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On the special values of monic polynomials of hypergeometric type

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Special values of monic polynomials $y_n(s)$, with leading coefficients of unity, satisfying the equation of hypergeometric type

$$\sigma(s)y_n'' + \tau(s)y_n' - n[\tau' + \tfrac{1}{2}(n-1)\sigma'']y_n = 0, \qquad n \in \mathbb{N}_0$$

have been examined in its full generality by means of a unified approach, where $\sigma(s)$ and $\tau(s)$ are at most quadratic and a linear polynomial in the complex variable s, respectively, both independent of n. It is shown, without actually determining the polynomials $y_n(s)$, that the use of particular solutions of a second order difference equation related to the derivatives $y_n^{(m)}(z)$ is sufficient to deduce special values for some appropriate $s = z$ points. Hence the special values of almost all polynomials and their derivatives can be generated by the universal formula

$$y_n^{(m)}(\theta_a) = m!\binom{n}{m}\frac{(\omega_a)_n(\omega_a + \omega_{-a} + n - 1)_m}{(\omega_a)_m(\omega_a + \omega_{-a} + n - 1)_n}\,(\theta_a - \theta_{-a})^{n-m},$$

in which $a = \mp\Delta \neq 0$ and $\theta_{\mp a}$ are the discriminant and the roots of $\sigma(s)$, respectively, and $\omega_{\mp a}$ denote a parameter depending on the coefficients of the differential equation. Furthermore, the interrelations that arise between $y_n^{(m)}(\theta_a)$ and $y_n^{(m)}(\theta_{-a})$ are also introduced. Finally, special values corresponding to the limiting and exceptional cases have been presented explicitly for completeness.

KEY WORDS: Differential equation of the hypergeometric type, polynomial solutions, special values, classical orthogonal polynomials, Bessel polynomials

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1. Introduction

It is well-known that the *equation of hypergeometric type* (EHT)

$$\sigma(s)y'' + \tau(s)y' + \lambda y = 0, \quad s \in \mathbb{C}, \tag{1.1}$$

in which $\sigma(s) \neq 0$ and $\tau(s)$ are polynomials of degree at most two and one, respectively, has a polynomial solution of exact degree n denoted by $y_n(s)$ if and only if

$$\lambda = \lambda_n = -n[\tau' + \tfrac{1}{2}(n-1)\sigma''], \qquad \tau' \neq 0 \tag{1.2}$$

for a prescribed nonnegative integer n [1]. The $y_n(s)$ are called here polynomials of hypergeometric (HG) type. In particular, when $s = x \in \mathbb{R}$, the classical orthogonal polynomials associated with the names Hermite, Laguerre and Jacobi belong to the class of polynomials of HG type, each of which is orthogonal over specific (a, b) interval relative to a weighting function $\rho(x)$ [2].

The coefficients of the EHT may be taken as

$$\sigma(s) = \sigma(0) + \sigma'(0)s + \tfrac{1}{2}\sigma'' s^2 \tag{1.3}$$

and

$$\tau(s) = \tau(0) + \tau' s, \tag{1.4}$$

however, we introduce their general Taylor polynomials

$$\sigma(s) = \sigma(z) + \sigma'(z)(s-z) + \tfrac{1}{2}(s-z)^2\sigma'' \tag{1.5}$$

and

$$\tau(s) = \tau(z) + (s-z)\tau' \tag{1.6}$$

about $s = z$, keeping the point z completely arbitrary to get a flexibility, where τ' and σ'' are independent of z. Then the polynomials of HG type are expressible accordingly as

$$y_n(s) = \sum_{k=0}^{n} \binom{n}{k} \Phi_k(z)(s-z)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \Phi_{n-k}(z)(s-z)^k, \tag{1.7}$$

in which the expansion coefficients can be defined *formally* by the relation

$$y_n^{(k)}(z) = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\, y_n(s)\Big|_{s=z} = k!\binom{n}{k}\Phi_{n-k}(z), \qquad y_n(z) = \Phi_n(z) \tag{1.8}$$

on using Taylor's theorem. Clearly, $\Phi_k(z)$ stands for a polynomial of degree k in z so that $\Phi_0(z)$ is a constant, which can be set to unity, $\Phi_0(z) = 1$ for all z, to consider a *monic* solution with a leading coefficient of unity. Meanwhile, by the substitution of $y_n(s)$ into (1.1) it is straightforward to show that the $\Phi_k(z)$ satisfy a second order difference equation

$$(\lambda_n - \lambda_{n-k})\Phi_k(z) + k\big[\tau(z) + (n-k)\sigma'(z)\big]\Phi_{k-1}(z) + k(k-1)\sigma(z)\Phi_{k-2}(z) = 0, \tag{1.9}$$

subject to the initial conditions

$$\Phi_{-1}(z) = 0 \quad \text{and} \quad \Phi_0(z) = 1 \tag{1.10}$$

for all $z \in \mathbb{C}$, $n \in \mathbb{N}_0$ and $k = 1, 2, \ldots, n$. The difference equation consists of the original coefficients of the EHT, which may be rewritten of the form

$$\begin{aligned}\left[\tau' + \tfrac{1}{2}(2n-k-1)\sigma''\right]\Phi_k(z) = &\left[\tau(z) + (n-k)\sigma'(z)\right]\Phi_{k-1}(z)\\ &+(k-1)\sigma(z)\Phi_{k-2}(z),\end{aligned} \tag{1.11}$$

where the eigenvalue parameter λ has been eliminated by virtue of (1.2).

The EHT is of importance due to its several applications in applied mathematics and theoretical physics and chemistry [1, 3]. For instance, the Schrödinger equation resulting from the solution of boundary value problems of quantum mechanics is often transformable to an EHT [1]. In addition, series of polynomials of HG type occur in the evaluation of certain integrals involving special functions. Such series also arise in the problem of finding the connection coefficients between two sequences of polynomials [4]. For these reasons it is desirable to have information about special values of solutions of the EHT, especially when $s = x$ is real. Generally, the required special values can be deduced by making use of the integral representations of solutions. In this paper, releasing the complex variable s and dropping the orthogonality restriction we follow an alternative and easier way to generate special values not only for $y_n(s)$ but also for $y_n^{(m)}(s)$. Actually, in section 2, we exploit a new parametrization of the main difference equation corresponding to the most general form of EHT, in which $\sigma(s)$ is of degree two with distinct roots, and solve it for a pair of special values of z. Section 3 is devoted to three limiting cases of (1.11), where $\sigma(s)$ is a linear polynomial, constant and a quadratic but having a double root. The last section concludes the paper with additional results and comments.

2. Alternative parametrization and special values

The EHT (1.1) can be put into the formally self-adjoint form

$$\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[\rho(s)\sigma(s)\frac{\mathrm{d}}{\mathrm{d}s}\right] + \lambda_n\rho(s)\right\} y_n(s) = 0, \tag{2.1}$$

where $\rho(s)$ is any non-trivial solution of the Pearson equation $[\sigma(s)\rho(s)]' = \tau(s)\rho(s)$ implying the integral

$$\ln\rho(s) = \int \frac{\tau(s) - \sigma'(s)}{\sigma(s)}\mathrm{d}s, \tag{2.2}$$

which is to be evaluated when the coefficients $\sigma(s)$ and $\tau(s)$ are specified. We call $\rho(s)$ a Pearson function keeping in mind that it turns out to be a weight in the particular case of orthogonal systems under auxiliary requirements.

Let us consider the usual form of the EHT whose $\sigma(s)$ is exactly a second degree polynomial with a nonzero determinant

$$\Delta = \sqrt{[\sigma'(0)]^2 - 2\sigma(0)\sigma''} \neq 0, \quad \sigma'' \neq 0 \tag{2.3}$$

having distinct roots. To determine $\rho(s)$, we first employ the factorization

$$\sigma(s) = \tfrac{1}{2}\sigma''\big(s - \theta_{-a}\big)\big(s - \theta_a\big) \tag{2.4}$$

without regard to the point z in (1.5), where

$$\theta_a := -\frac{1}{\sigma''}\big[\sigma'(0) + a\big] \tag{2.5}$$

is a parameter function of $a := \mp\Delta$. Then, in care of the partial fraction decomposition of the integrand, it is found from (2.2) that

$$\rho(s) = \big(s - \theta_{-a}\big)^{\omega_{-a}-1}\big(s - \theta_a\big)^{\omega_a - 1} \tag{2.6}$$

with the exponents

$$\omega_a := \frac{\tau'}{a}\big(\delta - \theta_a\big), \quad a = \mp\Delta \tag{2.7}$$

in which

$$\delta := -\frac{\tau(0)}{\tau'} \tag{2.8}$$

denotes the root of linear coefficient in (1.4), i.e,

$$\tau(s) = \tau'(s - \delta). \tag{2.9}$$

Recall that τ' can never be zero so that δ is always of finite modulus. On the other hand, since

$$\theta_a - \theta_{-a} = -\frac{2a}{\sigma''}, \tag{2.10}$$

we have

$$\omega_a + \omega_{-a} = \frac{2\tau'}{\sigma''} \tag{2.11}$$

being independent of a, which is true in the sense of a limiting value as well, when $a \to 0$. Note also the identity

$$\frac{2\tau'}{\sigma''}\delta = (\omega_a + \omega_{-a})\delta = \theta_a\omega_{-a} + \theta_{-a}\omega_a, \tag{2.12}$$

which follows from the definitions of $\theta_{\mp a}$ and $\omega_{\mp a}$.

Now, after some algebraic manipulation, it is not difficult to convert the original difference equation (1.11) into the more tractable form

$$\Phi_k(z) = \frac{(\omega_a + n - k)(z - \theta_{-a}) + (\omega_{-a} + n - k)(z - \theta_a)}{\omega_a + \omega_{-a} + 2n - k - 1} \Phi_{k-1}(z) + (k-1)\frac{(z - \theta_{-a})(z - \theta_a)}{\omega_a + \omega_{-a} + 2n - k - 1} \Phi_{k-2}(z), \tag{2.13}$$

containing the new parameters $\theta_{\mp a}$ and $\omega_{\mp a}$ that are the roots of $\sigma(s)$ and the exponents in $\rho(s)$, respectively. Observe that the new difference equation remains invariant under the replacement of a by $-a$. Furthermore, because of the flexibility of the point z, we can reduce it to a first order one

$$\Phi_k(\theta_a) = \frac{\omega_a + n - k}{\omega_a + \omega_{-a} + 2n - k - 1} (\theta_a - \theta_{-a}) \Phi_{k-1}(\theta_a), \quad \Phi_0(\theta_a) = 1 \tag{2.14}$$

on setting $z = \theta_a$, which is an analytically solvable recurrence relation for $k = 1, 2, \ldots, n$. Actually, choosing successive values of k, we have in general

$$\Phi_k(\theta_a) = \frac{(1 - n - \omega_a)_k}{(2 - 2n - \omega_a - \omega_{-a})_k} (\theta_a - \theta_{-a})^k, \quad k = 0, 1, \ldots, n, \tag{2.15}$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$ is the Pochhammer symbol. So, from (1.8), we obtain

$$y_n^{(m)}(\theta_a) = m!\tbinom{n}{m}\Phi_{n-m}(\theta_a) = m!\binom{n}{m}\frac{(1 - n - \omega_a)_{n-m}}{(2 - 2n - \omega_a - \omega_{-a})_{n-m}} (\theta_a - \theta_{-a})^{n-m} \tag{2.16}$$

for $m = 0, 1, \ldots, n$, which may be written in a more neatly form

$$y_n^{(m)}(\theta_a) = m!\binom{n}{m}\frac{(\omega_a)_n(\omega_a + \omega_{-a} + n - 1)_m}{(\omega_a)_m(\omega_a + \omega_{-a} + n - 1)_n} (\theta_a - \theta_{-a})^{n-m} \tag{2.17}$$

by using the links

$$(\alpha)_{n-k} = (\alpha)_n \frac{(-1)^k}{(1 - n - \alpha)_k}, \quad (\alpha)_n = (-1)^n(1 - n - \alpha)_n \tag{2.18}$$

between the Pochhammer symbols. Because a and $-a$ are interchangeable, we also have

$$y_n^{(m)}(\theta_{-a}) = m!\binom{n}{m}\frac{(\omega_{-a})_n(\omega_a + \omega_{-a} + n - 1)_m}{(\omega_{-a})_m(\omega_a + \omega_{-a} + n - 1)_n} (\theta_{-a} - \theta_a)^{n-m}, \tag{2.19}$$

standing for the solution of (2.13) at $z = \theta_{-a}$. The analytical results in (2.17) and (2.19) imply immediately the simple interrelation

$$y_n^{(m)}(\theta_a) = (-1)^{n-m} \frac{(\omega_a)_n (\omega_{-a})_m}{(\omega_a)_m (\omega_{-a})_n}\, y_n^{(m)}(\theta_{-a}), \tag{2.20}$$

among the two special values. Note that $y_n^{(n)}(z) = n!$ for all z including $\theta_{\mp a}$ due to the monic property of the polynomials.

In fact the EHT being considered, for which $\sigma(s)$ is quadratic with distinct roots, could be classified as the *first category*. Then, in general, we may denote a polynomial of the first category by

$$p_n(s) = p_n(\theta_a, \theta_{-a}, \omega_a, \omega_{-a}; s) \tag{2.21}$$

to distinguish different cases. It is seen that it contains four parameter functions $\theta_{\mp a}$ and $\omega_{\mp a}$, all appearing in $\rho(s)$, which are to be computed from (2.5) and (2.7), respectively, for $a = \mp\Delta$. Therefore, the special values of $p_n^{(m)}(s)$ at $s = \mp\theta_a$ are those of (2.17) and (2.19). In particular, from (2.17) and (2.20) we have

$$p_n(\theta_a) = p_n(\theta_a, \theta_{-a}, \omega_a, \omega_{-a}; s)\big|_{s=\theta_a} = \frac{(\omega_a)_n}{(\omega_a + \omega_{-a} + n - 1)_n}\, (\theta_a - \theta_{-a})^n \tag{2.22}$$

and

$$p_n(\theta_{-a}) = (-1)^n \frac{(\omega_{-a})_n}{(\omega_a)_n}\, p_n(\theta_a), \tag{2.23}$$

when $m = 0$. Moreover, the derivative values at $s = \theta_a$,

$$p_n^{(m)}(\theta_a) = m!\binom{n}{m} \frac{(\omega_a + \omega_{-a} + n - 1)_m}{(\omega_a)_m}\, (\theta_a - \theta_{-a})^{-m}\, p_n(\theta_a) \tag{2.24}$$

are expressible in terms of $p_n(\theta_a)$ for all $m = 1, 2, \ldots, n$. Remind that a similar result holds for $p_n^{(m)}(s)$ at $s = \theta_{-a}$ as well, if a is replaced by $-a$.

A little more careful inspection indicates that there is an exceptional case of the first category in which special values are obtainable at an additional point. Consider the difference equation (1.11) and choose z as a point representing the common root of $\tau(s)$ and $\sigma'(s)$,

$$\sigma'(s) = \tfrac{1}{2}\sigma''(2s - \theta_a - \theta_{-a}), \tag{2.25}$$

which are both linear polynomials in s. From (2.9) and (2.25), then we must have

$$z = \delta = \tfrac{1}{2}(\theta_a + \theta_{-a}), \tag{2.26}$$

describing the midpoint of the line connecting the points θ_a and θ_{-a} in the complex plane. In such a situation, the exponents of $\rho(s)$ are identical by definition (2.7) so that $\omega_{-a} = \omega_a$, and (2.13) reduces to

$$\Phi_k(\delta) = \frac{k-1}{k+1-2n-2\omega_a}\left[\tfrac{1}{2}(\theta_a - \theta_{-a})\right]^2 \Phi_{k-2}(\delta), \quad \Phi_{-1}(\delta) = 0, \quad \Phi_0(\delta) = 1 \tag{2.27}$$

for $z = \delta$, which is still second order but a simple recursion relating each coefficient to the second one before it. Thus, special values of polynomials of the specific form denoted and defined by

$$c_n(s) := c_n(\theta_a, \theta_{-a}, \omega_a; s) = p_n(\theta_a, \theta_{-a}, \omega_a, \omega_{-a}; s)\big|_{\omega_{-a}=\omega_a} \tag{2.28}$$

can be determined at $s = \delta$ as well. Actually, we find from (2.27) that the odd-numbered coefficients all vanish

$$\Phi_{2k-1}(\delta) = 0, \quad k = 1, 2, \ldots, [\![\tfrac{1}{2}(n+1)]\!] \tag{2.29}$$

and that

$$\Phi_{2k}(\delta) = \frac{(2k)!}{k!2^{2k}(\frac{3}{2} - n - \omega_a)_k}\left[\tfrac{1}{2}(\theta_a - \theta_{-a})\right]^{2k}, \quad k = 0, 1, \ldots, [\![\tfrac{1}{2}n]\!], \tag{2.30}$$

where $[\![\alpha]\!]$ denotes the greatest integer function. In particular, from (1.8), we obtain

$$c_{2n+1}(\delta) = 0 \tag{2.31}$$

and

$$c_{2n}(\delta) = c_{2n}(\theta_a, \theta_{-a}, \omega_a; s)\big|_{s=\delta} = \frac{(-1)^n(\frac{1}{2})_n}{(\omega_a + n - \frac{1}{2})_n}\left[\tfrac{1}{2}(\theta_a - \theta_{-a})\right]^{2n} \tag{2.32}$$

on replacing n by $2n+1$ and $2n$, respectively, by use of the last elements of (2.29) and (2.30). On the other hand, it follows from (2.22) and (2.23) that

$$c_n(\theta_a) = c_n(\theta_a, \theta_{-a}, \omega_a; s)\big|_{s=\theta_a} = \frac{(\omega_a)_n}{(2\omega_a + n - 1)_n}(\theta_a - \theta_{-a})^n \tag{2.33}$$

and

$$c_n(\theta_{-a}) = (-1)^n c_n(\theta_a) \tag{2.34}$$

due to the fact that $c_n(s)$ is no more than a polynomial $p_n(s)$ with $\omega_{-a} = \omega_a$.

3. Limit forms of the difference equation

Consider an EHT having a linear $\sigma(s)$,

$$\sigma(s) = \sigma(z) + (s-z)\sigma' = \sigma(0) + \sigma' s = \sigma'\big(s-\theta\big) \tag{3.1}$$

with the root

$$\theta := -\frac{\sigma(0)}{\sigma'} \tag{3.2}$$

in which $\sigma' \neq 0$ does not depend on the point z any more. Clearly, in view of (1.3), such an EHT may be regarded as a limiting case of the first category, where $\sigma'' \to 0$. Here, we classify it as the *second category* whose polynomial solutions will be denoted by $\ell_n(s)$.

In this category, by examining the new parameters of Section 2 it is shown, from (2.3) and (2.5) that $\Delta = \sigma'$ as $\sigma'' \to 0$ and, therefore, θ_{-a} has the limit which is θ in (3.2), whereas θ_a tends to infinity. Besides, from (2.7), we deduce that ω_a approaches also infinity, like

$$\omega_a \sim \kappa\theta_a \quad \text{as } \theta_a \to \infty \tag{3.3}$$

and define ω,

$$\omega := \lim_{\theta_{-a}\to\theta} \omega_{-a} = \frac{\tau'}{\sigma'}\left[\frac{\tau(0)}{\tau'} - \frac{\sigma(0)}{\sigma'}\right] = \kappa(\delta - \theta) \tag{3.4}$$

as the existing limit of ω_{-a}, where

$$\kappa := -\frac{\tau'}{\sigma'} \neq 0 \tag{3.5}$$

appears to be an additional parameter of finite modulus. Since, by definition, the Pochhammer symbol $(\omega_a + \omega_{-a} + n - 1)_k$ behaves like ω_a^k for large enough values of ω_a, we first find the asymptotic form of (2.19)

$$y_n^{(m)}(\theta_{-a}) \sim m!\binom{n}{m}\frac{(\omega_{-a})_n}{(\omega_{-a})_m}\left(\frac{\theta_a}{\omega_a}\right)^{n-m}\left(\frac{\theta_{-a}}{\theta_a} - 1\right)^{n-m} \tag{3.6}$$

as $\omega_a \to \infty$. Then passing to the limit we obtain

$$\ell_n^{(m)}(\theta) := \lim_{\substack{\theta_a\to\infty \\ \theta_{-a}\to\theta}} y_n^{(m)}(\theta_{-a}) = m!\binom{n}{m}\frac{(\omega)_n}{(\omega)_m}\left(-\frac{1}{\kappa}\right)^{n-m} \tag{3.7}$$

by the help of (3.3) and (3.4), which stands for the special value of $\ell_n^{(m)}(s)$ at $s = \theta$. In particular, the polynomials of the second category has the special value

$$\ell_n(\theta) = (-1)^n \kappa^{-n} (\omega)_n \tag{3.8}$$

at the root θ of linear $\sigma(s)$ implying the relation

$$\ell_n^{(m)}(\theta) = (-1)^m m! \binom{n}{m} \frac{\kappa^m}{(\omega)_m}\, \ell_n(\theta) \tag{3.9}$$

for all $m = 1, 2, \dots, n$.

Alternatively, if we proceed with the difference equation (1.11) then we would have

$$\kappa\, \Phi_k(z) = \big[k - n - \omega + \kappa(z - \theta)\big]\Phi_{k-1}(z) - (k-1)(z-\theta)\, \Phi_{k-2}(z) \tag{3.10}$$

in accordance with (2.8), (3.1), (3.4) and (3.5), which is equivalent to the asymptotic form of (2.13) as $\theta_a \to \infty$. Setting $z = \theta$, there follows

$$\Phi_k(\theta) = \frac{1}{\kappa}(k - n - \omega)\, \Phi_{k-1}(\theta), \quad \Phi_0(\theta) = 1, \tag{3.11}$$

provided that $\kappa \neq 0$. Thus it is an easy matter to find the general solution

$$\Phi_k(\theta) = \frac{1}{\kappa^k}(1 - n - \omega)_k, \quad k = 0, 1, \dots, n \tag{3.12}$$

of the resulting simple recursion, which leads to

$$\ell_n^{(m)}(\theta) = m! \binom{n}{m} \frac{(1 - n - \omega)_{n-m}}{\kappa^{n-m}} \tag{3.13}$$

with the relation (1.8). This is precisely the same as (3.7) if we reuse the connections between the Pochhammer symbols in (2.18). Difference equation (3.10) implies also that the polynomials of the second category will be of the form

$$\ell_n(s) = \ell_n\big(\theta, \omega, \kappa; s\big), \tag{3.14}$$

depending, in general, on three parameters. To identify these parameters, one can check again an appropriate Pearson function

$$\rho(s) = (s - \theta)^{\omega - 1}\, \mathrm{e}^{-\kappa s}, \tag{3.15}$$

corresponding to the present category, determined by (2.2).

Another limit form, namely the *third category*, is provided by an EHT for which σ'' and σ' both vanish, i.e., $\sigma(s) := \sigma \in \mathbb{C}$ is merely a nonzero constant. That is to say, the roots $\theta_{\mp a}$ in (2.5) are both undefined approaching infinity. An EHT in the third category may also be regarded as a limiting case of the second category in (3.1), where $\sigma' = 0$ and, hence, $\theta \to \infty$. As a result, it is impossible to derive special values of a polynomial of the third category, denoted by $h_n(s)$, from $p_n(\theta_{\mp a})$ or $\ell_n(\theta)$ by means of a limiting process. Nevertheless, it is still possible to make use of the difference equation (1.11) to this end. Indeed, if we define a new parameter

$$\mu := -\frac{\tau'}{2\sigma} \neq 0, \tag{3.16}$$

suggested by a Pearson function of the form

$$\rho(s) = \mathrm{e}^{-\mu(s-\delta)^2}, \tag{3.17}$$

which has been found immediately from (2.2), the main difference equation then becomes

$$2\mu\, \Phi_k(z) = (z-\delta)\, \Phi_{k-1}(z) - (k-1)\, \Phi_{k-2}(z) \tag{3.18}$$

for $k = 1, 2, \ldots, n$. Thus the polynomials of the third category will have the form

$$h_n(s) = h_n(\delta, \mu; s), \tag{3.19}$$

whose special values can be assigned for $s = \delta$, that is, at the root of $\tau(s)$ in (2.8). Actually, substituting $z = \delta$ into (3.18) we get a simple recursion with the initial conditions

$$\Phi_k(\delta) = -\frac{1}{2\mu}\,(k-1)\,\Phi_{k-2}(\delta), \quad \Phi_{-1}(\delta) = 0, \quad \Phi_0(\delta) = 1 \tag{3.20}$$

of type (2.27). Therefore, a similar treatment yields

$$\Phi_{2k-1}(\delta) = 0, \quad k = 1, 2, \ldots, [\![\tfrac{1}{2}(n+1)]\!] \tag{3.21}$$

and

$$\Phi_{2k}(\delta) = \frac{(-1)^k}{2^{2k}}\frac{(2k)!}{k!}\mu^{-k} = (-1)^k (\tfrac{1}{2})_k \mu^{-k}, \quad k = 0, 1, \ldots, [\![\tfrac{1}{2}n]\!] \tag{3.22}$$

from which we obtain

$$h_{2n+1}(\delta) = 0 \tag{3.23}$$

and

$$h_{2n}(\delta) = h_{2n}(\delta, \mu; s)\big|_{s=\delta} = (-1)^n (\tfrac{1}{2})_n \mu^{-n} = \frac{(-1)^n}{2^{2n}}\frac{(2n)!}{n!}\,\mu^{-n} \tag{3.24}$$

for all n.

Finally, let us introduce an EHT of the *fourth category* in which

$$\sigma(s) = \tfrac{1}{2}\sigma''\big(s-\theta_0\big)^2 \tag{3.25}$$

has a double root at $s = \theta_0$,

$$\theta_0 = \frac{\sigma'(0)}{\sigma''}, \tag{3.26}$$

calculated from (2.5) for $a = \mp\Delta = 0$. The difference equation corresponding to the present case is written as

$$\Phi_k(z) = \frac{\nu + (\varpi + 2n - 2k)(z - \theta_0)}{\varpi + 2n - k - 1}\,\Phi_{k-1}(z) + (k-1)\frac{(z-\theta_0)^2}{\varpi + 2n - k - 1}\,\Phi_{k-2}(z), \tag{3.27}$$

where the constants

$$\varpi := \frac{2\tau'}{\sigma''}, \qquad \nu := \frac{2\tau'}{\sigma''}\left[\frac{\tau(0)}{\tau'} - \frac{\sigma'(0)}{\sigma''}\right] = \varpi(\theta_0 - \delta) \tag{3.28}$$

have been defined for convenience. As a consequence, we infer that the polynomials of the fourth category may be denoted by

$$b_n(s) = b_n(\theta_0, \varpi, \nu; s), \tag{3.29}$$

depending on three parameters. The special values of these polynomials at $s = \theta_0$ are deduced from $y_n^{(m)}(\theta_a)$ in (2.17) by a simple asymptotic analysis as $a \to 0$. To this end, first notice from (2.7) that, for sufficiently small values of a, ω_a grows unboundedly so that $(\omega_a)_k \sim \omega_a^k$. Second, consider the asymptotic form

$$y_n^{(m)}(\theta_a) \sim m!\binom{n}{m}\frac{(\omega_a + \omega_{-a} + n - 1)_m}{(\omega_a + \omega_{-a} + n - 1)_n}\,\frac{\omega_a^n(\theta_a - \theta_{-a})^n}{\omega_a^m(\theta_a - \theta_{-a})^m} \tag{3.30}$$

of $y_n^{(m)}(\theta_a)$ as $\omega_a \to \infty$. Then combining this result with the limit relations

$$\lim_{a\to 0}(\omega_a + \omega_{-a}) = \varpi, \qquad \lim_{a\to 0}[\omega_a(\theta_a - \theta_{-a})] = \nu, \tag{3.31}$$

suggested by (2.11), (2.7), and (2.10), we obtain

$$b_n^{(m)}(\theta_0) = \lim_{a\to 0} y_n^{(m)}(\theta_a) = m!\binom{n}{m}\frac{(\varpi + n - 1)_m}{(\varpi + n - 1)_n}\,\nu^{n-m}, \tag{3.32}$$

provided that $\nu \neq 0$. In fact, ν should not be zero because the EHT of the fourth category with $\nu = 0$ is degenerate reducing to a Cauchy–Euler equation. For $m = 0$, we have

$$b_n(\theta_0) = \frac{\nu^n}{(\varpi + n - 1)_n} \tag{3.33}$$

and, therefore,

$$b_n^{(m)}(\theta_0) = m!\binom{n}{m}(\varpi + n - 1)_m \nu^{-m} b_n(\theta_0) \tag{3.34}$$

for $m = 1, 2, \ldots, n$. Note that the same formulas could be derived from the difference equation (3.27) with the initial condition $\Phi(\theta_0) = 1$, when $z = \theta_0$.

4. Concluding remarks

In this paper, we have elucidated the formulas generating the special values of the monic polynomials that satisfy the EHT in (1.1). Since the classical polynomial systems of Jacobi, Laguerre, Hermite and generalized Bessel consist of solutions of the same differential equation transformed into canonical forms, their special values can be reproduced by making use of the present 4-categories of polynomials for specific sets of parameters. For example, the Jacobi polynomials $P_n^{(\alpha,\beta)}(s)$ are solutions of an EHT of the first category

$$(1 - s^2)y'' + \big[\beta - \alpha - (\alpha + \beta + 2)s\big]y' + n(n + \alpha + \beta + 1)y = 0 \tag{4.1}$$

having the parameters

$$\theta_{\mp a} = \mp 1, \quad \omega_a = \alpha + 1, \quad \omega_{-a} = \beta + 1, \tag{4.2}$$

of section 2 with $a = \mp 2$ as $\Delta = 2$. Therefore, we find from (2.22) and (2.23) that

$$\begin{aligned} P_n^{(\alpha,\beta)}(1) = k_n(\alpha, \beta)\, p_n(1) &= k_n(\alpha, \beta)\, p_n(1, -1, \alpha + 1, \beta + 1; s)\big|_{s=1} \\ &= k_n(\alpha, \beta) \frac{(\alpha + 1)_n 2^n}{(\alpha + \beta + n + 1)_n} = \frac{1}{n!}(\alpha + 1)_n \end{aligned} \tag{4.3}$$

and

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{(\beta + 1)_n}{(\alpha + 1)_n} P_n^{(\alpha,\beta)}(1) = \frac{(-1)^n}{n!}(\beta + 1)_n, \tag{4.4}$$

where the monic polynomials of the first category have been multiplied by the coefficient

$$k_n(\alpha, \beta) = \frac{(\alpha + \beta + n + 1)_n}{2^n n!} \tag{4.5}$$

of the leading term s^n in $P_n^{(\alpha,\beta)}(s)$ according to the historical normalization of the Jacobi polynomials [5]. Furthermore, (2.24) leads to the interrelations

$$\frac{\mathrm{d}^m}{\mathrm{d}s^m} P_n^{(\alpha,\beta)}(s)\Big|_{s=1} = m! \binom{n}{m} \frac{(\alpha + \beta + n + 1)_m}{2^m (\alpha + 1)_m} P_n^{(\alpha,\beta)}(1) \tag{4.6}$$

and

$$\frac{\mathrm{d}^m}{\mathrm{d}s^m} P_n^{(\alpha,\beta)}(s)\Big|_{s=-1} = (-1)^m m! \binom{n}{m} \frac{(\alpha + \beta + n + 1)_m}{2^m (\beta + 1)_m} P_n^{(\alpha,\beta)}(-1) \tag{4.7}$$

between $P_n^{(\alpha,\beta)}(s)$ and its derivative values of any order at $s = \mp 1$.

Now, recall that the ultraspherical or Gegenbauer polynomials $C_n^{(\alpha)}(s)$ are the renormalized Jacobi polynomials of the form [5]

$$C_n^{(\alpha)}(s) = \frac{(2\alpha)_n}{(\alpha+\frac{1}{2})_n} P_n^{\left(\alpha-\frac{1}{2},\alpha-\frac{1}{2}\right)}(s), \quad \alpha \neq 0 \tag{4.8}$$

to find out the relation

$$\begin{aligned} C_n^{(\alpha)}(s) = \ & k_n(\alpha) p_n\left(1, -1, \alpha+\tfrac{1}{2}, \alpha+\tfrac{1}{2}; s\right) = k_n(\alpha) c_n\left(1, -1, \alpha+\tfrac{1}{2}; s\right), \\ & k_n(\alpha) = \frac{2^n}{n!}(\alpha)_n \end{aligned} \tag{4.9}$$

in view of (2.28). As a result, the exceptional case in the first category states that special values of the Gegenbauer polynomials can be assigned at the roots of $\sigma(s)$ as well as at the midpoint of the line joining the two roots. Specifically, the special values

$$C_n^{(\alpha)}(1) = \frac{2^{2n}}{n!}\frac{(\alpha)_n(\alpha+\frac{1}{2})_n}{(2\alpha+n)_n} = \frac{1}{n!}(2\alpha)_n, \qquad C_n^{(\alpha)}(-1) = (-1)^n C_n^{(\alpha)}(1) \tag{4.10}$$

at $s = \pm 1$ are found from (2.33) and (2.34). Also, it follows from (2.31) and (2.32) that

$$C_{2n+1}^{(\alpha)}(0) = 0, \qquad C_{2n}^{(\alpha)}(0) = \frac{(-1)^n 2^{2n}}{(2n)!}\frac{(\alpha)_{2n}(\frac{1}{2})_n}{(\alpha+n)_n} = \frac{(-1)^n}{n!}(\alpha)_n, \tag{4.11}$$

where $\delta = 0$ in (2.26) is clearly the center of the interval $[-1, 1]$.

In a similar fashion, it is seen that the Laguerre polynomials $L_n^{(\alpha)}(s)$ satisfying

$$sy'' + (\alpha+1-s)y' + ny = 0 \tag{4.12}$$

are the special case of polynomials $\ell_n(\theta, \omega, \kappa; s)$ of the second category

$$L_n^{(\alpha)}(s) = k_n \ell_n(0, \alpha+1, 1; s), \quad k_n = \frac{(-1)^n}{n!} \tag{4.13}$$

with the parameters $\theta = 0$, $\omega = \alpha+1$ and $\kappa = 1$. Hence, from (3.9), we have

$$\frac{\mathrm{d}^m}{\mathrm{d}s^m} L_n^{(\alpha)}(s)\Big|_{s=0} = m!\binom{n}{m}\frac{(-1)^m}{(\alpha+1)_m} L_n^{(\alpha)}(0), \tag{4.14}$$

in which

$$L_n^{(\alpha)}(0) = k_n \ell_n(0) = k_n \ell_n(0, \alpha+1, 1; s)\big|_{s=0} = \frac{1}{n!}(\alpha+1)_n \tag{4.15}$$

is defined by (3.8).

The Hermite differential equation

$$y'' - 2sy' + 2ny = 0 \tag{4.16}$$

is the canonical form of the EHT of the third category. In other words, we identify the Hermite polynomials

$$H_n(s) = k_n h_n(0, 1; s), \quad k_n = 2^n \tag{4.17}$$

by substituting $\delta = 0$ and $\mu = 1$ into (3.19), where k_n is again the coefficient of s^n in $H_n(s)$ [5]. Then the special values

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) = k_{2n} h_{2n}(0, 1; s)\big|_{s=0} = (-1)^n \frac{(2n)!}{n!} \tag{4.18}$$

at the root $s = \delta = 0$ of $\tau(s) = -2s$, follow easily from (3.23) and (3.24).

Finally, it is readily shown that the generalized Bessel polynomials $B_n^{(\alpha)}(s)$ satisfying [6, 7]

$$s^2 y'' + (2 + \alpha s)y' - n(n + \alpha - 1)y = 0, \tag{4.19}$$

fall into the fourth category with $\theta_0 = 0$, $\varpi = \alpha$ and $\nu = 2$, confirmed by (3.26) and (3.28). Next the expression

$$B_n^{(\alpha)}(0) = k_n(\alpha) b_n(0, \alpha, 2; s)\big|_{s=0} \tag{4.20}$$

is written at the double root $\theta_0 = 0$ of $\sigma(s)$, where, by the help of (3.33), we must require that

$$b_n(0, \alpha, 2; s)\big|_{s=0} = \frac{2^n}{(\alpha + n - 1)_n} = \frac{1}{k_n(\alpha)}, \tag{4.21}$$

because the generalized Bessel polynomials are normalized to give

$$B_n^{(\alpha)}(0) = 1 \tag{4.22}$$

for all n and α [8]. From (3.34), we also have

$$\frac{\mathrm{d}^m}{\mathrm{d}s^m} B_n^{(\alpha)}(s)\bigg|_{s=0} = m! \binom{n}{m} 2^{-m} (\alpha + n - 1)_m \tag{4.23}$$

for the derivative values at $s = 0$.

In a very recent paper [9], Koepf and Masjad-Jamei considered the EHT with a real variable and expressed the monic polynomial solutions $y_n(x)$ in terms of HG functions. Then, by the evaluation of the resulting HG functions, they presented a generic formula for the values at the boundary points of monic orthogonal polynomials. However, we have shown here that the general solution of the EHT and, hence, explicit definitions of polynomials of HG type are not needed to obtain their special values at several points. If we restrict ourselves to the classical orthogonal polynomial systems of Jacobi, Laguerre and Hermite, which are orthogonal on $[-1, 1]$, $[0, \infty)$ and $(-\infty, \infty)$ with respect to gamma, beta and normal distributions, respectively, then some of the special values coincide with values at the boundary points of orthogonality intervals. More specifically, $P_n^{(\alpha,\beta)}(\mp 1)$ in (4.3) and (4.4), yield the values of the Jacobi polynomials at the two boundary points of $[-1, 1]$. Also, $L_n^{(\alpha)}(0)$ in (4.15) stands for the left hand boundary value of the Laguerre polynomial. It is obvious that we can present only the special values at the origin for the Hermite polynomials. On the other hand, the generalized Bessel polynomials are not orthogonal on any real x-interval. Nevertheless, the importance of these polynomials was first demonstrated in [7] in their connection with the wave equation in spherical coordinates.

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