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DRAG MINIMIZATION IN STOKES FLOW

H. TAŞELI and M. DEMIRALP
TUBITAK, Research Institute for Basic Sciences, Applied Mathematics Department, 41470, Gebze-Kocaeli, Turkey

K. KAFALI
Istanbul Technical University, Faculty of Naval Architecture and Ocean Engineering, Maslak, Istanbul, Turkey

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Abstract—The recent analytical method for solving the axisymmetric Stokes flow past an arbitrary body is applied to minimize the viscous-drag of an elementary double-ship model which moves with a constant speed in an infinite fluid domain. The main framelines of the method are briefly explained, and the optimum forms, with the volume specified as a constant, are presented for illustrative purposes.

1. INTRODUCTION

Determination of the flow characteristics around the bodies moving on the free surface or infinitely under the surface of a liquid has attracted the attention of scientists for a long time. Free surface hydrodynamics extensively deals with the evaluation of the wave resistance of ships under certain assumptions [1–5]. In general, fluid is assumed to be inviscid and incompressible, and the fluid motion is irrotational. As is well known, the problem is then reduced to search for a velocity potential. The main reason for adopting certain assumptions and approximations is that the integration of the full Navier–Stokes equation, except a few simple systems, is yet lacking.

To determine and to minimize the viscous resistance, on the other hand, boundary layer theory is widely employed. For example, Nagamatsu [6, 7] and Nagamatsu *et al.* [8] proposed a method for the calculation and the minimization of viscous resistance based on a higher order boundary layer theory. Hess [9] also realized an optimization procedure for two-dimensional axisymmetric bodies by using the approximate solutions of turbulent boundary layer equations.

An alternative approach in this field is Stokes approximation which the inertial terms in Navier–Stokes equation can be neglected providing that Reynolds number is sufficiently small. In such a case, the governing equation of motion is the linear Stokes equation. An inverse approach for minimum drag in Stokes flow was discussed by Tuck [10] for slender bodies. Pironneau [11] proved that the necessary optimality condition to obtain smallest drag for a unit-volume body is a constant vorticity distribution on the body surface. Bourot [12] made a numerical computation of the optimum profile by making use of the theoretical results of Pironneau. Bessho and Himeno [13] developed a method like the boundary integral method, and presented an inverse procedure for the minimization of the hydrodynamic drag.

In this work, the results of a very recent theory established by Taşeli [14] and Taşeli *et al.* [15] are utilized in a numerical procedure to find the bodies of given volume with the smallest drag. In Section 2, the statement of the problem and the framework of the present theory are given. We derive a first approximation in closed form and present the minimization scheme in Section 3. The last section is devoted to the conclusions as usual.

2. METHODOLOGY

Stokes flow problem of the uniform motion of an inertialess unbounded viscous, incompressible fluid past an arbitrary axisymmetric body, which may be called as $F(\theta)$, in a domain $\mathscr{D}$ (Fig. 1), is described by

$$\nabla^2 \mathbf{v} = \nabla p, \qquad \nabla \cdot \mathbf{v} = 0 \tag{2.1}$$

where $\mathbf{v}$ and p are the velocity vector and the pressure function, respectively.

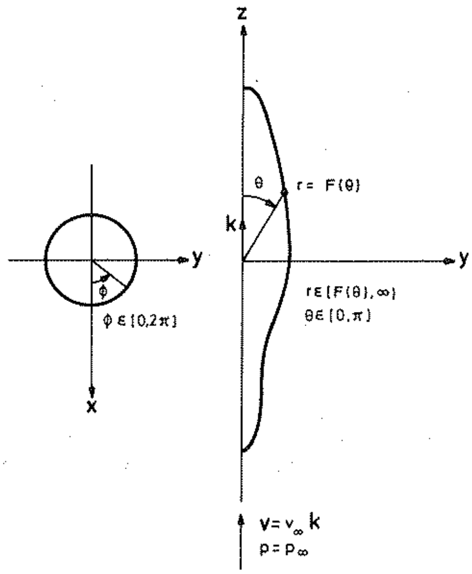

Fig. 1

For an axisymmetrical fluid motion, the mathematical problem is reduced to search for a scalar function which must satisfy the fourth-order partial differential equation,

$$E^4\Psi = 0 \tag{2.2}$$

where Ψ is known as the stream function. It is convenient to write (2.2) in the form

$$E^2\Phi = 0, \qquad E^2\Psi = \Phi. \tag{2.3}$$

Here, E^2 is a second order differential operator defined by

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \tag{2.4}$$

in spherical co-ordinates, and all flow characteristics can be expressed in terms of Φ and Ψ. The boundary conditions of the problem are no-slip condition on the body surface,

$$r = F(\theta) \Rightarrow v_r \equiv v_\theta \equiv 0, \tag{2.5}$$

and equality of the uniform flow at infinity,

$$v_r = \cos\theta, \qquad v_\theta = -\sin\theta \quad \text{as} \quad r \to \infty. \tag{2.6}$$

After solving the problem, the viscous-drag will be evaluated by

$$D = \frac{1}{3}\int_0^\pi \int_{(\theta)}^\infty \Phi^2(r,\theta)\frac{dr\,d\theta}{\sin\theta} \tag{2.7}$$

It should be noticed that velocities, lengths, pressure and drag have been non-dimensionalized by uniform fluid speed, v_∞, a characteristic length l, $l = (3V/4\pi)^{1/3}$, $\mu v_\infty/l$ and $6\pi\mu v_\infty l$ respectively, where V is the volume of body and μ is the dynamic viscosity of fluid. Obviously, if a sphere with unit radius is considered then $l = 1$, and drag becomes unity because of the Stokes' law.

After these basic statements of the problem, we can now give the mainlines of our method. By performing the co-ordinate transformations,

$$\eta = \cos\theta, \qquad \zeta = r - F(\eta); \qquad \eta \in [-1, 1], \qquad \zeta \in [0, \infty) \tag{2.8}$$

drag is expressible as

$$D = \frac{1}{3}\int_{-1}^1 \int_0^\infty \Phi^2(\zeta,\eta)\frac{d\zeta\,d\eta}{1-\eta^2}. \tag{2.9}$$

It is well-known that the physically and practically acceptable values of drag have to be finite, $D<\infty$. It may be stated from (2.9), that Φ and Ψ are the members of a Hilbert space of the square integrable functions, denoted by L_2 which is an interesting point of view of the treatment of the problem. Since the Gegenbauer polynomials generate an orthonormal basis set of the L_2 space on the interval $-1\leq\eta\leq 1$, both Φ and Ψ can be uniquely expanded into the following series:

$$\Phi(\zeta,\eta)=(1-\eta^2)\sum_{k=0}^{\infty}\mathcal{B}_k(\zeta)\mathcal{C}_k(\eta) \tag{2.10}$$

$$\Psi(\zeta,\eta)=(1-\eta^2)\sum_{k=0}^{\infty}\mathcal{A}_k(\zeta)\mathcal{C}_k(\eta) \tag{2.11}$$

where $\mathcal{A}_k$ and $\mathcal{B}_k$ are solely ζ-dependent linear combination coefficients, and $\mathcal{C}_k$ is the Gegenbauer function of order k and degree (3/2). By recalling some of the basic properties of the Hilbert space of the problem and the theory of differential equations, in addition to choosing a function of the type

$$F(\eta)=\alpha_0\left(1+\sum_{i=1}^{\infty}\alpha_i\eta^i\right) \tag{2.12}$$

as the shape function, after a considerably long derivation, one can arrive at the vector differential equations

$$\mathcal{T}\boldsymbol{\mathcal{B}}(\xi)=0 \tag{2.13}$$

$$\mathcal{T}\boldsymbol{\mathcal{A}}(\xi)=\alpha_0^2[\xi^2\mathbf{I}+\xi\boldsymbol{\lambda}^{(1)}+\mathbf{q}^{(2)}]\boldsymbol{\mathcal{B}}(\xi) \tag{2.14}$$

for the determination of the vector functions $\boldsymbol{\mathcal{A}}(\xi)$ and $\boldsymbol{\mathcal{B}}(\xi)$, that is,

$$\boldsymbol{\mathcal{A}}^T(\xi)=[\mathcal{A}_0(\xi),\mathcal{A}_1(\xi),\ldots,\mathcal{A}_k(\xi),\ldots] \tag{2.15}$$

$$\boldsymbol{\mathcal{B}}^T(\xi)=[\mathcal{B}_0(\xi),\mathcal{B}_1(\xi),\ldots,\mathcal{B}_k(\xi),\ldots]. \tag{2.16}$$

Here, α_i's are the design parameters, and the ordinary differential operator, $\mathcal{T}$, with matrix coefficients is defined by

$$\mathcal{T}=(\xi^2\mathbf{I}+\xi\boldsymbol{\lambda}^{(1)}+\mathbf{q}^{(1)})\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}-\boldsymbol{\lambda}^{(2)}\frac{\mathrm{d}}{\mathrm{d}\xi}-\boldsymbol{\gamma} \tag{2.17}$$

where $\mathbf{I}$ is the identity matrix and $\boldsymbol{\gamma}$ is a diagonal matrix,

$$\gamma_{kj}=(k+1)(k+2)\delta_{kj}. \tag{2.18}$$

The other matrices in (2.17) and (2.14), whose elements depend on the design parameters, are obtained such that

$$\lambda_{kj}^{(1)}=2\sum_{i=1}^{\infty}f_{kji}\alpha_i \tag{2.19}$$

$$\lambda_{kj}^{(2)}=\sum_{i=1}^{\infty}i\alpha_i\{(k-j)f_{kji}-(k+2)f_{k-1,j,i-1}\mathcal{N}_{k-1}/\mathcal{N}_k+(j+2)f_{j-1,k,i-1}\mathcal{N}_{j-1}/\mathcal{N}_j\} \tag{2.20}$$

$$q_{kl}^{(1)}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha_i\alpha_j\{(1-ij)f_{k,l,i+j}+ijf_{k,l,i+j-2}\} \tag{2.21}$$

$$q_{kl}^{(2)}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha_i\alpha_jf_{k,l,i+j} \tag{2.22}$$

where

$$f_{kji}=\int_{-1}^{1}\eta^i\mathcal{C}_k(\eta)\mathcal{C}_j(\eta)(1-\eta^2)\,\mathrm{d}\eta \tag{2.23}$$

and

$$\mathcal{N}_k=\left[\int_{-1}^{1}(1-\eta^2)\mathcal{C}_k^2(\eta)\,\mathrm{d}\eta\right]^{1/2}=[2(k+1)(k+2)/(2k+3)]^{1/2}. \tag{2.24}$$

Changing the variable ζ to ξ

$$\xi = \frac{1}{\alpha_0}\zeta + 1, \qquad \xi \in [1, \infty) \tag{2.25}$$

the arbitrary shape of body has been transformed to a circle with unit radius in the fluid domain, and now the geometrical effects are characterized by the above matrices (2.19–22).

The last step of the methodology is the reconstruction of the boundary conditions. Since the solution vector function $\mathscr{A}(\xi)$ in (2.14) produces four arbitrary constant vectors, the determination of them can be accomplished by investigating the vector forms of (2.5) and (2.6) compatible with the presented procedure. Without giving the intermediate algebra we, consequently, obtain

$$\mathscr{A}(1) = \mathscr{A}'(1) = \mathbf{0} \tag{2.26}$$

$$\lim_{\xi\to\infty} \frac{1}{\xi^2}\mathscr{A}(\xi) = -\alpha_0^2 N_0 \boldsymbol{\gamma}^{-1}\mathbf{e}_1, \qquad \lim_{\xi\to\infty}\frac{1}{\xi}\mathscr{A}'(\xi) = -\alpha_0^2\mathcal{N}_0\mathbf{e}_1 \tag{2.27}$$

where $\mathbf{e}_1$ stands for the unit vector,

$$\mathbf{e}_1^T = [1, 0, 0, \ldots]. \tag{2.28}$$

The general solution of the infinitely dimensioned vector differential equation in the N-dimensional finite subspace may be sought from [14] and [15]. In this work, we present an explicit analytic solution when it is considered the one-dimensional subspace, $N = 1$.

3. ANALYTIC SOLUTION IN ONE-DIMENSIONAL SUBSPACE AND DRAG MINIMIZATION

To investigate the truncated solution of the problem when $N = 1$, we assume that the function Φ and the stream function Ψ can be represented by the first terms of their serial expansions such as

$$\Phi(\xi, \eta) = (1 - \eta^2)\mathscr{B}_0(\xi)/\mathcal{N}_0 \tag{3.1}$$

and

$$\Psi(\xi, \eta) = (1 - \eta^2)\mathscr{A}_0(\xi)/\mathcal{N}_0 \tag{3.2}$$

where $\mathscr{A}_0$ and $\mathscr{B}_0$ are the solutions of the two simultaneous equations

$$\mathscr{T}_{00}\mathscr{B}_0(\xi) = 0 \tag{3.3}$$

$$\mathscr{T}_{00}\mathscr{A}_0(\xi) = \alpha_0^2(\xi^2 + \xi\lambda_{00}^{(1)} + q_{00}^{(2)})\mathscr{B}_0(\xi) \tag{3.4}$$

according to (2.13) and (2.14). In this case, the matrix-valued coefficients of $\mathscr{T}$ are 1×1 sub-matrices, that is, scalar quantities which are

$$\begin{gathered} \mathbf{I} \to \delta_{00} = 1, \qquad \boldsymbol{\lambda}^{(1)} \to \lambda_{00}^{(1)} = \lambda, \qquad \boldsymbol{\lambda}^{(2)} \to \lambda_{00}^{(2)} = 0 \\ \mathbf{q}^{(1)} \to q_{00}^{(1)} = q, \qquad \mathbf{q}^{(2)} \to q_{00}^{(2)} = \tilde{q}, \qquad \boldsymbol{\gamma} \to 2\delta_{00} = 2 \end{gathered} \tag{3.5}$$

and then

$$\mathscr{T}_{00} \equiv \mathscr{T} = (\xi^2 + \lambda\xi + q)\frac{d^2}{d\xi^2} - 2. \tag{3.6}$$

For brevity, if we transform the variable ξ to x,

$$x = (2\xi + \lambda)/\Delta_0, \qquad x \in [x_0, \infty] \tag{3.7}$$

wherein

$$x_0 = (2 + \lambda)/\Delta_0, \qquad \Delta_0 = (4q - \lambda^2)^{1/2}, \tag{3.8}$$

the differential operator is altered to

$$\mathscr{T} = (x^2 + 1)\frac{d^2}{dx^2} - 2. \tag{3.9}$$

The solution of the homogeneous equation $\mathscr{T}\mathscr{B}_0(x)=0$ is then

$$\mathscr{B}_0(x)=c_1u_1(x)+c_2u_2(x) \tag{3.10}$$

where c_1 and c_2 are arbitrary constants, and

$$u_1(x)=x^2+1 \tag{3.11}$$

$$u_2(x)=u_1(x)\operatorname{arc\,cot} x-x. \tag{3.12}$$

According to the change of variable (3.7), the inhomogeneous equation (3.4) becomes

$$(x^2+1)\mathscr{A}_0'(x)-2\mathscr{A}_0(x)=\frac{1}{4}\alpha_0^2\Delta_0^2(x^2+a)\mathscr{B}_0(x) \tag{3.13}$$

where the parameter a is

$$a=(4\tilde{q}-\lambda^2)/(4q-\lambda^2). \tag{3.14}$$

If $y_1(x)$ and $y_2(x)$ are the particular integrals of (3.13), we can provide a general solution of the type

$$\mathscr{A}_0(x)=\frac{1}{4}\alpha_0^2\Delta_0^2[c_1y_1(x)+c_2y_2(x)+c_3u_1(x)+c_4u_2(x)] \tag{3.15}$$

where c_3 and c_4 are additional unknown constants. With the assistance of homogeneous solutions we can observe that y_1 and y_2 should be of the form

$$y_1(x)=u_1(x)f_1(x) \tag{3.16}$$

$$y_2(x)=u_2(x)f_1(x)+f_2(x) \tag{3.17}$$

where

$$f_1(x)=\frac{1}{30}\{3u_1(x)+(5a-1)[\ln u_1(x)-2/u_1(x)]\} \tag{3.18}$$

and

$$f_2(x)=\frac{1}{15}\{(5a-1)[x\ln u_1(x)-x/u_1(x)+2(1-\ln 2)x-u_1(x)\mathscr{X}(x)]-6x\}. \tag{3.19}$$

The $\mathscr{X}(x)$ function in (3.19) is a special function which is known as the Clausen's integral [16].

In this case, the corresponding boundary conditions given in Section 2 are

$$\lim_{x\to\infty}\frac{1}{x^2}\mathscr{A}_0(x)=-\frac{1}{8}\mathscr{N}_0\alpha_0^2\Delta_0^2$$

$$\lim_{x\to\infty}\frac{1}{x}\mathscr{A}_0'(x)=-\frac{1}{4}\mathscr{N}_0\alpha_0^2\Delta_0^2$$

$$\mathscr{A}_0(x_0)=\mathscr{A}_0'(x_0)=0 \tag{3.20}$$

which yield

$$c_1=0$$

$$c_2=\mathscr{N}_0W^{-1}\{u_2(x_0),y_2(x_0)\}$$

$$c_3=-\frac{1}{2}\mathscr{N}_0$$

$$c_4=\frac{1}{2}c_2W\{u_1(x_0),y_2(x_0)\} \tag{3.21}$$

where $W\{F(t),G(t)\}$ denotes the Wronskian conventionally defined by

$$W=\det\begin{bmatrix}F(t) & G(t)\\ F'(t) & G'(t)\end{bmatrix}. \tag{3.22}$$

Consequently, the complete approximate solution of the flow problem which exactly satisfies the original boundary conditions (2.5) and (2.6) can be given by means of the stream function

Ψ (3.2) and the related function Φ (3.1). From (2.9), the viscous-drag is

$$D = \frac{1}{2}\alpha_0^2\Delta_0 \int_{x_0}^{\infty} \mathscr{B}_0^2(x)\,dx = \frac{1}{6}\alpha_0\Delta_0\mathcal{N}_0 c_2 I_0(x_0) \quad (3.23)$$

where the integral part of the expression can be evaluated in the form

$$I_0(x_0) = \int_{x_0}^{\infty} u_2^2(x)\,dx = -\frac{1}{15}x_0(3x_0^4 + 10x_0^2 + 15)\operatorname{arc\,cot}^2 x_0 + \frac{2}{15}\left\{u_1(x_0)(3x_0^2+1) - 4\ln\left[\frac{1}{4}u_1(x_0)\right]\right\}\operatorname{arc\,cot} x_0 - \frac{1}{15}x_0(3x_0^2-2) + \frac{8}{15}\mathscr{X}(x_0). \quad (3.24)$$

It is clear that drag is an implicit function of the design parameters except α_0. The shape function (2.12) has the capability of representing all body shapes which have continuous surfaces. Furthermore, the structure of it gives us a flexibility to adjust some or all of the α_i parameters to optimize viscous-drag. As an illustration, we attempt to find the optimum forms with volume specified.

The volume of body expressed in spherical co-ordinates is

$$V = 2\pi \int_0^{\pi} \int_0^{F(\theta)} r^2 \sin\theta\,dr\,d\theta \quad (3.25)$$

which is equivalent to

$$V = \frac{2}{3}\pi \int_{-1}^{1} F^3(\eta)\,d\eta = \frac{2}{3}\pi\alpha_0^3 \int_{-1}^{1}\left(1 + \sum_{i=1}^{\infty}\alpha_i\eta^i\right)^3 d\eta. \quad (3.26)$$

If we normalize the volume to one of a sphere with unit radius, that is $4\pi/3$, α_0 can be written in terms of the other design parameters such as

$$\alpha_0 = 2^{1/3}\Big/\left[\int_{-1}^{1}\left(1 + \sum_{i=1}^{\infty}\alpha_i\eta^i\right)^3 d\eta\right]^{1/3}. \quad (3.27)$$

Naturally, it is supposed for computational purposes that the number of parameters is finite, say M, so

$$F(\eta) = \alpha_0\left(1 + \sum_{i=1}^{M}\alpha_i\eta^i\right). \quad (3.28)$$

In order to obtain minimum drag, we employ the simplex method [17], which optimizes a function of M independent variables, for the evaluation of the optimum values of the design parameters, α_i ($i = 1, 2, \ldots, M$). Results are tabulated in Table 1 for various M, and some of the optimum forms are shown in Fig. 2.

Table 1

M	0 (sphere)	2	4	8	14	20	30	40	50
D_{opt}	1.0000	0.9849	0.9797	0.9751	0.9729	0.9711	0.9696	0.9690	0.9684

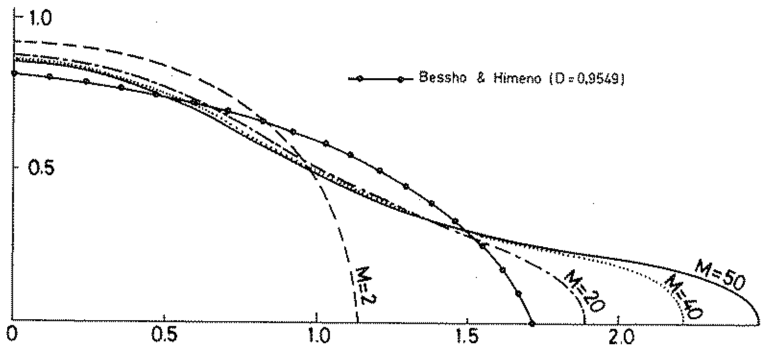

Fig. 2

4. CONCLUSIONS

In this work, we have attempted to introduce the recent analytic and exact method to determine the Stokes flow past axisymmetrical bodies of arbitrary shapes using the basis function expansion of the space of the square integrable functions. The most striking aspect of the theory is the validity for all bodies which have continuous surfaces in the sense of mathematical concepts.

The problem of the streaming flow past a stationary sphere with unit radius, when setting $F = 1$ in the equation (3.28) is a particular case of our presentation which was originally treated and solved by Stokes in 1851. If this well-known problem is taken up for checking purposes then one can easily deduce that the presented theory exactly yields Stokes' solution. This is, of course, a desired property which gives an idea about the validity of the theory.

The structure of the shape function which enables us to optimize a fluid mechanical problem is another useful aspect of the theory. Indeed, the application of the first approximate solution in closed form for obtaining the body shapes with minimum drag yields numerical results which are seen to be consistent with the results in the literature and optimum forms are realistic. It can be shown from Table 1 that the optimum value of drag converges as the number of design parameters increases. Another observation is that the optimum body shapes with constant volume and smallest drag have a centre of symmetry. This result is completely the same as the theoretical results of Pironneau [11].

On the other hand, the optimum drag value for the body where $M = 50$ is larger than the result of Bessho and Himeno [13]. It seems that this causes from using the first approximate solution in this work. The presented theory is therefore promising, and it is hoped that it can be used for the improvement of the actual ship forms at the preliminary design stage in the future works.

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