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A class of orthogonal polynomials suggested by a trigonometric Hamiltonian: Antisymmetric states

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This is the second in a series of papers dealing with the sets of orthogonal polynomials generated by a trigonometric Hamiltonian. In the first of this series, a subclass of the Jacobi polynomials denoted by $\mathcal{T}_n^{(\mu)}(x)$ and referred to as the $\mathcal{T}$-polynomial of the first kind, which arises in the investigation of the symmetric state eigenfunctions of the Hamiltonian under consideration, was examined. Another subclass of the Jacobi polynomials denoted by $\mathcal{U}_n^{(\mu)}(x)$ is introduced here representing the antisymmetric states, and is called in accordance the $\mathcal{T}$-polynomial of the second kind. Moreover, by the derivation of the ultraspherical polynomial wavefunctions, interrelations between the $\mathcal{T}$-polynomials of the first and second kinds as well as the other orthogonal polynomial systems are also emphasized.

KEY WORDS: Schrödinger equation, exactly solvable Hamiltonians, special functions, classical orthogonal polynomials

AMS subject classification: 33C45, 81Q05, 33C05

1. Introduction

The investigation of exactly solvable systems in quantum mechanics, such as the Schrödinger equation $H\Psi = E\Psi$ with a trigonometric Hamiltonian

$$H(\theta;\mu) = -\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + v(\theta;\mu), \quad v(\theta;\mu) = \tfrac{1}{4}\mu(\mu+1)\sec^2\tfrac{1}{2}\theta, \quad \mu > 0 \tag{1}$$

over $\theta \in (-\pi, \pi)$, is interesting due to their several applications [1–3]. In most cases, analytical solutions are expressible in terms of special functions of mathematical physics and chemistry [3,4]. Therefore, they are also important in the study and use of special functions. In the first of this series [1] (hereafter referred to as PI), the symmetric state eigenfunctions $\Psi_{2n}(\theta;\mu)$ and eigenvalues $E_{2n}(\mu)$ of the trigonometric Hamiltonian (1) were examined in detail by starting with the particle-in-a-box potential for which $\mu = 0$. For this limit potential, the normalised antisymmetric state eigenfunctions are written as

$$\Psi_{2n+1}(\theta; 0) = \frac{1}{\sqrt{\pi}} \sin(n+1)\theta = \frac{1}{\sqrt{\pi}} \sin\theta\, U_n(\cos\theta) \tag{2}$$

corresponding to the eigenvalues

$$E_{2n+1}(0) = (n+1)^2 = \frac{1}{4}[(2n+1)+1]^2, \tag{3}$$

where

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta \tag{4}$$

denotes the Chebyshev polynomial of the second kind, which is indeed a polynomial of degree n in x when $x = \cos\theta$ [5]. With the help of simple trigonometric manipulations, this eigensolution may be put in an alternative form

$$\begin{aligned}
\sqrt{\pi}\,\Psi_{2n+1}(\theta; 0) &= \sin(n+1)\theta = \sin\left[\left(n+\frac{1}{2}\right)\theta + \frac{1}{2}\theta\right] \\
&= \sin\tfrac{1}{2}\theta \cos\tfrac{1}{2}\theta \left[\frac{\cos(n+(1/2))\theta}{\cos(1/2)\theta} + \frac{\sin(n+(1/2))\theta}{\sin(1/2)\theta}\right] \\
&= \tfrac{1}{2}\sin\theta\,[V_n(\cos\theta) + W_n(\cos\theta)],
\end{aligned} \tag{5}$$

where

$$V_n(x) = \frac{\cos(n+(1/2))\theta}{\cos(1/2)\theta} \quad \text{and} \quad W_n(x) = \frac{\sin(n+(1/2))\theta}{\sin(1/2)\theta} \tag{6}$$

stand for Chebyshev polynomials of the *third* and *fourth* kinds, respectively, which are both polynomials of degree n in $x = \cos\theta$ [6]. Comparing (2) and (5), we get easily the connection formula

$$U_n(x) = \tfrac{1}{2}\,[V_n(x) + W_n(x)] \tag{7}$$

between the Chebyshev polynomials of the second, third and fourth kinds.

The particle-in-a-box wavefunction in (2) now suggests the transformation of the dependent variable

$$\Psi_{\mathrm{as}}(\theta; \mu) = \sin\theta\, \Phi(\theta; \mu) \tag{8}$$

and then the change of the independent variable from θ to $x = \cos\theta$, for an appropriate treatment of the odd-parity state eigenfunctions Ψ_{as}. Note here that Φ should remain bounded at the boundaries $\theta = \pm\pi$, or at $x = -1$. After standard calculations, we see that $\Phi(x; \mu)$ satisfies the equation

$$\left[(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}}{\mathrm{d}x} + E - 1 - \frac{\mu(\mu+1)}{2(1+x)}\right]\Phi(x; \mu) = 0 \tag{9}$$

and that a more tractable differential equation

$$(1-x^2)y'' + [\mu - (\mu+3)x]y' + \left[E - \tfrac{1}{4}(\mu+2)^2\right]y = 0 \tag{10}$$

is obtainable if

$$\Phi(x;\mu) = (1+x)^{\mu/2}y(x). \tag{11}$$

By completely the same mathematical arguments made in PI, it is possible to deduce that the required solution of the quantum mechanical problem can be established if and only if

$$E - \tfrac{1}{4}(\mu+2)^2 = n(n+2+\mu), \quad n = 0, 1, \ldots, \tag{12}$$

which implies the analytical formula

$$E(\mu) := E_{2n+1}(\mu) = \tfrac{1}{4}[(2n+1)+1+\mu]^2 \tag{13}$$

for the antisymmetric state energy eigenvalues.

The condition (12) is equivalent to seek polynomial solutions of the differential equation (10). By introducing (13) into (10), we have

$$(1-x^2)y'' + [\mu - (\mu+3)x]y' + n(n+2+\mu)y = 0, \tag{14}$$

whose polynomial solutions are nothing but the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with the parameter values $\alpha = 1/2$ and $\beta = \mu + 1/2$ [5]. In fact, we may adapt one of the following variables:

$$\xi = \tfrac{1}{2}(1+x) = \tfrac{1}{2}(1+\cos\theta) = \cos^2\tfrac{1}{2}\theta = u^2 \tag{15}$$

or

$$1-\xi = \tfrac{1}{2}(1-x) = \tfrac{1}{2}(1-\cos\theta) = \sin^2\tfrac{1}{2}\theta = t^2 \tag{16}$$

to express the analytical solutions in terms of various special functions. In accordance with (11), (14) and (15), we first specialize Φ to the form

$$\Phi(\theta;\mu) := \Phi_{2n+1}(\theta;\mu) = B_n(\mu)\cos^{\mu}\tfrac{1}{2}\theta\, P_n^{(\frac{1}{2},\mu+\frac{1}{2})}(\cos\theta), \tag{17}$$

where B_n is some normalisation constant. Hence the odd function in (8)

$$\Psi_{\rm as}(\theta;\mu) := \Psi_{2n+1}(\theta;\mu) = \sin\theta\,\Phi_{2n+1}(\theta;\mu) \tag{18}$$

now describes an eigenfunction of the original Hamiltonian (1) corresponding to an eigenvalue E_{2n+1} given by (13), for every n. By making use of the variable ξ we find, after some manipulation, that

$$\Phi_{2n+1}(\xi;\mu) = B_n(\mu)\frac{(-1)^n}{n!}(\mu+3/2)_n\,\xi^{\mu/2}\,{}_2F_1(-n, n+2+\mu; \mu+3/2; \xi) \tag{19}$$

and that

$$\Phi_{2n+1}(\xi;\mu) = B_n(\mu)\frac{(3/2)_n}{n!}\,\xi^{\mu/2}\,{}_2F_1(-n, n+2+\mu; 3/2; 1-\xi), \tag{20}$$

wherein each Gauss hypergeometric function ${}_2F_1$ reduces to a polynomial, for each has a first parameter equal to a negative integer $-n$.

In section 2, the basic properties of the polynomials so obtained are studied under the name $\mathcal{T}$-polynomials of the second kind. In section 3, we employ the variable t in (16) and show that the polynomial factors of solutions with this variable are expressible in terms of the ultraspherical or Gegenbauer polynomials. The last section includes the derivations of further functional equations, which exist between the $\mathcal{T}$-polynomials of the two kinds and other classical orthogonal polynomials (COPs) as well as the concluding remarks.

2. The $\mathcal{T}$-polynomials $\mathcal{U}_n^{(\mu)}$ of the second kind

In analogy with the case of the symmetric states in PI, we update the normalisation constant to be

$$B_n(\mu) := \frac{n!}{(\mu+3/2)_n}\,\mathcal{B}_n(\mu) \tag{21}$$

and express the antisymmetric wavefunctions in the form

$$\Psi_{2n+1}(\theta;\mu) = \mathcal{B}_n(\mu)\sin\theta\,\cos^{\mu}\tfrac{1}{2}\theta\,\mathcal{U}_n^{(\mu)}(\cos\theta) \tag{22}$$

or

$$\Psi_{2n+1}(x;\mu) = \mathcal{B}_n(\mu)\sqrt{1-x^2}\,[(1+x)/2]^{\mu/2}\,\mathcal{U}_n^{(\mu)}(x) \tag{23}$$

in which

$$\mathcal{U}_n^{(\mu)}(x) := \frac{n!}{(\mu+3/2)_n}\,P_n^{\left(\frac{1}{2},\,\mu+\frac{1}{2}\right)}(x) \tag{24}$$

has been recognized as the $\mathcal{T}$-polynomials of the second kind of order μ and degree n. The hypergeometric representations of the $\mathcal{U}_n^{(\mu)}(x)$,

$$\mathcal{U}_n^{(\mu)}(x) = (-1)^n\,{}_2F_1(-n, n+2+\mu; \mu+3/2; \xi) \tag{25}$$

and

$$\mathcal{U}_n^{(\mu)}(x) = \frac{(3/2)_n}{(\mu+3/2)_n}\,{}_2F_1(-n, n+2+\mu; 3/2; 1-\xi) \tag{26}$$

with argument $\xi = (1+x)/2$ introduced by (15), follow directly from (19) and (20), respectively. Using for example (25), we may also define these polynomials explicitly

$$\mathcal{U}_n^{(\mu)}(x) = (-1)^n \sum_{k=0}^{n} \frac{(-n)_k (n+2+\mu)_k}{(\mu+3/2)_k 2^k k!} (1+x)^k \tag{27}$$

the first few of which are

$$\begin{aligned} \mathcal{U}_0^{(\mu)}(x) &= 1, \\ \mathcal{U}_1^{(\mu)}(x) &= \frac{1}{2\mu+3}[(\mu+3)x-\mu], \\ \mathcal{U}_2^{(\mu)}(x) &= \frac{1}{(2\mu+3)(2\mu+5)}[(\mu+4)(\mu+5)x^2 - 2\mu(\mu+4)x + \mu^2 - \mu - 5]. \end{aligned} \tag{28}$$

On the other hand, if we consider the inner product

$$\begin{aligned} \langle \Psi_{2m+1}, \Psi_{2n+1} \rangle &= \mathcal{B}_m \mathcal{B}_n \int_{-\pi}^{\pi} \sin^2\theta \, \cos^{2\mu} \tfrac{1}{2}\theta \, \mathcal{U}_m^{(\mu)}(\cos\theta) \, \mathcal{U}_n^{(\mu)}(\cos\theta) \mathrm{d}\theta \\ &= 2^{1-\mu} \mathcal{B}_m \mathcal{B}_n \int_{-1}^{1} \rho(x) \, \mathcal{U}_m^{(\mu)}(x) \, \mathcal{U}_n^{(\mu)}(x) \mathrm{d}x \end{aligned} \tag{29}$$

then the orthogonality of the eigenfunctions allows us to compute the normalisation constant

$$\mathcal{B}_n^2(\mu) = \frac{(\mu+3/2)_n}{n!(3/2)_n} \frac{\Gamma(\mu+2+n)}{\Gamma(1/2)\Gamma(\mu+1/2)} \frac{(2+\mu+2n)}{2(1+2\mu)}, \tag{30}$$

where

$$\rho(x) = (1-x)^{1/2}(1+x)^{\mu+1/2}. \tag{31}$$

Furthermore, the Rodrigues' formula for $\mathcal{U}_n^{(\mu)}(x)$

$$\mathcal{U}_n^{(\mu)}(x) = \frac{(-1)^n}{2^n(\mu+3/2)_n} \frac{1}{\rho(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[\rho(x)(1-x^2)^n\right] \tag{32}$$

can be determined in a straightforward manner.

In the particular case of the particle-in-a-box potential where $\mu = 0$, we see that the normalisation constant simplifies into

$$\mathcal{B}_n^2(0) = (n+1)^2/\pi \tag{33}$$

and the wavefunction (22) takes the form

$$\Psi_{2n+1}(\theta; 0) = \frac{1}{\sqrt{\pi}}(n+1) \sin\theta \, \mathcal{U}_n^{(0)}(\cos\theta). \tag{34}$$

Thus a comparison with (2) shows that

$$U_n(x) = (n+1)\mathcal{U}_n^{(0)}(x) \tag{35}$$

i.e., the $\mathcal{T}$-polynomials of the second kind reduces to a multiple of the Chebyshev polynomials of the second kind when $\mu = 0$. Alternatively, this relation may also be verified directly from the Rodrigues' formula (32) by putting $\mu = 0$ throughout.

A typical three term relation of the form

$$\mathcal{U}_{n+1}^{(\mu)}(x) = [a_n(\mu)x - b_n(\mu)]\mathcal{U}_n^{(\mu)}(x) - c_n(\mu)\mathcal{U}_{n-1}^{(\mu)}(x), \tag{36}$$

which is necessarily satisfied by each COP can derived by means of the explicit formula (27), or otherwise, where the coefficients $a_n(\mu)$, $b_n(\mu)$ and $c_n(\mu)$ are

$$a_n(\mu) = \frac{(2n+\mu+2)(2n+\mu+3)}{(n+\mu+2)(2n+2\mu+3)}, \tag{37}$$

$$b_n(\mu) = \frac{\mu(\mu+1)(2n+\mu+2)}{(n+\mu+2)(2n+\mu+1)(2n+2\mu+3)} \tag{38}$$

and

$$c_n(\mu) = \frac{n(2n+1)(2n+\mu+3)}{(n+\mu+2)(2n+\mu+1)(2n+2\mu+3)}, \tag{39}$$

respectively. Again, when $\mu = 0$, we have

$$(n+2)\mathcal{U}_{n+1}^{(0)}(x) = 2x(n+1)\mathcal{U}_n^{(0)}(x) - n\mathcal{U}_{n-1}^{(0)}(x) \tag{40}$$

leading to the simple result $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ on using (35), which is the usual recurrence relation valid for Chebyshev polynomials of all kinds.

3. Ultraspherical polynomial wavefunctions and interrelations

The symmetric

$$\Psi_{2n}(x;\mu) = \mathcal{A}_n(\mu)[(1+x)/2]^{(\mu+1)/2}\,\mathcal{T}_n^{(\mu)}(x) \tag{41a}$$

and antisymmetric state wavefunctions

$$\Psi_{2n+1}(x;\mu) = \mathcal{B}_n(\mu)\sqrt{1-x^2}\,[(1+x)/2]^{\mu/2}\,\mathcal{U}_n^{(\mu)}(x) \tag{41b}$$

of the trigonometric Hamiltonian have been introduced in PI and here, in terms of the $\mathcal{T}$-polynomials depending on $x = \cos\theta$, which does not represent a one-to-one transformation from the original domain of θ, $\theta \in (-\pi, \pi)$, to the

x-interval $(-1, 1)$. If we look at the alternative variables in (16), it is an easy matter to see that

$$t = \sin \tfrac{1}{2}\theta, \quad t \in (-1, 1) \tag{42}$$

provides a one-to-one mapping with which the Hamiltonian $H(\theta; \mu)$ becomes

$$H(t; \mu) = -\frac{1}{4}\left[(1-t^2)\frac{\mathrm{d}^2}{\mathrm{d}t^2} - t\frac{\mathrm{d}}{\mathrm{d}t} - \frac{\mu(\mu+1)}{1-t^2}\right], \tag{43}$$

so that the reflection symmetry $H(\theta; \mu) = H(-\theta; \mu)$ of the system is preserved since

$$H(t; \mu) = H(-t; \mu) \tag{44}$$

also holds. As a result, we may deal with a wavefunction of the type

$$\Psi(t; \mu) = (1-t^2)^{(\mu+1)/2}\, G(t) \tag{45}$$

satisfying the boundary conditions $\Psi(\pm 1; \mu) = 0$ and seek bounded solutions of the differential equation

$$(1-t^2)G'' - (2\mu+3)tG' + \left[4E - (\mu+1)^2\right]G = 0 \tag{46}$$

for $G(t)$. Thus polynomial solutions are the required solutions existing actually under the assumption that

$$4E - (\mu+1)^2 = k(k+2+2\mu), \tag{47}$$

which generates the complete spectrum

$$E(\mu) := E_k(\mu) = \tfrac{1}{4}(k+1+\mu)^2, \quad k = 0, 1, \ldots \tag{48}$$

i.e. both even and odd indexed spectral points, of the eigenvalue problem. Clearly, this formula agrees correctly with E_{2n} of PI (see equation (26) therein) and E_{2n+1} in (13).

The substitution of (47) into (46) leads to the well-known equation

$$(1-t^2)G'' - (2\lambda+1)tG' + k(k+2\lambda)G = 0, \quad \lambda = \mu + 1, \tag{49}$$

whose polynomial solutions are constant multiples of the Jacobi polynomials $P_k^{(\alpha,\beta)}(t)$ having equal parameter values $\alpha = \beta = \lambda - 1/2 = \mu + 1/2$. Moreover, a rescaling of these polynomials chosen for historical reasons makes it possible to introduce the celebrated ultraspherical or Gegenbauer polynomials denoted by $C_k^{(\lambda)}$ [7]. To be more specific, recalling the fact that [5]

$$G(t) \simeq P_k^{(\mu+(1/2),\mu+(1/2))}(t) = \frac{(\mu+3/2)_k}{(2\mu+2)_k} C_k^{(\mu+1)}(t) \tag{50}$$

there follows:

$$\Psi_k(t;\mu) = (-1)^{[\![k/2]\!]}\mathcal{N}_k(\mu)(1-t^2)^{(\mu+1)/2}\,\mathcal{C}_k^{(\mu+1)}(t), \tag{51}$$

where $[\![z]\!]$ denotes the integer part of z, and $\mathcal{N}_k$ is a normalisation constant to be determined by the condition

$$\langle \Psi_j, \Psi_k\rangle = \int_{-\pi}^{\pi} \Psi_j(\theta;\mu)\Psi_k(\theta;\mu)\mathrm{d}\theta = \delta_{jk}. \tag{52}$$

With the variable t in (42), the inner product integral

$$\langle \Psi_j, \Psi_k\rangle = 2\mathcal{N}_j(\mu)\mathcal{N}_k(\mu)\int_{-1}^{1}(1-t^2)^{\mu+1/2}\,\mathcal{C}_j^{(\mu+1)}(t)\,\mathcal{C}_k^{(\mu+1)}(t)\mathrm{d}t \tag{53}$$

on the left-hand side of (52) can easily be evaluated, so that

$$\mathcal{N}_k^2(\mu) = (1+\mu+k)\,2^{2\mu}\,k!\,\frac{\Gamma^2(1+\mu)}{\Gamma^2(1/2)\Gamma(2+2\mu+k)}, \tag{54}$$

where we have used the $\mathcal{L}_2$-norm of the Gegenbauer polynomials [5].

It is worth mentioning that $\Psi_k(t;\mu)$ in (51) consists in both symmetric and antisymmetric state eigenfunctions. Therefore, we deduce from (41a) and (51) with $k = 2n$, that

$$\begin{aligned}\Psi_{2n}(t;\mu) &= (-1)^n\mathcal{N}_{2n}(\mu)(1-t^2)^{(\mu+1)/2}\,\mathcal{C}_{2n}^{(\mu+1)}(t)\\ &= \mathcal{A}_n(\mu)[(1+x)/2]^{(\mu+1)/2}\,\mathcal{T}_n^{(\mu)}(x), \quad x = 1-2t^2,\end{aligned} \tag{55}$$

which leads to the functional equations

$$\mathcal{C}_{2n}^{(\mu+1)}(t) = \frac{(-1)^n}{(2n)!}\,(2\mu+1)_{2n}\,\mathcal{T}_n^{(\mu)}(1-2t^2) \tag{56}$$

$$= \frac{(-1)^n n!}{(2n)!}2^{2n}(\mu+1)_n\,P_n^{(-(1/2),\mu+(1/2))}(1-2t^2) \tag{57}$$

among orthogonal polynomials of different families. Notice here that $\mathcal{A}_n(\mu)$ is taken as the positive root of $\mathcal{A}_n^2(\mu)$ in equation (43) of PI. Likewise $\mathcal{N}_{2n}(\mu)$ stands for the positive root of $\mathcal{N}_k^2(\mu)$ in (54) with $k = 2n$. In addition, by making appropriate use of the formula (56), we obtain

$$\mathcal{T}_n^{(\mu)}(x) = \frac{(-1)^n(2n)!}{(2\mu+1)_{2n}}\,\mathcal{C}_{2n}^{(\mu+1)}(t), \quad t = \sqrt{(1-x)/2}, \tag{58}$$

which is, precisely, an interrelation between the $\mathcal{T}$-polynomial of the first kind of order μ and degree n and the Gegenbauer polynomial of order $\mu+1$ and degree $2n$, each depending on a different argument.

On the other hand, the odd-indexed eigenfunctions $\Psi_{2n+1}(t;\mu)$ in (51) are related to the $\mathcal{T}$-polynomials of the second kind. Actually, we may find out in a similar fashion that

$$\mathcal{C}_{2n+1}^{(\mu+1)}(t) = \frac{(-1)^n}{(2n+1)!}(2\mu+2)_{2n+1}\, t\, \mathcal{U}_n^{(\mu)}(1-2t^2) \tag{59}$$

$$= \frac{(-1)^n n!}{(2n+1)!} 2^{2n+1}(\mu+1)_{n+1}\, t\, P_n^{\left(\frac{1}{2},\mu+\frac{1}{2}\right)}(1-2t^2) \tag{60}$$

and that

$$\mathcal{U}_n^{(\mu)}(x) = \frac{(-1)^n(2n+1)!}{(2\mu+2)_{2n+1}}\frac{1}{t}\,\mathcal{C}_{2n+1}^{(\mu+1)}(t), \quad t=\sqrt{(1-x)/2} \tag{61}$$

connecting the $\mathcal{T}$-polynomials of the second kind and others. Notice once more that all of such functional equations are the consequences of taking into account various forms of the normalised eigensolutions of the same differential operator.

In the special case of $\mu=0$, the normalisation constant (54) is simply $\mathcal{N}_k^2(0)=1/\pi$, as well as $\mathcal{A}_n^2(0)$, so that particle-in-a-box wavefunctions can also be expressed in the form

$$\Psi_k(t;0) = \frac{1}{\sqrt{\pi}}(-1)^{[\![k/2]\!]}\sqrt{1-t^2}\,\mathcal{C}_k^{(1)}(t), \tag{62}$$

which could be used to reproduce the known relationships between certain Chebyshev polynomials and the Gegenbauer polynomial of the first order.

4. More on the functional equations and remarks

In this series of two papers, we discuss the orthogonal polynomials encountered in the exact analytical solution of the quantum mechanical system with Hamiltonian (1). To sum up, the polynomials $\mathcal{T}_n^{(\mu)}(x)$,

$$\begin{aligned}\mathcal{T}_n^{(\mu)}(x) &= (-1)^n\frac{(\mu+3/2)_n}{(\mu+1/2)_n}\,{}_2F_1(-n,n+1+\mu;\mu+3/2;\xi)\\ &= \frac{(1/2)_n}{(\mu+1/2)_n}\,{}_2F_1(-n,n+1+\mu;1/2;1-\xi)\\ &= \frac{n!}{(\mu+1/2)_n}P_n^{\left(-\frac{1}{2},\mu+\frac{1}{2}\right)}(x)\end{aligned} \tag{63}$$

and $\mathcal{U}_n^{(\mu)}(x)$,

$$\begin{aligned}\mathcal{U}_n^{(\mu)}(x) &= (-1)^n\,{}_2F_1(-n, n+2+\mu; \mu+3/2; \xi)\\ &= \frac{(3/2)_n}{(\mu+3/2)_n}\,{}_2F_1(-n, n+2+\mu; 3/2; 1-\xi)\\ &= \frac{n!}{(\mu+3/2)_n}P_n^{\left(\frac{1}{2},\mu+\frac{1}{2}\right)}(x),\end{aligned} \tag{64}$$

which are, evidently, rescalings of two particular subclasses of Jacobi polynomials, have been investigated under the titles of $\mathcal{T}$-polynomials of the first and second kinds, respectively. If use is made of the derivative formula of Gauss hypergeometric function [5]

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\,{}_2F_1(a, b; c; z) = \frac{(a)_n(b)_n}{(c)_n}\,{}_2F_1(a+n, b+n; c+n; z) \tag{65}$$

and the recurrance relation $(a)_n = a(a+1)_{n-1}$ for the Pochhammer's symbol, the differential-difference equation

$$\frac{n(n+\mu+1)(2n+2\mu+1)}{(2\mu+1)(2\mu+3)}\mathcal{U}_{n-1}^{(\mu+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{T}_n^{(\mu)}(x) \tag{66}$$

is obtained, which establishes a link between the $\mathcal{T}$-polynomials. Also, making use of the hypergeometric representations of $\mathcal{T}_n^{(\mu)}$, $\mathcal{U}_n^{(\mu)}$ and $\mathcal{U}_{n-1}^{(\mu)}$ in connection with the so-called *contiguity* relation [5]

$$(a-b)\,{}_2F_1(a, b; c; z) = a\,{}_2F_1(a+1, b; c; z) - b\,{}_2F_1(a, b+1; c; z) \tag{67}$$

we may derive

$$(n+\mu+1)\,\mathcal{U}_n^{(\mu)}(x) - n\,\mathcal{U}_{n-1}^{(\mu)}(x) = \frac{(2\mu+1)(2n+\mu+1)}{2n+2\mu+1}\mathcal{T}_n^{(\mu)}(x) \tag{68}$$

as being another functional equation. For $\mu = 0$, we get

$$\mathcal{T}_n^{(0)}(x) = (n+1)\,\mathcal{U}_n^{(0)}(x) - n\,\mathcal{U}_{n-1}^{(0)}(x) \tag{69}$$

or, after using (35) and identity (53) of PI,

$$\mathcal{T}_n^{(0)}(x) = U_n(x) - U_{n-1}(x) = V_n(x) \tag{70}$$

that is, a connection between the zeroth-order $\mathcal{T}$-polynomials and the Chebyshev polynomials of the second and third kinds. As a matter of fact, if we recall the equation $2T_n = U_n - U_{n-2}$ [5], we write (70) in the form $\mathcal{T}_n^{(0)} = [U_n - U_{n-2}] - [U_{n-1} - U_{n-2}]$ to obtain

$$2T_n(x) = \mathcal{T}_n^{(0)}(x) + \mathcal{T}_{n-1}^{(0)}(x), \tag{71}$$

which is now a link between the Chebyshev polynomials $T_n(x)$ of the first kind and $\mathcal{T}$-polynomials of the first kind of the zeroth-order.

Finally, we introduce a limit relationship with the set of associated Laguerre polynomials defined by

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n;\alpha+1;x), \qquad \alpha > -1, \tag{72}$$

which constitutes an independent system of COPs over the semi-infinite real axis $x \in (0,\infty)$ with respect to the weight $\mathrm{e}^{-x}x^{\alpha}$, where ${}_1F_1(a;c;z)$ stands for the *confluent* hypergeometric function. The limit formula

$$\lim_{b\to\infty} {}_2F_1(a,b;c;z/b) = {}_1F_1(a;c;z) \tag{73}$$

constructs a bridge between the two fundamental hypergeometric functions [4]. Now let us consider the hypergeometric form of $\mathcal{T}_n^{(\mu)}$ in (63)

$$\mathcal{T}_n^{(\mu)}(1-2t^2) = \frac{(1/2)_n}{(\mu+1/2)_n}\,{}_2F_1(-n,n+1+\mu;1/2;t^2) \tag{74}$$

involving the argument t. If we replace μ by $\mu-n-1$ and put $s=t^2$, it is found that

$$\mathcal{T}_n^{(\mu-n-1)}(1-2s) = \frac{(-1)^n(1/2)_n}{(3/2-\mu)_n}\,{}_2F_1(-n,\mu;1/2;s), \quad s\in[0,1), \tag{75}$$

where we have employed the identity $(a)_n = (-1)^n(1-n-a)_n$ which follows directly from the definition of the Pochhammer's symbol. Furthermore, setting s/μ in place of s and letting μ tends to infinity, we get

$$\begin{aligned}\lim_{\mu\to\infty}\left[(3/2-\mu)_n\,\mathcal{T}_n^{(\mu-n-1)}(1-2s/\mu)\right] &= (-1)^n(1/2)_n\,{}_1F_1(-n,1/2;s)\\ &= (-1)^n n!L_n^{(-1/2)}(s)\end{aligned} \tag{76}$$

in which the Laguerre polynomials at the last step have been identified from (72). A similar limit relationship

$$\lim_{\mu\to\infty}\left[(3/2-\mu)_n\,\mathcal{U}_n^{(\mu-n-2)}(1-2s/\mu)\right] = (-1)^n n!L_n^{(1/2)}(s) \tag{77}$$

for the $\mathcal{T}$-polynomials of the second kind may be derived in the same way. These limits are useful in finding asymptotic expansions for the $\mathcal{T}$-polynomials. Note that they could be verified easily for a few low-lying states, for instance, for $n=0,1$ and 2 for illustration.

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