

Bessel Basis with Applications: *N*-Dimensional Isotropic Polynomial Oscillators

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ABSTRACT: The efficient technique of expanding the wave function into a Fourier–Bessel series to solve the radial Schrödinger equation with polynomial potentials, $V(r) = \sum_{i=1}^K v_{2i} r^{2i}$, in two dimensions is extended to *N*-dimensional space. It is shown that the spectra of two- and three-dimensional oscillators cover the spectra of the corresponding *N*-dimensional problems for all *N*. Extremely accurate numerical results are presented for illustrative purposes. The connection between the eigenvalues of the general anharmonic oscillators and the confinement potentials of the form $V(r) = -Z/r + \sum_{i=1}^{K-1} c_i r^i$ is also discussed. © 1997 John Wiley & Sons, Inc. *Int J Quant Chem* 63: 935–947, 1997

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1. Introduction

The dimensionless form of the Schrödinger equation written in *N*-dimensional Cartesian coordinates

$$\left[-\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + V(x_1, x_2, \dots, x_N) - E \right] \Psi(x_1, x_2, \dots, x_N) = 0, \quad (1.1)$$

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where

$$\lim_{|x_i| \rightarrow \infty} \Psi(x_1, x_2, \dots, x_N) = 0 \quad (1.2)$$

has attracted the attention of scientists for a long time due to its application in quantum field theory and molecular physics. The corresponding equation in one dimension has been studied very extensively. However, there is a limited number of works on multidimensional systems. Before entering the complexity of the general *N*-dimensional oscillators it is perhaps more reasonable to deal with the spherically symmetric states for which the

potential function is of the form

$$V(x_1, x_2, \dots, x_N) = V\left(\sqrt{x_1^2 + x_2^2 + \dots + x_N^2}\right). \tag{1.3}$$

Therefore, introducing the N -dimensional spherical coordinates

$$r^2 = \sum_{i=1}^N x_i^2 \tag{1.4}$$

the radial part of the Schrödinger equation (1.1) becomes

$$\left[-\frac{d^2}{dr^2} - \frac{N-1}{r} \frac{d}{dr} + \frac{l(l+N-2)}{r^2} + V(r) \right] \times \Psi(r) = E\Psi(r), \quad r \in [0, \infty), \tag{1.5}$$

where

$$\lim_{r \rightarrow \infty} \Psi(r) = 0. \tag{1.6}$$

Additionally, the wave function must behave like r^l as $r \rightarrow 0$ to have a regular solution.

We study here the generalized anharmonic oscillator in N dimensions described by the potentials of the form

$$V(r) = \sum_{i=1}^K v_{2i} r^{2i}, \quad v_{2K} > 0 \tag{1.7}$$

with $K = 1, 2, \dots$, where the v_{2i} are the coupling constants. As is well known, the trivial case of the harmonic oscillator when $K = 1$, i.e., $V(r) = v_2 r^2$, admits exact solutions in terms of the associated Laguerre polynomials,

$$\Psi_{n,l}^{(N)}(r) = r^l e^{-(1/2)\sqrt{v_2}r^2} L_n^{(1/2)N+l-1}(\sqrt{v_2} r^2) \tag{1.8}$$

for the eigenfunctions while the eigenvalues are specified by

$$E_{n,l}^{(N)} = \sqrt{v_2} (N + 4n + 2l), \quad n, l = 0, 1, \dots \tag{1.9}$$

in N -dimensional space. Moreover, it can be deduced that, if $K = 2K' + 1$, $K' = 1, 2, \dots$, explicit solutions in closed form may be constructed whenever certain algebraic relations between the coupling constants hold [1, 2]. For instance, the ground-state wave function of the sextic oscillator, $V(r) = v_2 r^2 + v_4 r^4 + v_6 r^6$, is expressible as

$$\Psi_{0,0}^{(N)}(r) = e^{-(1/4)ar^4 - (1/2)br^2}, \quad a > 0, \tag{1.10}$$

which satisfies the boundary condition (1.6). Substituting into (1.5), we see that (1.10) is indeed an exact eigenfunction with the corresponding eigenvalue,

$$E_{0,0}^{(N)} = Nb, \tag{1.11}$$

subject to a constraint on v_2 that

$$v_2 = b^2 - (N + 2)a, \tag{1.12}$$

where

$$a = \sqrt{v_6}, \quad b = \frac{1}{2} \frac{v_4}{\sqrt{v_6}}. \tag{1.13}$$

The other states are characterized exactly by exponentially weighted polynomial wave functions. This class of exact solutions is enumerably infinite but not complete. However, if $K = 2K'$, such a construction of exact eigensolutions is not possible. The quartic anharmonic oscillator, $V(r) = v_2 r^2 + v_4 r^4$ is the simplest example to this case [3–5].

An exponential part in the solution is necessary owing to the essential singularity at infinity, which is controlled mainly by the dominant coupling r^{2K} . It is a simple matter to find out the correct asymptotic form of the wave function as $r \rightarrow \infty$, but the introduction of an orthonormal basis set reflecting the required asymptotic behavior is the problem. In a recent study [6] by the same authors, it is shown that the modification of the asymptotic boundary condition (1.6) such that $\Psi(L) = 0$, where L is finite, enables to use Bessel functions of the first kind as a basis set in the Rayleigh–Ritz variational method for solving very accurately the two-dimensional radial Schrödinger equation with any polynomial potential in (1.7).

In this article we truncate again the semi-infinite interval of r and consider the Dirichlet-type boundary condition

$$\Psi(r)|_{r=L} = 0 \tag{1.14}$$

over a spherical surface of radius L . The unperturbed Schrödinger equation is defined as

$$\left[-\frac{d^2}{dr^2} - \frac{N-1}{r} \frac{d}{dr} + \frac{l(l+N-2)}{r^2} \right] \Psi(r) = \lambda^2 \Psi(r) \tag{1.15}$$

in which a zero potential energy, i.e., $V(r) \equiv 0$, is assumed, where λ is a constant. When $N = 2$ the unperturbed equation so defined is directly equiv-

alent to Bessel's differential equation whose normalized eigenfunctions form a complete orthonormal set over the interval $0 \leq r \leq L$ with respect to the weighting function r [6]. In the general case of N dimensions we introduce a new dependent variable $\Phi(r)$ such that

$$\Psi(r) = r^{-(1/2)(N-1)}\Phi(r) \quad (1.16)$$

where $\Phi(r)$ satisfies the differential equation

$$\frac{d^2\Phi}{dr^2} + \left(\lambda^2 - \frac{\nu^2 - \frac{1}{4}}{r^2} \right) \Phi(r) = 0 \quad (1.17)$$

with

$$\nu = l + \frac{1}{2}(N - 2), \quad (1.18)$$

which can be solved analytically as well. It is clear that if $N = 1$, then $\Phi(r)$ is identical to $\Psi(r)$. As a result of (1.16), the solution of the transformed full Schrödinger equation

$$\left[\frac{d^2}{dr^2} + E - \frac{\nu^2 - \frac{1}{4}}{r^2} - V(r) \right] \Phi(r) = 0, \quad r \in [0, L] \quad (1.19)$$

may be postulated to be an expansion in terms of the exact eigenfunctions of (1.17). Such an expansion is presented in Section 2. Note that the accompanying conditions with (1.19) are now

$$\Phi(r) \approx r^{\nu+(1/2)} \text{ as } r \rightarrow 0, \quad \Phi(L) = 0, \quad (1.20)$$

and that (1.5) and (1.19) have completely the same eigenvalues.

As a matter of fact, truncating the interval of r we consider an enclosed quantum mechanical system of multidimensional isotropic oscillators which also has received great interest in different fields of physics and astronomy [7]. Our strategy in this work, however, is to determine the spectrum of the corresponding unbounded system, where $L \rightarrow \infty$, making use of the finite boundary as a nonlinear optimization parameter. This approach is justified analytically in Section 3.

Another remark is that the transformed equation (1.19) is concerned explicitly with the parameter ν rather than the angular quantum number l and the dimension of the space N , separately. In other words, Eq. (1.18) implies that any eigensolution of (1.19) denoted by $\{\Phi_{n,l}^{(N)}, E_{n,l}^{(N)}\}$ remains unchanged for a prescribed value of $2l + N$. For instance, $\Phi_{n,1}^{(2)} = \Phi_{n,0}^{(4)}$ and $E_{n,1}^{(2)} = E_{n,0}^{(4)}$. However,

the situation with the eigenfunctions $\Psi_{n,l}^{(N)}$ of the original problem (1.5) is different. More specifically, we see from (1.6) that $\Psi_{n,1}^{(2)} = r\Psi_{n,0}^{(4)}$ and hence $\Psi_{n,1}^{(2)} \neq \Psi_{n,0}^{(4)}$, although the corresponding eigenvalues are equal to each other. The last section is, therefore, devoted to the discussion of these interesting degeneracies of the spectrum of N -dimensional polynomial potentials, and the numerical applications. Furthermore, we establish functional relationships between the anharmonic oscillators considered in this work and the perturbed Coulomb problem, which was studied especially from the view-point of the perturbation theory and Lie algebra technology [8, 9].

2. Eigenfunction Expansion

As is readily seen the Bessel functions of the first kind multiplied by $r^{1/2}$

$$\phi(r) = cr^{1/2}J_\nu(\lambda r) \quad (2.1)$$

stands for the exact solution of (1.17) which behaves like $r^{\nu+(1/2)}$ at the origin. Then the second condition in (1.20) is fulfilled if λL is a positive root of the equation

$$J_\nu(\xi) = 0 \quad (2.2)$$

having a countable infinite set of distinct zeros, $\lambda L \equiv \alpha = \alpha_1, \alpha_2, \dots, \alpha_n, \dots$ [10]. Furthermore, when c is properly chosen, the functions in (2.1) may be normalized so that

$$\langle \phi_n, \phi_m \rangle = \int_0^L \phi_n(r)\phi_m(r) dr = \delta_{nm}, \quad (2.3)$$

where

$$\phi_n(r) = \frac{\sqrt{2r}}{LJ_{\nu+1}(\alpha_n)} J_\nu(\alpha_n r/L), \quad n = 1, 2, \dots \quad (2.4)$$

and δ_{nm} is the Kronecker delta. Now according to the theory of orthogonal functions, an eigenfunction expansion of the form

$$\Phi(r) = \sum_{n=1}^{\infty} a_n \phi_n(r) \quad (2.5)$$

is proposed to characterize the solution of (1.19) satisfying the conditions. The only thing which remains is the determination of the coefficients a_n .

To this end, the standard Rayleigh–Ritz method leads to the algebraic equations

$$\sum_{n=1}^{\infty} (H_{nm} - E\delta_{nm})a_n = 0, \quad m = 1, 2, \dots \quad (2.6)$$

where the matrix elements H_{nm} are defined by

$$H_{nm} = \lambda_n^2 \delta_{nm} + \sum_{j=1}^K v_{2j} L^{2j} I_{nm}^{(\nu, j)} \quad (2.7)$$

for the polynomial potentials given in (1.7). Here the $I_{nm}^{(\nu, j)}$ denote the integrals of the type

$$I_{nm}^{(\nu, j)} = c_{nm}^{(\nu)} \int_0^1 \xi^{2j+1} J_{\nu}(\alpha_n \xi) J_{\nu}(\alpha_m \xi) d\xi, \quad (2.8)$$

$$c_{nm}^{(\nu)} = \frac{2}{J_{\nu+1}(\alpha_n) J_{\nu+1}(\alpha_m)}$$

to be evaluated, provided $J_{\nu}(\alpha_n) = 0$ for any ν fixed. In the exceptional case of $j = 0$, $I_{nm}^{(\nu, 0)} = \delta_{nm}$ because of the orthonormality relation (2.3).

If the dimension of the space is even, $N = 2, 4, \dots$, then ν becomes an integer from (1.18), and we encounter integrals containing Bessel functions of integer order which have been treated in [6]. Actually, it has been shown that such integrals may be evaluated analytically by means of recursive relations. On the other hand, whenever N is odd, $N = 3, 5, \dots$, Bessel functions of fractional order related to spherical Bessel functions appear in the integrand of (2.8). These functions can simply be expressed in terms of circular functions, the first three of which are

$$J_{1/2}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2} \sin \xi, \quad (2.9)$$

$$J_{3/2}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2} \left(\frac{\sin \xi}{\xi} - \cos \xi\right), \quad (2.10)$$

$$J_{5/2}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2} \left[\left(\frac{3}{\xi^2} - 1\right)\sin \xi - \frac{3}{\xi} \cos \xi\right]. \quad (2.11)$$

In general, for $m = 0, 1, \dots$, we have

$$J_{m+(1/2)}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2} [f_m(\xi)\sin \xi - g_m(\xi)\cos \xi], \quad (2.12)$$

where $f_m(\xi)$ and $g_m(\xi)$ are polynomials in inverse powers of ξ [11].

As specific examples, let us first consider $\nu = \frac{1}{2}$, i.e., $N = 3$ and $l = 0$, so that, from (2.8),

$$I_{nm}^{(1/2, j)} = c_{nm}^{(1/2)} \int_0^1 \xi^{2j+1} J_{1/2}(\alpha_n \xi) J_{1/2}(\alpha_m \xi) d\xi, \quad (2.13)$$

where $J_{1/2}(\alpha_n) = 0$. From (2.9) we have $\alpha_n = n\pi$, for $n = 1, 2, \dots$, and therefore $I_{nm}^{(1/2, j)}$ reduces to

$$I_{nm}^{(1/2, j)} = \frac{2}{\pi^2 \sqrt{nm}} c_{nm}^{(1/2)} \times \int_0^1 \xi^{2j} \sin(n\pi\xi) \sin(m\pi\xi) d\xi. \quad (2.14)$$

Using (2.10) to calculate $c_{nm}^{(1/2)}$ it follows that

$$I_{nm}^{(1/2, j)} = (-1)^{n+m} [S_j(\alpha_n - \alpha_m) - S_j(\alpha_n + \alpha_m)], \quad (2.15)$$

$$\alpha_k = k\pi$$

wherein the integral denoted by $S_j(\alpha)$,

$$S_j(\alpha) = \int_0^1 \xi^{2j} \cos \alpha \xi d\xi, \quad S_j(0) = \frac{1}{2j+1} \quad (2.16)$$

can be evaluated both recursively and explicitly. The recursion relation

$$\alpha^2 S_j(\alpha) + 2j(2j-1)S_{j-1}(\alpha) = \alpha \sin \alpha + 2j \cos \alpha, \quad j = 1, 2, \dots \quad (2.17)$$

with the initial condition $S_0(\alpha) = \sin \alpha / \alpha$ is the computationally more useful one.

For $\nu = \frac{3}{2}$, i.e., $N = 3$ and $l = 1$ or $N = 5$ and $l = 0$, we face the integral

$$I_{nm}^{(3/2, j)} = c_{nm}^{(3/2)} \int_0^1 \xi^{2j+1} J_{3/2}(\alpha_n \xi) J_{3/2}(\alpha_m \xi) d\xi, \quad (2.18)$$

where the α_n 's are now the positive roots of $J_{3/2}(\xi) = 0$. Equation (2.10) implies that the α_n 's should satisfy the equation

$$\tan \xi = \xi. \quad (2.19)$$

In connection with (2.10) and (2.11), the integral in (2.18) may be written as

$$I_{nm}^{(3/2, j)} = \frac{2}{\sin \alpha_n \sin \alpha_m} \times \int_0^1 \xi^{2j} \left(\frac{\sin \alpha_n \xi}{\alpha_n \xi} - \cos \alpha_n \xi \right) \times \left(\frac{\sin \alpha_m \xi}{\alpha_m \xi} - \cos \alpha_m \xi \right) d\xi \quad (2.20)$$

resulting again an expression in terms of $S_j(\alpha)$,

$$I_{nm}^{(3/2, j)} = \frac{1}{\sin \alpha_n \sin \alpha_m} \times \left\{ S_j(\alpha_n - \alpha_m) + S_j(\alpha_n + \alpha_m) + \frac{2j}{\alpha_n \alpha_m} [S_{j-1}(\alpha_n - \alpha_m) - S_{j-1}(\alpha_n + \alpha_m)] \right\} - \frac{2}{\alpha_n \alpha_m}, \quad (2.21)$$

with $j \geq 1$.

The integral $I_{nm}^{(\nu, j)}$ with a larger ν involves more labor in the calculations but can be evaluated in a similar fashion. However, a straightforward recursive evaluation of $I_{nm}^{(\nu, j)}$ is preferable. Therefore, using the general recurrence relations for the Bessel functions [12], we have derived that if $n \neq m$, then

$$I_{nm}^{(\nu, j)} = \frac{c_{nm}^{(\nu)}}{\alpha_n^2 - \alpha_m^2} [\alpha_n J_{\nu+1}(\alpha_n) J_\nu(\alpha_m) - \alpha_m J_{\nu+1}(\alpha_m) J_\nu(\alpha_n)] + \frac{2j c_{nm}^{(\nu)}}{(\alpha_n^2 - \alpha_m^2)^2} \times [(\alpha_n^2 + \alpha_m^2) J_\nu(\alpha_n) J_\nu(\alpha_m) + 2\alpha_n \alpha_m J_{\nu+1}(\alpha_n) J_{\nu+1}(\alpha_m)] - \frac{4j}{(\alpha_n^2 - \alpha_m^2)^2} \times [(\alpha_n^2 + \alpha_m^2)^2 (\nu + j) I_{nm}^{(\nu, j-1)} + 2\alpha_n \alpha_m (j - \nu - 1) I_{nm}^{(\nu+1, j-1)}] \quad (2.22)$$

with

$$I_{nm}^{(\nu, 0)} = \frac{c_{nm}^{(\nu)}}{\alpha_n^2 - \alpha_m^2} [\alpha_n J_{\nu+1}(\alpha_n) J_\nu(\alpha_m) - \alpha_m J_{\nu+1}(\alpha_m) J_\nu(\alpha_n)], \quad (2.23)$$

and if $n = m$, then

$$I_{nn}^{(\nu, j)} = \frac{c_{nn}^{(\nu)}}{2(2j + 1)\alpha_n^2} \times \{ [\alpha_n^2 + 2j(j - \nu)] J_\nu^2(\alpha_n) + \alpha_n^2 J_{\nu+1}^2(\alpha_n) + j(j - \nu) \alpha_n J_\nu(\alpha_n) J_{\nu+1}(\alpha_n) \} + \frac{2j(\nu^2 - j^2)}{(2j + 1)\alpha_n^2} I_{nn}^{(\nu, j-1)} \quad (2.24)$$

with

$$I_{nn}^{(\nu, 0)} = \frac{c_{nn}^{(\nu)}}{2} [J_\nu^2(\alpha_n) + J_{\nu+1}^2(\alpha_n)], \quad (2.25)$$

where $\nu \geq 0$. It should be noted that evaluating such integrals even numerically is not an easy task, and the above formulas being quite general in the sense that no assumption is made about α_n can be effectively used. As in our case, if α_n 's are zeros of $J_\mu(x)$ with a specific $\mu \geq 0$, then $J_\nu(\alpha_n)$ can be written in terms of $J_{\mu+1}(\alpha_n)$ for $\nu = \mu, \mu + 1, \dots, \mu + k - j + 1$ and $j = 1, 2, \dots, k$, where k is a fixed positive integer, and therefore the integrals $I_{nm}^{(\mu, k)}$ are expressible in terms of the zeros α_n of J_μ . Thus, no evaluation of any Bessel function is required, which is of course computationally very efficient.

At the numerical side of the present study, (2.6) is truncated to a homogeneous system of M equations. Therefore the roots of the determinantal equation

$$\det[H_{nm} - E \delta_{nm}]_{M \times M} = 0 \quad (2.26)$$

then give approximately the eigenvalues $E_{n,l}^{(N)}$ of N -dimensional oscillators as a function of the boundary parameter L . The numerical algorithm requires only the zeros α_n , which are calculated on *Mathematica*, of the Bessel functions depending on ν and the potential coefficients v_{2i} as its input data.

3. Variation of Eigenvalues with Respect to L

In this section, we examine briefly the qualitative behavior of the energy eigenvalues with respect to the boundary parameter L . Regarded as $E = E(L)$ and $\Phi = \Phi(r, L)$, Eq. (1.19), satisfied by such a pair of eigensolution, may be reconsidered in the form

$$\begin{aligned} \mathcal{L}\Phi &= 0, \\ \mathcal{L} &= \frac{\partial^2}{\partial r^2} - \frac{\nu^2 - \frac{1}{4}}{r^2} - V(r) + E(L) \end{aligned} \quad (3.1)$$

subject to

$$\Phi(r, L)|_{r=0} = \text{constant}, \quad \Phi(r, L)|_{r=L} = 0. \quad (3.2)$$

In fact, the constant in (3.2) is zero from (1.20) unless the parameter ν is equal to $\pm \frac{1}{2}$. If we differentiate both sides of (3.1) with respect to L , there follows

$$\mathcal{L}\Phi_L + \Phi \frac{dE}{dL} = 0, \quad (3.3)$$

where $\Phi_L = \partial\Phi/\partial L$. The result of multiplying (3.3) by Φ and integrating over the interval $r \in [0, L]$ takes the form

$$\langle \mathcal{L}\Phi_L, \Phi \rangle + \frac{dE}{dL} = 0, \quad (3.4)$$

where we have assumed that the wave function is normalized to have $\langle \Phi, \Phi \rangle = 1$. On integrating by parts and using (3.2) the last equation may be written as

$$\frac{dE}{dL} = \frac{\partial\Phi}{\partial L} \frac{\partial\Phi}{\partial r} \Big|_{r=0}^L - \left\langle \frac{\partial\Phi}{\partial L}, \mathcal{L}\Phi \right\rangle \quad (3.5)$$

in which the inner product on the right-hand side is zero from (3.1). The conditions in (3.2) imply that $d\Phi = 0$ at $r = 0$ and $r = L$. Thus we have

$$\begin{aligned} d\Phi|_{r=0} &= \left(\frac{\partial\Phi}{\partial r} dr + \frac{\partial\Phi}{\partial L} dL \right) \Big|_{r=0} \\ &= \frac{\partial\Phi}{\partial L} \Big|_{r=0} dL = 0 \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} d\Phi|_{r=L} &= \left(\frac{\partial\Phi}{\partial r} dr + \frac{\partial\Phi}{\partial L} dL \right) \Big|_{r=L} \\ &= \left(\frac{\partial\Phi}{\partial r} + \frac{\partial\Phi}{\partial L} \right) \Big|_{r=L} dL = 0 \end{aligned} \quad (3.7)$$

to obtain the relations

$$\frac{\partial\Phi}{\partial L} \Big|_{r=0} = 0, \quad \frac{\partial\Phi}{\partial L} \Big|_{r=L} = -\frac{\partial\Phi}{\partial r} \Big|_{r=L} \quad (3.8)$$

which make it possible to express (3.5) in a more neat form

$$\frac{dE}{dL} = -\left(\frac{\partial\Phi}{\partial r} \right)^2 \Big|_{r=L} \quad (3.9)$$

showing that dE/dL is always negative.

On the other hand, the limit energy $E(L)$ as L approaches zero may be determined analytically. Indeed, making use of the linear transformation, $r = L\xi$, Eq. (1.19) becomes

$$\begin{aligned} \left[\frac{d^2}{d\xi^2} + L^2 E(L) - \frac{\nu^2 - \frac{1}{4}}{\xi^2} - L^2 V(L\xi) \right] \\ \times \Phi(\xi) = 0, \quad \xi \in [0, 1]. \end{aligned} \quad (3.10)$$

When $L \rightarrow 0$, we revert to the unperturbed Schrödinger equation (1.17) for which

$$\lim_{L \rightarrow 0} L^2 E(L) = \alpha^2, \quad (3.11)$$

where α is any root of (2.2). Thus $E(L)$ goes to infinity like $1/L^2$ as L tends to zero and decreases monotonically, since $dE/dL < 0$ from (3.9), as L increases. However, $E(L)$ must converge to a limit for sufficiently large values of L owing to the well-known fact that a discrete spectrum for each ν exists in the case of the corresponding unbounded problem. Therefore, the eigenvalues $E(L)$ of a confined oscillator are upper bounds for the asymptotic eigenvalues $E(\infty)$.

From a computational point of view it is essential to find a specific boundary value, which may be referred to as the critical boundary parameter L_{cr} , such that for a given $\varepsilon > 0$ as small as pleased,

$$E(L_{cr}) - E(L_{cr} + \delta) < \varepsilon \quad (3.12)$$

for all (large) values of $\delta > 0$. The strictly decreasing property of eigenvalues then implies that the absolute error in determining asymptotic eigen-

values $E(\infty)$ is less than ε , that is,

$$E(L_{\text{cr}}) - E(\infty) < \varepsilon. \quad (3.13)$$

It is clear that a critical value of L so defined represents numerically the infinity depending on ε , the quantum numbers and the potential function in question. Hence the present approach based mainly on estimating such L_{cr} values by numerical experiments.

4. Results and Discussion

In this work, we explored a new way of computing the energy eigenvalues of N -dimensional polynomial oscillators. For a complete analysis of the system, let us first consider the exceptional case of one dimension. For $N = 1$, it is shown from (1.5) that l is either 0 or 1, and the equation reduces to the usual one-dimensional Schrödinger equation:

$$\left[-\frac{d^2}{dx^2} + V(x) \right] \Psi(x) = E\Psi(x),$$

$$x \in [-L, L], \quad \Psi(\pm L) = 0, \quad (4.1)$$

where r has been replaced by x , $x = \pm r$. Corresponding to $l = 0$ and $l = 1$, we see from (1.18) that $\nu = \pm \frac{1}{2}$, respectively, and hence that the unperturbed equation (1.17) is altered to the differential equation satisfied by the circular functions. Therefore, the one-dimensional problem may be treated applying trigonometric basis sets, as was done in three earlier studies of the first author [13–15]. Notice that such simple trigonometric functions can be derived within our general formalism since, for $\nu = \pm \frac{1}{2}$,

$$\sqrt{\frac{\pi}{2}} \lambda x J_{-1/2}(\lambda x) = \cos \lambda x,$$

$$\sqrt{\frac{\pi}{2}} \lambda x J_{1/2}(\lambda x) = \sin \lambda x \quad (4.2)$$

representing the symmetric and antisymmetric states, respectively.

It is important to observe that the symmetric energy levels $E_{2n}^{(1)}$ of the one-dimensional problem are single in N -dimensional system. However, the antisymmetric levels $E_{2n+1}^{(1)}$ are doubly degenerate since $\nu = \frac{1}{2}$ stands also for the three-dimensional oscillators with $l = 0$. As outlined in the introduc-

tion, we deduce from (1.18) and (1.19) that

$$\begin{aligned} E_{n,1}^{(3)} &= E_{n,0}^{(5)} \\ E_{n,2}^{(3)} &= E_{n,1}^{(5)} = E_{n,0}^{(7)} \\ &\vdots \\ E_{n,l}^{(3)} &= E_{n,l-1}^{(5)} = E_{n,l-2}^{(7)} = \dots \\ &= E_{n,2}^{(2l-1)} = E_{n,1}^{(2l+1)} = E_{n,0}^{(2l+3)} \end{aligned} \quad (4.3)$$

when N is odd. In particular, we have $E_{n,0}^{(3)} = E_{2n+1}^{(1)}$.

If N is even, it can be shown that $E_{n,0}^{(2)}$ is single in the system, and the remaining eigenvalues are degenerate as follows:

$$\begin{aligned} E_{n,1}^{(2)} &= E_{n,0}^{(4)} \\ E_{n,2}^{(2)} &= E_{n,1}^{(4)} = E_{n,0}^{(6)} \\ &\vdots \\ E_{n,l}^{(2)} &= E_{n,l-1}^{(4)} = E_{n,l-2}^{(6)} = \dots \\ &= E_{n,2}^{(2l-2)} = E_{n,1}^{(2l)} = E_{n,0}^{(2l+2)} \end{aligned} \quad (4.4)$$

As a significant consequence of (4.4) and (4.3), the full spectra of N -dimensional oscillators can be furnished by determining the spectra of two- and three-dimensional problems. The former rests on an expansion of the wave function in terms of the Bessel functions of integer order, whereas the latter is treated by an expansion of the spherical Bessel functions. We now understand that the spectrum of a two-dimensional system considered numerically in [6] is representative for $2N$ -dimensional spaces as well. We, therefore, present computer results here for $N = 3$ to cope with also the $(2N + 1)$ -dimensional spaces according to (4.3), for completion.

In Tables I–V, the eigenvalues accurate to 30-digits have been tabulated in a systematic manner to illustrate the efficiency of the present basis set, for a wide class of polynomial potentials of degree up to 20. In the tables, n and l stand for the quantum numbers of the state, and L_{cr} denotes the radius of the finite spherical surface which represents numerically the unbounded domain to the accuracy quoted. The number of basis elements or the truncation order M of the variational matrix required to get the prescribed accuracy is also included in the tables.

Within the framework of the above general remarks, Table I tests the rate of convergence of the method in calculating the ground-state eigenvalue $E_{0,0}^{(3)}$ of the harmonic oscillator, which is known

TABLE I
Convergence rate of the method as functions of L and M for the ground-state energy $E_{0,0}^{(3)}$ of the three-dimensional harmonic oscillator, $V(r) = r^2$.

L	M	$E_{0,0}^{(3)}$
5	8	3.000 000 007 359
	10	3.000 000 007 344
	12	3.000 000 007 343
7	14	3.000 000 000 000 000 054
	16	3.000 000 000 000 000 000 786 5
	18	3.000 000 000 000 000 000 786 3
9	20	3.000 000 000 000 000 000 014 071
	22	3.000 000 000 000 000 000 000 431
	24	3.000 000 000 000 000 000 000 000 00000
11	26	3.000 000 000 000 000 000 000 050
	28	3.000 000 000 000 000 000 000 000 00691
	30	3.000 000 000 000 000 000 000 000 00000

exactly from (1.9). It is shown that 30 significant digits have been achieved when $M = 24$ and $L = 9$. If we continue to increase L , we obtain the same accuracy at the cost of using a higher truncation size M . Thus, in this case, we identify 9 as the critical or optimum boundary value L_{cr} . In a general situation, the behavior of L_{cr} is virtually the same, further discussion on which can be found in [6].

In Table II, we consider a polynomial potential of degree $2K$ containing only the dominant term,

$$V(r) = v_{2K} r^{2K} \tag{4.5}$$

for $K = 1, 2, 3, 4,$ and 10 . Since a linear scaling transformation transforms any case to $v_{2K} = 1$ via the relation,

$$E(v_{2K}) = v_{2K}^{1/(K+1)} E(1), \tag{4.6}$$

TABLE II
Critical values L_{cr} and the energy eigenvalues $E_{n,l}^{(3)}$ of the potential, $V(r) = r^{2K}$, as a function of K .

K	n	l	$E_{n,l}^{(3)}$	L_{cr}	M
1	0	0	3.000 000 000 000 000 000 000 00000	9	24
	0	5	13.000 000 000 000 000 000 000 00000	9.5	28
	0	10	23.000 000 000 000 000 000 000 00000	10.5	30
	5	0	23.000 000 000 000 000 000 000 00000	10.5	36
	5	5	33.000 000 000 000 000 000 000 00000	11	38
	10	0	43.000 000 000 000 000 000 000 00000	12	46
2	0	0	3.799 673 029 801 394 168 783 094 18851	4.75	28
	0	5	23.940 622 097 894 264 116 215 207 4430	5	30
	0	10	50.305 736 167 707 034 998 003 310 8137	5.25	32
	5	0	56.734 214 055 173 036 047 301 671 0335	5.25	34
	5	5	90.373 823 037 450 379 389 305 630 0458	5.5	38
	10	0	130.642 068 748 629 978 827 208 481 361	6	46
3	0	0	4.338 598 711 513 981 191 647 336 88806	3.6	34
	0	5	31.719 280 900 827 550 416 566 946 3749	3.6	34
	0	10	72.020 071 790 814 934 528 204 397 1274	3.6	34
	5	0	88.392 375 769 030 215 606 669 098 8364	3.7	38
	5	5	148.364 911 801 181 853 867 431 981 880	3.8	42
	10	0	225.852 006 903 743 593 994 689 769 811	3.8	46
4	0	0	4.755 874 413 960 759 869 312 240 99904	2.9	42
	0	5	37.386 714 985 516 424 289 823 232 5515	2.9	42
	0	10	88.512 091 769 467 303 189 964 067 9302	2.9	42
	5	0	115.120 801 108 759 852 450 770 305 251	3	46
	5	5	199.378 311 104 884 152 381 275 535 826	3	48
	10	0	313.042 006 220 240 519 655 199 877 759	3	52
10	0	0	6.219 360 528 618 843 784 646 893 95199	1.72	80
	0	5	54.133 868 954 795 890 507 106 798 6092	1.72	80
	0	10	137.372 769 001 823 278 207 626 189 356	1.72	80
	5	0	204.369 762 904 256 022 069 915 366 685	1.73	84
	5	5	378.613 187 208 892 251 863 487 972 759	1.73	84
	10	0	635.690 810 800 680 389 162 356 077 133	1.74	86

TABLE III
Critical values l_{cr} and the energy eigenvalues $E_{n,l}^{(3)}$ of the potential, $V(r) = r^2 + v_4 r^4$, as a function of v_4 .

v_4	n	l	$E_{n,l}^{(3)}$	L_{cr}	M
10^{-3}	0	0	3.003 739 748 168 729 116 488 725 07463	9	24
	0	5	13.048 378 068 080 421 254 187 583 3067	9.5	28
	1	4	15.074 020 969 202 344 245 961 029 6237	9.8	30
	2	3	17.101 569 170 561 308 854 770 534 9872	10	30
	3	2	19.131 017 513 855 497 934 607 460 5195	10.3	32
	4	1	21.162 360 877 068 068 839 761 723 9594	10.4	34
	0	10	23.141 963 064 797 673 774 804 145 3094	10	34
	5	0	23.195 594 174 058 120 738 137 527 1473	10.5	36
	5	5	33.385 109 542 173 632 991 924 769 4468	11	38
	10	0	43.673 755 145 812 020 962 037 125 0611	12	44
1	0	0	4.648 812 704 212 077 536 377 032 91726	4.75	30
	0	5	26.528 917 558 123 923 941 508 340 2800	5	30
	1	4	33.212 194 662 929 070 579 600 705 8230	5.1	32
	2	3	39.889 929 660 067 122 409 265 080 0608	5.25	32
	3	2	46.589 466 063 451 956 061 405 768 4506	5.3	34
	4	1	53.331 114 296 954 178 530 932 191 5861	5.4	36
	0	10	54.184 984 610 454 439 924 123 480 1757	5.25	34
	5	0	60.129 522 959 157 771 315 848 016 0599	5.5	36
	5	5	94.797 175 049 923 730 961 266 375 4573	5.5	38
	10	0	135.818 417 325 610 373 340 451 430 113	6	46
10^3	0	0	38.086 833 459 382 264 084 978 363 2123	1.6	32
	0	5	239.672 274 524 698 345 359 912 040 056	1.6	32
	1	4	305.336 770 890 603 425 275 221 109 146	1.63	32
	2	3	370.654 732 557 085 375 806 448 040 350	1.65	32
	3	2	435.984 209 509 541 907 095 088 099 667	1.68	34
	4	1	501.595 871 272 387 416 849 121 622 403	1.69	34
	0	10	503.452 899 010 694 512 429 587 216 895	1.6	34
	5	0	567.686 243 636 190 200 546 860 732 547	1.7	36
	5	5	904.185 674 358 283 932 192 842 208 734	1.8	38
	10	0	1306.942 911 074 395 820 900 455 242 36	1.9	48

we present eigenvalues of r^{2K} for some pairs of (n, l) . It is seen that there is no accuracy loss for higher values of K although the required truncation sizes increase.

Tables III and IV are concerned with the potentials of the form, $V(r) = v_2 r^2 + v_{2K} r^{2K}$, with two limiting values of K , 2 and 10. The eigenvalues $E(v_2, v_{2K})$ has the scaling property

$$E(v_2, v_{2K}) = \sqrt{v_2} E(1, v_2^{-K-1} v_{2K}) \quad (4.7)$$

so that we set $v_2 = 1$ in our calculations and tabulate the results for a wide range of v_{2K} from 10^{-3} to 10^3 .

The last table is devoted to a general polynomial potential, $V(r) = v_2 r^2 + v_4 r^4 + v_6 r^6 + v_8 r^8$, with more than one effective coupling constants. Of course, there is a huge number of combinations

of the set of parameters $\{v_2, v_4, v_6, v_8\}$ so that only the ground-state energies of some selected potentials are recorded in order not to overfill the content of the study with tabular material anymore. Further results are available from the authors.

The potentials ①–④ and ⑤–⑩ in Table V are sextic and octic oscillators, respectively, in which the coupling constants differ from 10^{-3} to 10^3 . It is noteworthy that the method is constantly efficient for any polynomial having small or large parameters. Furthermore, results with such a high accuracy are being reported for the first time, which may be used as a guide for future numerical calculations. The tabulated eigenvalues are in good agreement with those already available in the literature [2, 5, 16]. For a specific example, if we recall that the ground-state eigenvalue for the potential ④ was given to seven digits in Ref. [2], employ-

TABLE IV
Critical values L_{cr} and the energy eigenvalues $E_{n,l}^{(3)}$ of the potential, $V(r) = r^2 + v_{20}r^{20}$, as a function of v_{20} .

v_{20}	n	l	$E_{n,l}^{(3)}$	L_{cr}	M
10^{-3}	0	0	4.137 469 780 373 265 686 313 598 80986	2.42	80
	0	5	30.599 446 675 081 585 897 426 650 1230	2.42	80
	1	4	45.655 148 163 437 269 170 353 340 0425	2.42	80
	2	3	60.912 272 479 007 174 094 873 229 4744	2.42	80
	0	10	75.384 522 962 114 054 459 554 561 2489	2.42	80
	3	2	76.711 559 372 667 931 323 402 503 3454	2.43	82
	4	1	93.149 750 288 227 298 165 553 288 6540	2.43	82
	5	0	110.260 067 772 132 375 655 271 759 226	2.44	84
	5	5	203.414 937 607 089 093 579 671 854 365	2.44	86
	10	0	340.582 874 351 791 797 703 497 042 000	2.44	88
1	0	0	6.667 629 449 728 714 435 493 312 58924	1.72	80
	0	5	55.050 045 936 608 940 288 654 546 2166	1.72	80
	1	4	83.699 045 453 764 288 517 530 921 8092	1.72	80
	2	3	112.457 302 584 932 499 410 693 143 433	1.72	80
	0	10	138.480 749 482 548 539 743 384 754 808	1.72	80
	3	2	142.130 750 936 868 757 438 060 708 422	1.73	82
	4	1	172.952 375 511 687 286 868 832 829 988	1.73	82
	5	0	205.006 515 756 644 844 577 430 284 862	1.74	84
	5	5	379.339 096 998 690 925 288 804 104 999	1.74	86
	10	0	636.402 431 535 269 368 066 728 648 309	1.74	88
10^3	0	0	11.894 969 152 837 305 083 224 495 2663	1.26	80
	0	5	101.926 481 877 639 669 968 098 594 566	1.26	80
	1	4	155.847 964 039 227 173 701 422 055 510	1.26	80
	2	3	209.827 540 039 145 130 410 923 458 331	1.26	80
	0	10	258.003 054 286 820 201 493 213 318 902	1.26	80
	3	2	265.467 200 514 777 898 330 488 852 910	1.27	82
	4	1	323.231 787 898 919 911 920 270 564 849	1.27	82
	5	0	383.291 363 931 949 564 424 566 919 323	1.28	84
	5	5	709.839 363 104 820 778 721 061 853 928	1.28	86
	10	0	1191.548 258 103 368 707 993 267 904 98	1.28	88

ing a Hill determinant of size 150×150 , the success of our method becomes quite impressive as it uses only a truncation order of 34 in calculating the same eigenvalue to 30 digits.

The potentials (11)–(16) in Table V are different in nature since they describe multiwell quartic, sextic, and octic oscillators in Cartesian coordinates. It should be mentioned that the variational method with Bessel basis set attains the same high accuracy for these potentials as well.

As another remark, the trivial eigenvalue ordering properties $E_{n,l_2} > E_{n,l_1} \Leftrightarrow l_2 > l_1$ and $E_{n_2,l} > E_{n_1,l} \Leftrightarrow n_2 > n_1$ for all N have been confirmed by our calculations. Therefore, Eqs. (4.3) and (4.4) now

imply that

$$E_{n,l}^{(N_2)} > E_{n,l}^{(N_1)} \Leftrightarrow N_2 > N_1 \tag{4.8}$$

for all pairs of (n, l) . Furthermore, we deduce from computer experiments that, if we characterize the energy levels $E_{n,l}$ as groups denoted by the number m , where $m = n + l$, then the eigenvalues in such a group may be ordered according to the rule

$$E_{0,m} < E_{1,m-1} < \dots < E_{m-1,1} < E_{m,0} \tag{4.9}$$

for all N .

The connection between hydrogen atom in three dimensions and the isotropic harmonic oscillator in two and four dimensions has been a subject

TABLE V
Specimen calculations for the ground-state eigenvalues $E_{0,0}^{(3)}$ of polynomial potentials $V(r) = v_2 r^2 + v_4 r^4 + v_6 r^6 + v_8 r^8$ as a function of the set $\{v_2, v_4, v_6, v_8\}$ of coupling constants.

Potential	$\{v_2, v_4, v_6, v_8\}$	$E_{0,0}^{(3)}$	L_{cr}	M
①	$\{1, 1, 10^{-3}, 0\}$	4.650 533 375 345 472 534 590 203 20042	5	38
②	$\{1, 1, 1, 0\}$	5.656 437 054 972 568 576 869 392 40107	3.5	32
③	$\{1, 1, 10^3, 0\}$	24.549 807 962 232 392 567 382 912 4853	1.52	36
④	$\{1, 0, 1, 0\}$	5.033 395 937 720 266 476 828 385 45349	3.5	34
⑤	$\{1, 0, 0, 1\}$	5.368 778 061 748 129 766 350 976 01368	3	42
⑥	$\{1, 1, 1, 10^{-3}\}$	5.657 449 899 594 538 806 906 662 36724	4	28
⑦	$\{1, 1, 1, 1\}$	6.329 519 831 389 824 189 257 054 61348	2.9	40
⑧	$\{1, 1, 1, 10^3\}$	19.138 122 248 280 439 855 397 688 23479	1.42	40
⑨	$\{1, 10^{-3}, 1, 1\}$	5.904 741 200 915 130 066 101 684 63870	2.9	40
⑩	$\{1, 10^3, 1, 1\}$	38.089 715 173 559 645 738 649 927 8700	2	40
⑪	$\{-1, 1, 0, 0\}$	2.834 536 202 119 304 214 654 676 20875	5	30
⑫	$\{-1, 0, 1, 0\}$	3.594 957 770 823 873 044 875 895 03249	3.6	36
⑬	$\{-1, 0, 0, 1\}$	4.113 468 271 041 643 593 317 885 09539	2.9	42
⑭	$\{-1, 10^3, 0, 0\}$	37.906 512 170 363 362 310 589 641 9268	1.58	30
⑮	$\{-1, 0, 10^3, 0\}$	24.269 876 516 816 955 726 323 836 6043	1.52	36
⑯	$\{-1, 0, 0, 10^3\}$	18.775 672 346 537 343 545 623 492 4729	1.45	40

of interest in the past [17]. In order to obtain explicit passage formulas between the spectra of hydrogen-like and anharmonic oscillator systems, we now recall the perturbed Coulomb problem with a general confinement potential defined by the equation

$$\left[-\frac{1}{2} \frac{d^2}{dr^2} - \frac{N-1}{2r} \frac{d}{dr} + \frac{l(l+N-2)}{2r^2} - \frac{Z}{r} + \sum_{i=1}^{K-1} c_i r^i \right] R(r) = \mathcal{E} R(r), \quad r \in [0, \infty) \quad (4.10)$$

where Z and the c_i are the nuclear charge parameter and the potential constants, respectively. In (4.10), \mathcal{E} denotes the eigenvalue of the problem in N -dimensional space for an angular-momentum state l . Transforming the independent variable from r to s , where

$$s = \sqrt{\mu r}, \quad \mu > 0, \quad s \in [0, \infty) \quad (4.11)$$

we obtain

$$\left[-\frac{d^2}{ds^2} - \frac{2N-3}{s} \frac{d}{ds} + \frac{4l(l+N-2)}{s^2} + \sum_{i=1}^K v_{2i} s^{2i} \right] R(s) = ER(s) \quad (4.12)$$

in which a new eigenvalue parameter E has been defined by the parameter μ such that

$$\mu = 8Z/E, \quad (4.13)$$

and that the potential has been rearranged with the coefficients

$$v_2 = -8\mathcal{E}/\mu^2, \quad v_{2i} = 8c_{i-1}/\mu^{i+1}, \quad i = 2, 3, \dots, K. \quad (4.14)$$

Note also that the sign, positive or negative, of μ depends on E . Thus, for negative values of E we may replace μ by $-\mu$ in (4.11) without any further modification. Now, introducing a new wave function $\Psi(s)$ such that $\Psi(s) = sR(s)$, Eq. (4.12) takes

the form

$$\left[-\frac{d^2}{ds^2} - \frac{N' - 1}{s} \frac{d}{ds} + \frac{l'(l' + N' - 2)}{s^2} + \sum_{i=1}^K v_{2i} s^{2i} \right] \Psi(s) = E\Psi(s), \quad (4.15)$$

which is equivalent completely to (1.5) with

$$N' = 2N - 4, \quad l' = 2l + 1. \quad (4.16)$$

This connection in a higher-dimensional space between the anharmonic oscillators and the confinement potentials suggests evidently that, after solving Eq. (4.15), we may generate accordingly some solutions for the perturbed Coulomb potential (4.10) as well. Actually, from (4.13), (4.14), and (4.16), we see that any eigenvalue $E_{n,2l+1}^{(2N-4)}$ of a polynomial oscillator of degree $2K$ is related explicitly to an eigenvalue $\mathcal{E}_{n,l}^{(N)}$ of a confinement potential of degree $K - 1$ by the expression.

$$\mathcal{E}_{n,l}^{(N)}(Z; c_1, c_2, \dots, c_{K-1}) = -8v_2 Z^2 \times [E_{n,2l+1}^{(2N-4)}(v_2; v_4, v_6, \dots, v_{2K})]^{-2} \quad (4.17)$$

with

$$c_{i-1}(E) = 8^i v_{2i} (Z/E)^{i+1}, \quad i = 2, 3, \dots, K. \quad (4.18)$$

The relation in (4.17) needs a correct interpretation. It should be noticed that every eigenvalue $E_{n,2l+1}^{(2N-4)}$ of a polynomial oscillator with a fixed set of parameters $\{v_2, v_4, \dots, v_{2K}\}$ corresponds only to one eigenvalue $\mathcal{E}_{n,l}^{(N)}$ of various perturbed Coulomb problems whose potential constants are calculated by (4.18), except the charge parameter Z . Thus, there is no one-to-one correspondence between the eigenvalues of a prescribed general anharmonic oscillator and those of a perturbed Coulomb potential with a definite set of parameters $\{Z, c_1, c_2, \dots, c_{K-1}\}$. The sole exceptional case is that of the harmonic oscillator, i.e., $v_4 = v_6 = \dots = v_{2K} = 0$ for which the hydrogen atom limit, $c_1 = c_2 = \dots = c_{K-1} = 0$, is determined uniquely from (4.18). In this case $E_{n,2l+1}^{(2N-4)}(v_2; 0, 0, \dots, 0) \equiv E_{n,2l+1}^{(2N-4)}(v_2) = 2\sqrt{v_2} (2n + 2l + N - 1)$ from (1.9), and hence the discrete spectrum of N -dimensional

hydrogen atom,

$$\mathcal{E}_{n,l}^{(N)}(Z; 0, 0, \dots, 0) \equiv \mathcal{E}_{n,l}^{(N)}(Z) = -\frac{Z^2}{2[n + l + (N - 1)/2]^2} \quad (4.19)$$

is obtainable exactly from (4.17). In particular, we reproduce the energy eigenvalues,

$$\mathcal{E}_{n,l} = -\frac{1}{2(n + l + 1)^2}, \quad n, l = 0, 1, \dots \quad (4.20)$$

of the standard three-dimensional case when $Z = 1$.

Finally, it should be stated that our main goal is to attack the more challenging problem of unisotropic potentials mentioned in the introduction. The first nontrivial problems of this kind are the two-dimensional

$$V(x, y) = v_2(x^2 + y^2) + v_4(x^4 + y^4 + 2ax^2y^2), \quad a > -1 \quad (4.21)$$

and three-dimensional

$$V(x, y, z) = v_2(x^2 + y^2 + z^2) + v_4(x^4 + y^4 + z^4 + 2ax^2y^2 + 2ax^2z^2 + 2ay^2z^2), \quad a > -\frac{1}{2} \quad (4.22)$$

quartic oscillators. It is apparent that the case $a = 1$ corresponds to the isotropic oscillators dealt with in the present article. Writing, for instance, (4.21) in the cylindrical polar coordinates

$$V(r, \theta) = v_2 r^2 + v_4 r^4 + 2v_4(a - 1)r^4 \cos^2\theta \sin^2\theta \quad (4.23)$$

we may notice the importance of developing such an exact method for the isotropic case, which is necessary and very promising in the investigation of unisotropic potentials along this line as well. Work is in progress toward this aim and will be reported in due course.

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