}$$

subject to

$$\Phi(0) = \text{constant}, \quad \lim_{r\to\infty} \Phi(r) = 0 . \tag{1.4}$$

In general, the number $l$ denotes the angular dependence of the system in a global sense. In the one-dimensional case of $N = 1$, however, $l$ takes on only the special values 0 and 1 for which eq. (1.3), where $r$ is replaced by $x \in (-\infty, \infty)$, corresponds to the equation of the symmetric and antisymmetric states, respectively.

As can be readily shown, the problem of $N$-dimensional isotropic harmonic oscillator, where $\beta = 0$, admits exact solutions in the form

$$\Phi_n(r) = e^{-\frac{1}{2}r^2} L_n^{(\frac{1}{2}N+l-1)}(r^2), \quad n = 0, 1, \ldots . \tag{1.5}$$

Here, the $L_n^{(p)}(z)$ are the associated Laguerre polynomials satisfying the differential equation [10]

$$z\frac{d^2y}{dz^2} + (p+1-z)\frac{dy}{dz} + \frac{1}{4}(E-N-2l)y = 0 \tag{1.6}$$

when the unperturbed energies are defined to be

$$E \equiv E_{nl} = 2(\tfrac{1}{2}N + 2n + l) \tag{1.7}$$

for all values of the quantum numbers $n$ and $l$. It is easily seen, in one dimension, that the Laguerre polynomials $L_n^{(-1/2)}(r^2)$ and $L_n^{(1/2)}(r^2)$ for $l = 0$ and 1 turn out to be the Hermite polynomials $H_{2n}(x)$ and $H_{2n+1}(x)/x$, respectively, where $x = \mp r$, i.e. $x \in (-\infty, \infty)$.

The exact solution of the harmonic oscillator may be regarded as a guide to solving the full equation (1.3) rescaled as

$$H\Phi = \nu E\Phi, \quad H = -\frac{d^2}{dr^2} - \frac{2l+N-1}{r}\frac{d}{dr} + \nu^2 r^2 + (1-\nu^3)r^4 \tag{1.8}$$

for the sake of dealing with a bounded potential. Now the parameter $\nu$ defined by

$$\nu = (1+\beta)^{-1/3}, \quad 0 \leqslant \nu \leqslant 1, \tag{1.9}$$

stands for the anharmonic interaction such that $\nu = 1$ and $\nu = 0$ represent, respectively, the harmonic and the purely quartic oscillators. In what follows, an exponentially weighted eigensolution similar to that of the harmonic oscillator may be suggested. An exponential factor in the solution is necessary owing to the essential singularity of the problem at infinity. It is a simple matter to determine the asymptotic form of the wave function $\Phi(r)$, which may be written as

$$\Phi(r) \approx e^{-\frac{1}{3}(1-\nu^3)^{1/2}r^3 - \frac{1}{2}(1-\nu^3)^{-1/2}r} \tag{1.10}$$

as $r \to \infty$, satisfying the boundary condition. So an appropriate solution weighted by (1.10) is obviously bounded but $H\Phi/\Phi$ is not, since $H\Phi = O(r)\Phi$ as $r \to \infty$. We notice, from (1.8), that the $H^m\Phi/\Phi$, for $m = 0, 1, \ldots$ are all bounded due to the linearity of the operator $H$. Therefore, the weighting factor could be taken as

$$\Phi(r) \approx r^{-\frac{1}{2}(2l+N+1)} e^{-\frac{1}{3}(1-\nu^3)^{1/2}r^3 - \frac{1}{2}(1-\nu^3)^{-1/2}r} \tag{1.11}$$

to make $H\Phi/\Phi$ bounded, i.e. $H\Phi = O(1)\Phi$ as $r \to \infty$, so do $H^2\Phi/\Phi$, $H^3\Phi/\Phi, \cdots$. Hence we might propose a complete solution of the form

$$\Phi(r) = r^{-\frac{1}{2}(2l+N+1)} e^{-\frac{1}{3}(1-\nu^3)^{1/2}r^3 - \frac{1}{2}(1-\nu^3)^{-1/2}r} F(r) \tag{1.12}$$

in terms of a transformed dependent variable $F(r)$ to be determined. However, a solution of this type would still be defective because of two main reasons. Firstly, if we wish to satisfy the constancy of $\Phi(r)$ as $r \to 0$, $F(r)$ must then behave like $r^{\frac{1}{2}(2l+N+1)}$ for sufficiently small values of $r$, which destroys again the boundedness of $H\Phi/\Phi$ at infinity. Secondly, it does not converge to the harmonic oscillator solution (1.5) as $\nu \to 1$, or $\beta \to 0$.

In section 2, following the ideas developed for the one- and two-dimensional problems [9,11] corresponding to (1.8) we construct a solution which reflects every desired property of the exact wave function. We show in section 3 that Hill's determinant of the problem leads to a matrix eigenvalue problem, where the coefficient matrix is of an upper Hessenberg form. The last section contains the numerical applications and the discussion of certain aspects of the energy spectrum of $N$-dimensional oscillators.

## 2. Reformulation of the problem

In this section, we start with the introduction of the transformation motivated by the success of [9]

$$\xi = (1+\alpha r^2)^{-1/2}, \quad 0 < \alpha < \infty, \tag{2.1}$$

which maps the semi-infinite interval of the original variable $r$ into $\xi \in [0, 1]$, where $\alpha$ is a finite positive parameter. Thus the problem alters to the form

$$T\Phi(\xi) = \nu E\Phi(\xi), \quad \Phi(0) = 0, \quad \Phi(1) = \text{constant}, \tag{2.2}$$

with the transformed Hamiltonian $T$,

$$T = \alpha\xi^4(\xi^2-1)\frac{d^2}{d\xi^2} + \alpha\xi^3(3\xi^2+2l+N-3)\frac{d}{d\xi} + \alpha^{-1}\nu^2(\xi^{-2}-1)$$
$$+\alpha^{-2}(1-\nu^3)(\xi^{-2}-1)^2. \tag{2.3}$$

To take care of the essential singularity located now at $\xi = 0$ we require that

$$\Phi(\xi) \approx \xi^c e^{-\frac{1}{3}a\xi^{-3}+b\xi^{-1}}, \quad a > 0, \tag{2.4}$$

upon substitution of which into (2.2) we get the result

$$\alpha^2 T\Phi \approx \{(1-\nu^3-\alpha^3a^2)\xi^{-4} + [2\alpha^3ab + \alpha^3a^2 - 2(1-\nu^3) + \alpha\nu^2]\xi^{-2}$$
$$+\alpha^3a(2l+N+1-2c)\xi^{-1} + O(1)\}\Phi \tag{2.5}$$

as $\xi \to 0$. We infer, from the last equation, that $T\Phi/\Phi$ remains finite at $\xi = 0$ if

$$1-\nu^3-\alpha^3a^2 = 0, \tag{2.6}$$

$$2\alpha^3ab + \alpha^3a^2 - 2(1-\nu^3) + \alpha\nu^2 = 0 \tag{2.7}$$

and

$$c = \tfrac{1}{2}(2l+N+1). \tag{2.8}$$

Therefore, the true wave function is assumed to be of the form

$$\Phi(\xi) = \xi^{\frac{1}{2}(2l+N+1)}e^{-\frac{1}{3}a\xi^{-3}+b\xi^{-1}}F(\xi) \tag{2.9}$$

for which the boundary conditions in (2.2) and the requirement that $T^m\Phi = O(1)\Phi$ as $\xi \to 0$ are fulfilled, provided that

$$F(\xi) = O(1) \quad \text{as } \xi \to 0, \quad F(1) = \text{constant}. \tag{2.10}$$

Substituting (2.9) into (2.2) and using the relations (2.6)–(8) we see that $F(\xi)$ satisfies the differential equation

$$(\mathcal{L} - \nu E)F(\xi) = 0, \quad \xi \in [0, 1], \tag{2.11}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L} = \alpha\xi^4(\xi^2-1)\frac{d^2}{d\xi^2} + \alpha[(2c+3)\xi^5 - 2b\xi^4 - 4\xi^3 + 2(a+b)\xi^2 - 2a]\frac{d}{d\xi} + \alpha\{c(c+2)\xi^4 - b(2c+1)\xi^3 + (b^2+c^2-3c)\xi^2 + [a(2c-1)+2b]\xi - b^2\}. \tag{2.12}$$

In particular case of the harmonic oscillator, where $\nu = 1$, the parameter $\alpha$ might be taken as zero from (2.6) and (2.7). However, the transformed equation seems to be indeterminate if $\alpha = 0$. Thus we should reformulate the problem as the determination of $\alpha$ in such a way that the harmonic oscillator is obtained whenever $\alpha \to 0$. To this end, letting $r^2 = z$ we find, from (2.1), that

$$\xi = 1 - \tfrac{1}{2}\alpha z + O(\alpha^2) \tag{2.13}$$

and that

$$\xi^m = 1 - \frac{m}{2}\alpha z + O(\alpha^2) \tag{2.14}$$

as $\alpha \to 0$. If we make use of the operational relations

$$\alpha\frac{d}{d\xi} = -2\frac{d}{dz} + O(\alpha^2), \quad \alpha^2\frac{d^2}{d\xi^2} = 4\frac{d^2}{dz^2} + O(\alpha^2), \tag{2.15}$$

eq. (2.11), with $\nu = 1$, may be written in the form

$$-4z(1-2\alpha z)\frac{d^2F}{dz^2} - [2(2c-1) - \alpha(10c+3)z - 4\alpha(a-b)z]\frac{dF}{dz} + [\alpha c(2c-1) + \alpha(a-b)(2c-1) - E]F + O(\alpha^2) = 0 \tag{2.16}$$

for sufficiently small values of $\alpha$. Now imposing the condition that

$$\alpha(a-b) = 1 \tag{2.17}$$

identically, we then arrive at the differential equation

$$-4z\frac{d^2F}{dz^2} - 2(2l+N-2z)\frac{dF}{dz} + (2l+N-E)F = 0 \tag{2.18}$$

for $\alpha = 0$, which is nothing but the Laguerre's differential equation (1.6). With (2.17) it is not difficult to see that $\xi^c e^{-\frac{1}{3}a\xi^{-3}+b\xi^{-1}}$ tends to a constant multiple of $e^{-\frac{1}{2}z}$, and hence that the wave function in (2.9) converges to the harmonic oscillator solution (1.5) as $\alpha \to 0$. This is an interesting analysis of the anharmonic oscillators because there is no analytic solution which yields the exact eigenfunctions of the unperturbed Schrödinger equation in this manner.

The determination of the structure parameters is accomplished by means of the nonlinear algebraic equations (2.6), (2.7) and (2.17). Fortunately, they provide us with nice mathematical formulas for expressing $\alpha$, $a$ and $b$,

$$\alpha = (1 - \nu^3)[1 + (1 - \nu^2)^{1/2}]^{-2}, \tag{2.19}$$

$$a = (1 - \nu^3)^{1/2}\alpha^{-3/2} \tag{2.20}$$

and

$$b = (1 - \nu^2)^{1/2}\alpha^{-1}, \tag{2.21}$$

respectively. Note that $\alpha$ lies entirely between the values of 0 and 0.25 as $\nu$ varies from 0 to 1.

Finally, we may verify that the resulting differential equation (2.11) for $F(\xi)$ has asymptotic solutions of the form

$$F_1(\xi) \approx \text{constant}, \quad F_2(\xi) \approx e^{\frac{2}{3}a\xi^{-3} - 2b\xi^{-1}}, \tag{2.22}$$

as $\xi \to 0$. So the second solution $F_2(\xi)$ must be rejected since it grows faster than the decay of the exponential term in (2.9) as $\xi \to 0$. In other words, it is important to ensure that $F(\xi)$ behaves correctly, like $F_1(\xi)$, as $\xi \to 0$ to satisfy the first condition in (2.10). On the other hand, we will show in section 3 that eq. (2.11) possesses a regular solution about the singular point at $\xi = 1$.

## 3. Hill's determinant

We seek solutions of eq. (2.11) in the form of power series about $\xi = 1$. Since the singularities of the differential equation are located at $\xi = \mp 1$ and 0, the series solutions are surely convergent inside a unit circle centered at $\xi = 1$, i.e. $|\xi - 1| < 1$. The exponents of the singularity at $\xi = 1$ are 0 and $-(2l + N - 2)/2$. Therefore, we have a solution starting with a constant which may be written as

$$F(\xi) = \sum_{k=0}^{\infty} \left(-\frac{2}{\alpha}\right)^k f_k(\xi - 1)^k, \tag{3.1}$$

where the factor $(-2/\alpha)^k$ in the coefficients of the series has been introduced for convenience. The second power series solution corresponding to the root $-(2l + N - 2)/2$ of indicial equation does not fulfill the condition at $\xi = 1$, which is neglected. In fact, it represents a physical solution only for the one-dimensional oscillators [9].

Substituting (3.1) into (2.11) we obtain the fifth-order difference equations

$$-2(2k + 2c - 1)(k + 1)f_{k+1}$$

$$+\{9\alpha k(k - 1) + [5(2c + 3)\alpha - 12\alpha + 4]k + (2c - 1)(1 + \alpha c) - \nu E\}f_k$$

$$-\tfrac{1}{2}\alpha^2\{16(k - 1)(k - 2)$$

$$+2[a-5b+5(2c+3)-6](k-1)+(2c-1)a$$

$$-(6c+1)b+2b^2+2c(3c+1)\}f_{k-1}+\tfrac{1}{4}\alpha^3\{14(k-2)(k-3)+2[5(2c+3)$$

$$-4b-2](k-2)+b^2-3(2c+1)b+c(7c+9)\}f_{k-2}$$

$$-\tfrac{1}{8}\alpha^4\{6(k-3)(k-4)+[5(2c+3)-2b](k-3)+4c(c+2)-(2c+1)b\}f_{k-3}$$

$$+\tfrac{1}{16}\alpha^5\{(k-4)(k-5)+(2c+3)(k-4)+c(c+2)\}f_{k-4}=0 \tag{3.2}$$

with $k=0,1,\ldots$, for the determination of the coefficients $f_k$, where

$$f_{-4}\equiv\ldots\equiv f_{-1}\equiv 0. \tag{3.3}$$

The series in (3.1) leads to the bounded solution required as $\xi\to 0$ or $F_1(\xi)$ in (2.22) only if it contains a finite number of terms. As a result, the square integrability of the wave function should be guaranteed by assuming that

$$f_{M+1}\equiv f_{M+2}\equiv\ldots\equiv 0, \tag{3.4}$$

where $M$ is the truncation order of $F(\xi)$. By (3.4) the linear recurrences (3.2) are reduced to a finite-dimensional homogeneous system whose coefficient matrix of order $M$, $L(E)$ say, is of a banded Hessenberg form. It is evident that there exist nontrivial solutions if and only if the so-called Hill's determinant is zero,

$$\det L(E)=0, \tag{3.5}$$

which are compatible with (3.3). The last condition allows the numerical evaluations of the truncated energy eigenvalues using available routines [12]. From a computational point of view, in the asymptotic domain of $k\gg 1$ a backward recursion for (3.2) with the initial conditions (3.4) should be proceeded. Such a procedure is known as Miller's algorithm [13], and is suitable for the stable calculation of the coefficients of the series yielding the nondominant asymptotic solution $F_1(\xi)$.

Reconsidering the harmonic oscillator, i.e. $\nu=1$ and $\alpha\to 0$, we find, in conjunction with (2.8) and (2.13), that the power series (3.1) and the recursions (3.2) for its coefficients are reduced to

$$F(r)=\sum_{k=0}^{\infty}f_k r^{2k} \tag{3.6}$$

and

$$-2(2k+2l+N)(k+1)f_{k+1}+(4k+2l+N-E)f_k=0,\quad k=0,1,\ldots,n,\ldots, \tag{3.7}$$

respectively. If the eigenvalues are specified as those in (1.7) it follows then that

$$f_{n+1}\equiv f_{n+2}\equiv\ldots\equiv 0, \tag{3.8}$$

and that the series terminates resulting simply in polynomials of degree $2n$. These are the associated Laguerre polynomials $L_n^{(\frac{1}{2}N+l-1)}$ with argument $r^2$ [14]. Observe that (3.8) is equivalent to (3.4).

## 4. Results and discussion

We solve two- and three-dimensional systems to test the numerical efficiency of the present method. Nevertheless, these calculations are representative for higher-dimensional spaces as well. We may deduce this interesting feature of $N$-dimensional isotropic oscillators directly from eq. (1.3). Actually, it is shown that the spectrum of the Schrödinger equation remains invariant for all fixed values of the positive integer number $2l+N$. As a result, denoting the eigenvalues in $N$ dimensions by $E_{n,l}^{(N)}$, we formulate the following degeneracies:

$$
\begin{aligned}
&E_{n,1}^{(2)} \equiv E_{n,0}^{(4)}\,,\\
&E_{n,2}^{(2)} \equiv E_{n,1}^{(4)} \equiv E_{n,0}^{(6)}\,,\\
&\vdots\\
&E_{n,l}^{(2)} \equiv E_{n,l-1}^{(4)} \equiv E_{n,l-2}^{(6)} \equiv \cdots \equiv E_{n,2}^{(2l-2)} \equiv E_{n,1}^{(2l)} \equiv E_{n,0}^{(2l+2)}
\end{aligned}
\tag{4.1}
$$

when $N$ is even, where $E_{n,0}^{(2)}$ is single in the system. Similarly, we have

$$
\begin{aligned}
&E_{n,1}^{(3)} \equiv E_{n,0}^{(5)}\,,\\
&E_{n,2}^{(3)} \equiv E_{n,1}^{(5)} \equiv E_{n,0}^{(7)}\,,\\
&\vdots\\
&E_{n,l}^{(3)} \equiv E_{n,l-1}^{(5)} \equiv E_{n,l-2}^{(7)} \equiv \cdots \equiv E_{n,2}^{(2l-1)} \equiv E_{n,1}^{(2l+1)} \equiv E_{n,0}^{(2l+3)}
\end{aligned}
\tag{4.2}
$$

if $N$ is odd. In the exceptional case of $N=1$, the eigenvalues can be characterized by $E_{2n+l}$ rather than two quantum numbers, with $l=0$ and 1. As was indicated earlier in the introduction, $l=0$ implies the symmetric states $E_{2n}$ and $l=1$ the anti-symmetric states $E_{2n+1}$. Furthermore, it may be seen that $E_{n,0}^{(3)} \equiv E_{2n+1}$ since $2l+N=3$ for each case.

The numerical results are given in Tables 1 and 2 so as to cover the whole range of the original coupling constant $\beta$. The case of $\beta \to \infty$ corresponds to the infinite-field limit Hamiltonian where the eigenvalue problem is

Table 1
Specimen calculations in two dimensions as a function of $\beta$. The eigenvalues $E_{n,l}^{(2)}$ represent also the eigenvalues in $2N$ dimensions according to eq. (4.1). The last column includes, where possible, Witwit's results for comparison.

| $\beta$ | $n$ | $l$ | $M$ | $E_{n,l}^{(2)}$ | $E_{n,l}^{(2)}$ in ref. [2] |
|---|---|---|---|---|---|
| 0.0001 | 0 | 0 | 7 | 2.000 199 955 022 234 348 149 930 | |
| | 1 | 1 | 9 | 8.002 398 591 662 386 403 444 616 | |
| | 0 | 5 | 8 | 12.004 196 959 526 950 540 562 41 | |
| | 3 | 1 | 12 | 16.009 589 044 588 030 969 292 34 | |
| 1 | 0 | 0 | 35 | 2.952 050 091 962 874 287 056 570 | 2.952 050 091 962 8 |
| | 0 | 2 | 39 | 10.390 627 295 503 782 127 355 48 | 10.390 627 295 503 6 |
| | 1 | 1 | 40 | 15.482 771 577 251 666 476 734 91 | |
| | 0 | 4 | 39 | 19.217 523 495 888 984 906 822 74 | 19.217 523 495 888 |
| 10000 | 0 | 0 | 37 | 50.548 044 948 103 018 662 533 82 | 50.548 044 948 102 |
| | 1 | 0 | 39 | 205.377 745 742 563 428 720 578 2 | 205.377 745 742 56 |
| | 0 | 4 | 41 | 368.030 082 448 256 517 258 408 0 | 368.030 082 448 24 |
| | 1 | 2 | 41 | 394.577 407 137 436 105 423 675 5 | 394.577 407 137 42 |
| $\infty$ | 0 | 0 | 37 | 2.344 829 072 744 275 209 808 996 | |
| | 0 | 2 | 39 | 8.928 082 199 849 951 179 552 253 | |
| | 0 | 4 | 40 | 17.077 681 440 978 319 569 540 21 | |

$$\left(-\frac{d^2}{dr^2}-\frac{2l+N-1}{r}\frac{d}{dr}+r^4\right)\Phi(r)=\mathcal{E}\Phi(r) \tag{4.3}$$

with $\mathcal{E}=\beta^{-1/3}E$, showing the correct asymptotic relation for the eigenvalues, $\mathcal{E}\approx\beta^{-1/3}E$, for large enough values of $\beta$. Therefore, the energy eigenvalues in our tables as $\beta\to\infty$ are tabulated in terms of $\mathcal{E}$. We confine ourselves to present only some specimen calculations in order not to overfill the contents of the paper with tabular material. Further results are available from the author.

We report eigenvalues to twenty-five significant digits for all values of the anharmonicity constant. The accuracy of the results is checked by employing the successive approximations of $M$. In the tables, $M$ stands for the truncation order of Hill's determinant for which the recorded accuracy is achieved. The eigenvalues by Witwit [2] are also included for comparison. It should be noted that in Table 1 the eigenvalues and the coupling constants taken from [2] are twice their original values because of a different notation adopted. The precision of Witwit's results is not uniform, and differs from nine- to fifteen-digits. In any case, however, the results are in a good agreement with each other to the accuracy quoted.

The trivial eigenvalue ordering properties

Table 2
Specimen calculations in three dimensions as a function of $\beta$. The eigenvalues $E_{n,l}^{(3)}$ represent also the eigenvalues in $(2N+1)$ dimensions according to eq. (4.2). The last column includes, where possible, Witwit's results for comparison.

| $\beta$ | $n$ | $l$ | $M$ | $E_{n,l}^{(3)}$ | $E_{n,l}^{(3)}$ in ref. [2] |
|---|---|---|---|---|---|
| 0.0001 | 0 | 0 | 7 | 3.000 374 896 936 121 098 337 847 | |
| | 0 | 10 | 9 | 23.014 356 719 158 913 502 993 75 | |
| | 5 | 5 | 15 | 33.039 285 168 376 526 523 592 58 | |
| 1 | 0 | 0 | 36 | 4.648 812 704 212 077 536 377 033 | |
| | 0 | 1 | 36 | 8.380 342 530 101 586 016 658 644 | 8.380 342 530 0 |
| | 0 | 5 | 43 | 26.528 917 558 123 923 941 508 39 | 26.528 917 558 123 9 |
| | 1 | 3 | 41 | 27.898 417 760 008 253 924 619 02 | 27.898 417 760 082 [a] |
| 20 | 0 | 1 | 39 | 19.783 251 911 335 357 058 067 14 | 19.783 251 90 |
| | 0 | 2 | 41 | 30.057 199 045 323 927 972 631 36 | 30.057 199 0 |
| | 1 | 1 | 41 | 44.209 279 973 156 846 774 816 28 | 44.209 279 973 15 |
| | 0 | 5 | 45 | 65.961 500 030 678 143 050 225 38 | 65.961 500 030 67 |
| 1000 | 0 | 0 | 37 | 38.086 833 459 382 264 084 978 36 | |
| | 0 | 3 | 41 | 149.439 045 580 769 578 290 941 8 | |
| $\infty$ | 0 | 0 | 38 | 3.799 673 029 801 394 168 783 094 | |
| | 0 | 5 | 42 | 23.940 622 097 894 264 116 215 20 | |

[a] There is most likely a misprint in this result.

$$E_{n,l_2}^{(N)} > E_{n,l_1}^{(N)} \Leftrightarrow l_2 > l_1, \quad E_{n_2,l}^{(N)} > E_{n_1,l}^{(N)} \Leftrightarrow n_2 > n_1 , \tag{4.4}$$

have been confirmed by our calculations. Now (4.4) and (4.1)–(4.2) imply that

$$E_{n,l}^{(N_2)} > E_{n,l}^{(N_1)} \Leftrightarrow N_2 > N_1 . \tag{4.5}$$

Thus the eigenvalues with the same quantum numbers increase as the dimension $N$ of the space increases, and vice versa. Finally, we characterize the energy levels $E_{n,l}^{(N)}$ as groups denoted by the number $m$, where $m = n + l$. We deduce, from the numerical experiments, that the eigenvalues in such a group may be ordered according to the rule

$$E_{0,m}^{(N)} < E_{1,m-1}^{(N)} < \ldots < E_{m-1,1}^{(N)} < E_{m,0}^{(N)} \tag{4.6}$$

independent of the coupling constant and of the dimension.

As a concluding remark, we should point out once more that an asymptotically correct wave function which approaches the classical harmonic oscillator solution as a special case has been presented. The intelligence substitution (2.1) plays a significant role to this end. The importance of this remark is discussed by Znojil [15], who concludes that 'we never get any elementary or exact particular solution as a special case of asymptotically correct wave functions'. It is the removal of such an undesired situation which explains both the complicated looking structure of the resulting differential equation (2.11) and the reason for referring to the simple harmonic oscillator throughout the text.

On the other hand, one might certainly use a standard power series method, having no novelty, which leads to a 3-term recurrence relation instead of the 5-term one in (3.2). At first sight, the usage of the former may be considered to be much more preferable than of the latter one. In the former case, however, it would be necessary to introduce certain appropriate optimization parameters in order to achieve satisfactory numerical results [2]. Note that we have never used in this study a specific numerical technique such as optimization, or otherwise, in getting extremely accurate results reported in the tables for all values of the coupling constant. Note also that there are some other variants of Hill's determinant approach that do not require such parameters as well. For instance, we may cite here the Riccati–Hill method, which yields accurate eigenvalues and eigenfunctions for anharmonic oscillators (see [16] and references therein). Consequently, if we remind how much labor is involved in the determination of a flexible convergence parameter properly even for the one-dimensional quartic oscillator [17], the numerical efficiency of our solution becomes much clearer. This is due to the fact that the present solution behaves exactly like the true wave function everywhere for all $\beta$.

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