JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 78 (1997) 233–254

A new approach to the classical Stokes flow problem: Part II Series solutions and higher-order applications

H. Taşeli*, R. Eid

Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey

Received 25 March 1996; revised 25 September 1996

Abstract

This is the second in a series of papers on the Stokes flow past an arbitrary axisymmetrical body. The truncated series solutions of the two infinite systems of simultaneous ordinary differential equations with variable coefficients are obtained for an arbitrary truncation order N. Each series solution together with logarithmic terms is shown to be convergent in the entire physical interval of interest. By the construction of the complete solutions of the systems, the corresponding hydrodynamical problem formulated in terms of the stream function has been solved. As a specimen numerical application, the drag on a prolate spheroid is computed and compared with the exact one. Highly accurate numerical results have been achieved depending on the b/a ratio of the spheroid.

Keywords: Stokes flow, Vector differential equations, Series solutions, Drag

AMS classification: 34A30, 76D07

1. Introduction

In the first paper of this series [5] (hereafter referred to as PI), the original mathematical problem has been reduced to solving the two vector differential equations

$$\sum_{j=0}^{N-1} T_{k,j}X_j(\xi) = 0 \tag{1.1}$$

and

$$\sum_{j=0}^{N-1} T_{k,j}Y_j(\xi) = \alpha_0^2 \sum_{j=0}^{N-1} (\xi^2\delta_{k,j} + \xi A_{k,j} + E_{k,j})X_j(\xi) \tag{1.2}$$

* Corresponding author. E-mail: taseli@rorqual.cc.metu.edu.tr

PII S0377-0427(96)00146-X

for $k = 0, 1, \ldots, N-1$, simultaneously, where the differential operator $T_{k,j}$ with matrix coefficients is of the form

$$T_{k,j} = (\xi^2 \delta_{k,j} + \xi A_{k,j} + B_{k,j}) \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - C_{k,j} \frac{\mathrm{d}}{\mathrm{d}\xi} - D_{k,j}. \tag{1.3}$$

The independent variable ξ lies in the interval $\xi \in [1, \infty)$, and the vector-valued function $\boldsymbol{Y}(\xi)$ in (1.2) obeys the boundary conditions

$$\boldsymbol{Y}(1) = \mathbf{0}, \qquad \boldsymbol{Y}'(1) = \mathbf{0} \tag{1.4}$$

and

$$\lim_{\xi \to \infty} \frac{\boldsymbol{Y}(\xi)}{\xi^2} = -\tfrac{1}{2} \alpha_0^2 \sqrt{\mathcal{N}_0}\, \boldsymbol{e}_1, \qquad \lim_{\xi \to \infty} \frac{\boldsymbol{Y}'(\xi)}{\xi} = -\alpha_0^2 \sqrt{\mathcal{N}_0}\, \boldsymbol{e}_1. \tag{1.5}$$

Notations and definitions of all quantities in the above formulas may be found in PI. It should be noted that N actually tends to infinity so that we have infinite systems of differential equations. However, we deal mainly with the truncated problem in which N is finite.

There is, unfortunately, no general method of finding closed-form solutions of such systems of differential equations, even for a single differential equation with variable coefficients. Therefore, it seems that the only efficient analytic way is to seek solutions in the form of power series.

The linearity of the systems (1.1) and (1.2) is an important advantage since $2N$ linearly independent solutions of the homogeneous system (1.1) are sure to exist from the basic theory. The homogeneous system of equations corresponding to (1.2) is the same as (1.1) so that the crucial step lies in the problem of the construction of the so-called fundamental matrix consisting of those $2N$ solutions. A particular solution of (1.2) may then be determined by the variation of parameters, or otherwise. Hence, we will focus on finding $2N$ linearly independent solutions.

However, the extension of the classical Frobenius power series method to a system of differential equations is not trivial. Up to our knowledge, there is no such a procedure in the literature. In this work, we will proceed analogous to the procedure of obtaining power series solutions of a single differential equation.

In the frame of these introductory remarks the paper is organized as follows: In Section 2, we examine some additional properties of the differential operator $T_{k,j}$ which form a base of constructing convergent power series solutions of the problem. The complete solutions of (1.1) and (1.2) are presented in Sections 3 and 4, respectively. The modification of the method for centrally symmetric bodies is given in Section 5. Section 6 includes a numerical application, and the last section is devoted to a fairly detailed discussion of the results as usual.

2. Singularities of the system

In the classical Frobenius theory the point about which a series solution is proposed gains a lot of importance. As is well known the radius of convergence of a power series solution about a specific point is closely related to the distance between two adjacent singular points in the complex plane of the independent variable. Therefore, the validity and convergence properties of power series solutions completely depends on the singularities of the differential operator $T_{k,j}$ in the ξ-complex plane. In this section, we determine the location of the singular points of the differential operator. To this end, we need some further investigations on the coefficient, $\boldsymbol{L}(\xi) = \xi^2\boldsymbol{I} + \xi\boldsymbol{A} + \boldsymbol{B}$, of the second-order derivative term in the system of equations.

Proposition 1. *The matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $4\boldsymbol{B} - \boldsymbol{A}^2$ are positive definite.*

Proof. Let Q_1,

$$Q_1 = \frac{1}{2\|\boldsymbol{a}\|^2}\boldsymbol{a}^{\mathrm{t}}\boldsymbol{A}\boldsymbol{a}, \tag{2.1}$$

be a quadratic form which is called Rayleigh quotient. Here $\boldsymbol{a}$ is a nonzero vector with real elements such that its norm $\|\boldsymbol{a}\| = (\boldsymbol{a}^{\mathrm{t}}, \boldsymbol{a})^{1/2} < \infty$. By the definition of the matrix $\boldsymbol{A}$ in PI, it follows that Q_1 is expressible as

$$Q_1 = \frac{1}{\|\boldsymbol{a}\|^2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} a_k a_j \int_{-1}^{1} f(\eta)(1-\eta^2)\mathscr{G}_k(\eta)\mathscr{G}_j(\eta)\,\mathrm{d}\eta. \tag{2.2}$$

By inserting

$$\frac{1}{\|\boldsymbol{a}\|}\sum_{k=0}^{\infty} a_k\mathscr{G}_k(\eta) = g(\eta), \tag{2.3}$$

we obtain that

$$Q_1 = \int_{-1}^{1} g^2(\eta)f(\eta)(1-\eta^2)\,\mathrm{d}\eta \tag{2.4}$$

which is always positive since the integrand is zero at only the points $\eta = -1, 0$ and 1 and otherwise positive. Therefore $\boldsymbol{A}$ is positive-definite.

By defining another quadratic form, Q_2 say,

$$Q_2 = \frac{1}{\|\boldsymbol{a}\|^2}\boldsymbol{a}^{\mathrm{t}}\boldsymbol{B}\boldsymbol{a} \tag{2.5}$$

it is similarly observed that

$$Q_2 = \int_{-1}^{1} \{f^2(\eta) + (1-\eta^2)[f'(\eta)]^2\}g^2(\eta)(1-\eta^2)\,\mathrm{d}\eta, \tag{2.6}$$

and hence $\boldsymbol{B}$ is also positive definite since $Q_2 > 0$.

Finally, we find the integral form of the Rayleigh quotient of A^2. If

$$Q_3 = \frac{1}{4\|\boldsymbol{a}\|^2}\boldsymbol{a}^{\mathrm{t}}\boldsymbol{A}^2\boldsymbol{a}, \tag{2.7}$$

then it is written as

$$Q_3 = \sum_{m=0}^{\infty}\int_{-1}^{1}\int_{-1}^{1} f(\eta)g(\eta)f(t)g(t)(1-\eta^2)(1-t^2)\mathscr{G}_m(\eta)\mathscr{G}_m(t)\,\mathrm{d}\eta\,\mathrm{d}t. \tag{2.8}$$

If we now consider the expansion of the product $f(x)g(x)$

$$f(x)g(x) = \sum_{k=0}^{\infty} h_k \mathscr{G}_k(x), \quad x \in [-1, 1] \tag{2.9}$$

in terms of Gegenbauer polynomials, Q_3 is expressible in the form

$$Q_3 = \int_{-1}^{1} f^2(\eta) g^2(\eta)(1 - \eta^2)\, \mathrm{d}\eta. \tag{2.10}$$

Therefore, the quadratic form, Q_4 say,

$$Q_4 = \frac{1}{4\|\boldsymbol{a}\|^2} \boldsymbol{a}^{\mathrm{t}}(4\boldsymbol{B} - \boldsymbol{A}^2)\boldsymbol{a} \tag{2.11}$$

related to the matrix $4\boldsymbol{B} - \boldsymbol{A}^2$ takes the form

$$Q_4 = Q_2 - Q_3 \tag{2.12}$$

which is, subtracting (2.10) from (2.6), equivalent to

$$Q_4 = \int_{-1}^{1} [(1 - \eta^2) f'(\eta) g(\eta)]^2\, \mathrm{d}\eta. \tag{2.13}$$

This completes the proof of the positive definiteness of the matrix $4\boldsymbol{B} - \boldsymbol{A}^2$, since Q_4 is obviously a positive quantity. □

Proposition 2. *The roots of the determinantal equation* $\det \boldsymbol{L}(\xi) = 0$ *have negative real parts.*

Proof. If $\det \boldsymbol{L}(\xi) = 0$, the nullspace of the matrix

$$\boldsymbol{L}(\xi) = \xi^2 \boldsymbol{I} + \xi \boldsymbol{A} + \boldsymbol{B} \tag{2.14}$$

is not empty, and there exists some nonzero vectors $\boldsymbol{u}_n$ satisfiying the equation

$$\boldsymbol{L}(\xi)\boldsymbol{u} = \boldsymbol{0} \tag{2.15}$$

at some specific points, $\xi = \xi_n$ say, in the ξ-complex plane. Premultiplying (2.15) by $(1/\|\boldsymbol{u}_n\|^2)\boldsymbol{u}_n^{\mathrm{t}}$, it is found that

$$\left(\xi + \frac{1}{2\|\boldsymbol{u}_n\|^2} \boldsymbol{u}_n^{\mathrm{t}} \boldsymbol{A} \boldsymbol{u}_n\right)^2 + \frac{1}{4\|\boldsymbol{u}_n\|^2} \boldsymbol{u}_n^{\mathrm{t}} \left(4\boldsymbol{B} - \boldsymbol{A} \frac{\boldsymbol{u}_n \boldsymbol{u}_n^{\mathrm{t}}}{\|\boldsymbol{u}_n\|^2} \boldsymbol{A}\right) \boldsymbol{u}_n = 0. \tag{2.16}$$

Here, although the vectors $\boldsymbol{u}_n$ are restricted as the homogeneous solutions of $\boldsymbol{L}(\xi)$, the collection of them is a subset of the set of arbitrary vectors which are denoted by $\boldsymbol{a}$ in Proposition 1. So, from (2.1), we may write

$$\frac{1}{2\|\boldsymbol{u}_n\|^2} \boldsymbol{u}_n^{\mathrm{t}} \boldsymbol{A} \boldsymbol{u}_n = r_n, \tag{2.17}$$

where r_n is a positive number. On the other hand, a real-symmetric matrix $\boldsymbol{P}$ for which $\boldsymbol{P}^2 = \boldsymbol{P}$ is said to be a projection [3]. In (2.16), it is easy to deduce that the matrix $\boldsymbol{u}_n \boldsymbol{u}_n^{\mathrm{t}} / \|\boldsymbol{u}_n\|^2$ is a projection.

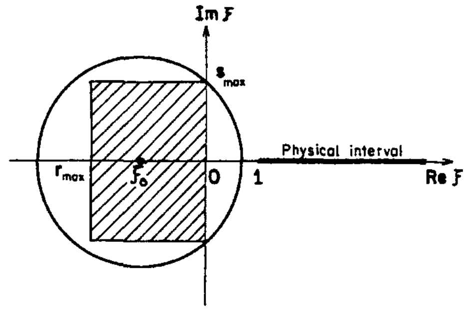

Fig. 1. Location of the singular points.

Therefore, (2.11) and the property that the norm of a projection operator is bounded by one imply the inequality

$$s_n^2 \geqslant \frac{1}{4\|\boldsymbol{u}_n\|^2}\boldsymbol{u}_n^{\mathrm{t}}(4\boldsymbol{B}-\boldsymbol{A}^2)\boldsymbol{u}_n > 0, \tag{2.18}$$

where the substitution

$$s_n^2 = \frac{1}{4\|\boldsymbol{u}_n\|^2}\boldsymbol{u}_n^{\mathrm{t}}\left(4\boldsymbol{B}-\boldsymbol{A}\frac{\boldsymbol{u}_n\boldsymbol{u}_n^{\mathrm{t}}}{\|\boldsymbol{u}_n\|^2}\boldsymbol{A}\right)\boldsymbol{u}_n \tag{2.19}$$

has been made. Hence, (2.16) can be expressed as a quadratic equation

$$(\xi + r_n)^2 + s_n^2 = 0 \tag{2.20}$$

having complex roots, ξ_n,

$$\xi_n = -r_n \pm \mathrm{i}s_n, \quad n = 0, 1, \ldots \tag{2.21}$$

with negative real parts, where i denotes the complex unit number $\sqrt{-1}$. □

The last proposition has very important consequences about the convergence properties of power series solutions. Since r_n and s_n in (2.21) depend on the shape function, finiteness of body implies that

$$\sqrt{r_{\max}^2 + s_{\max}^2} < \infty \tag{2.22}$$

and that all singular points can be enclosed in a circle with finite radius. In such a case a serial expansion at infinity converges in a domain outside of the smallest circle surrounding the singularities according to Abel's theorem [2].

It is very easy to show that "the point at infinity" is a regular singular point of our system of equations. Therefore, series solutions can be constructed in terms of the inverse powers of $\xi - \xi_0$, if we choose a circle centered at an appropriate point ξ_0 on the negative part of the real axis which excludes the point $\xi = 1$ (Fig. 1). Hence, these types of solutions are valid and convergent in the whole physical domain of $\xi \in [1, \infty)$.

We complete this section by introducing a very useful relation

$$(n+1)A_{n,n-1} + C_{n,n-1} = 0, \quad n = 1,2,\ldots \tag{2.23}$$

between subdiagonal entries of the matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ which was shown in [4].

3. Solution of the homogeneous system of equations

Regular singularity of infinity implies that any truncated solution vector $\boldsymbol{X}_k(\xi)$ of order N may be proposed to be of the form

$$\boldsymbol{X}_k(\xi) = \sum_{i=0}^{\infty} \boldsymbol{b}_{k,i}(\xi - \xi_0)^{-i-r_k}, \tag{3.1}$$

where the subscript k indicates the kth solution, and we expect totally $2N$ linearly independentsolutions. For convenience, we introduce the shifted variable x such that

$$x = \xi + \xi_0, \quad x \in [x_0, \infty) \tag{3.2}$$

where $x_0 = 1 + \xi_0$, and ξ_0 is replaced by $-\xi_0$ so that henceforth ξ_0 is regarded to be a positive parameter. In terms of the new variable x, the mathematical problem introduced in Section 1 is completely unaltered in form; only the original matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{E}$ are redefined as

$$\boldsymbol{A} \to -2\xi_0 \boldsymbol{I} + \boldsymbol{A}, \tag{3.3a}$$

$$\boldsymbol{B} \to \xi_0^2 \boldsymbol{I} - \xi_0 \boldsymbol{A} + \boldsymbol{B}, \tag{3.3b}$$

and

$$\boldsymbol{E} \to \xi_0^2 \boldsymbol{I} - \xi_0 \boldsymbol{A} + \boldsymbol{E}, \tag{3.3c}$$

respectively. It is important to note that we have used the same symbols here for symbol economization. Obviously, for $\xi_0 = 0$ we have the original matrices defined previously. Otherwise, the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{E}$, from now on, will denote the matrices defined in (3.3) involving ξ_0 dependent extra terms.

Now the solution (3.1) takes the form

$$\boldsymbol{X}_k(x) = \sum_{i=0}^{\infty} \boldsymbol{b}_{k,i} x^{-i-r_k}, \tag{3.4}$$

and since it is convergent by Proposition 2, $X_k'(x)$ and $X_k''(x)$ may be determined by term-by-term differentiation. Substitution of $X_k(x)$ and its derivatives into Eq. (1.1) leads to

$$\sum_{i=0}^{\infty}\{[(i+r_k)(i+r_k+1)\boldsymbol{I}-\boldsymbol{D}]\boldsymbol{b}_{k,i}+(i+r_k-1)[((i+r_k)\boldsymbol{A}+\boldsymbol{C})\boldsymbol{b}_{k,i-1} + (i+r_k-2)\boldsymbol{B}\boldsymbol{b}_{k,i-2}]\}x^{-i-r_k}=\boldsymbol{0}, \tag{3.5}$$

from which the three-term recurrence equations

$$[(i+r_k)(i+r_k+1)\boldsymbol{I}-\boldsymbol{D}]\boldsymbol{b}_{k,i}+(i+r_k-1)\{[(i+r_k)\boldsymbol{A}+\boldsymbol{C}]\boldsymbol{b}_{k,i-1} + (i+r_k-2)\boldsymbol{B}\boldsymbol{b}_{k,i-2}\}=\boldsymbol{0} \tag{3.6}$$

are obtained for the calculation of the coefficient vectors $\boldsymbol{b}_{k,i}$, for $i\geqslant 0$ where $\boldsymbol{b}_{k,-1}$ and $\boldsymbol{b}_{k,-2}$ are both identically zero by definition. By a direct analogy, we identify the case of $i=0$ as the "indicial equation" which determines the possible values of the r_k in the proposed solution (3.1). Therefore, the indicial equation is of the form

$$\begin{pmatrix} r_k^2+r_k-2 & 0 & \cdots & 0 \\ 0 & r_k^2+r_k-6 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k^2+r_k-N^2-N \end{pmatrix}\boldsymbol{b}_{k,0}=\boldsymbol{0}. \tag{3.7}$$

By choosing the $\boldsymbol{b}_{k,0}$ parallel to the unit vector $\boldsymbol{e}_k$, $2N$ roots of the indicial equation have been found that

$$r_k=k+1, \quad k=0,1,\ldots,N-1 \tag{3.8}$$

and

$$r_k=-(k+2), \quad k=0,1,\ldots,N-1. \tag{3.9}$$

As is known, these roots are called the "exponents of the singularity". Note that equation (3.4) gives $2N$ linearly independent series solutions corresponding to each r_k determined by (3.8) and (3.9) if the coefficient vector $\boldsymbol{b}_{k,i}$ is obtained successfully from the recurrence equation (3.6). However, if $\boldsymbol{b}_{k,i}$ cannot be determined from the recurrence equation (3.6) for any r_k, the corresponding series is then not a solution of the homogeneous system. This is probably the case due to the singularities of the diagonal matrix $[(i+r_k)(i+r_k+1)\boldsymbol{I}-\boldsymbol{D}]$. In such a case, we have to introduce some additional series multiplying by certain powers of $\ln(x)$ according to the theory of differential equations. This can be seen clearly from solving recurrence equations (3.6) for the $\boldsymbol{b}_{k,i}$. Let us first consider the equation with $r_k=k+1$. For $i=0$

$$[(k+1)(k+2)\boldsymbol{I}-\boldsymbol{D}]\boldsymbol{b}_{k,0}=\boldsymbol{0} \tag{3.10}$$

in which the matrix has zero entries on the kth row. The equation has been satisfied by the choice of $\boldsymbol{b}_{k,0} = c_0\boldsymbol{e}_k$, where c_0 is an arbitrary constant. However, for $i = 1$, we have the singular equation

$$[(k+2)(k+3)\boldsymbol{I} - \boldsymbol{D}]\boldsymbol{b}_{k,1} + (k+1)[(k+2)\boldsymbol{A} + \boldsymbol{C}]\boldsymbol{b}_{k,0} = \boldsymbol{0} \tag{3.11}$$

since the $(k+1)$th row of $[(k+2)(k+3)\boldsymbol{I} - \boldsymbol{D}]$ is identically zero. The equation has a solution for $\boldsymbol{b}_{k,1}$ if and only if the $(k+1)$th row of $[(k+2)\boldsymbol{A} + \boldsymbol{C}]\boldsymbol{b}_{k,0}$ is zero. Replacing $\boldsymbol{b}_{k,0}$ by $c_0\boldsymbol{e}_k$, $\boldsymbol{b}_{k,1}$ can be determined subject to the condition that

$$(k+2)A_{k+1,k} + C_{k+1,k} = 0. \tag{3.12}$$

It is obvious from (2.23) that (3.12) is fulfilled so that (3.11) is solvable, and $\boldsymbol{b}_{k,1}$ has the form

$$\boldsymbol{b}_{k,1} = c_0\bar{\bar{\boldsymbol{b}}}_{k,1} + c_1\boldsymbol{e}_{k+1} \tag{3.13}$$

where the components $\bar{\bar{b}}_{k,1,l}$, $0 \leqslant l \leqslant N-1$, of the vector $\bar{\bar{\boldsymbol{b}}}_{k,1}$ are found to be

$$\bar{\bar{b}}_{k,1,l} = -\frac{(k+1)[(k+2)A_{l,k} + C_{l,k}]}{(k+2)(k+3) - (l+1)(l+2)} \tag{3.14}$$

for $l \neq k+1$ and

$$\bar{\bar{b}}_{k,1,k+1} = 0. \tag{3.15}$$

for $l = k+1$. In Eq. (3.13) c_1 is a constant resulting from the equality of $0 = 0$ on the $(k+1)$th row. Now, for $i = 2$, we obtain the equation

$$[(k+3)(k+4)\boldsymbol{I} - \boldsymbol{D}]\boldsymbol{b}_{k,2} + (k+2)\{[(k+3)\boldsymbol{A} + \boldsymbol{C}]\boldsymbol{b}_{k,1} + (k+1)\boldsymbol{B}\boldsymbol{b}_{k,0}\} = \boldsymbol{0}. \tag{3.16}$$

The singularity of $[(k+3)(k+4)\boldsymbol{I} - \boldsymbol{D}]$ is at the $(k+2)$th row, so $\boldsymbol{b}_{k,2}$ can be determined if and only if the $(k+2)$th row of

$$\{[(k+3)\boldsymbol{A} + \boldsymbol{C}]\boldsymbol{b}_{k,1} + (k+1)\boldsymbol{B}\boldsymbol{b}_{k,0}\} \tag{3.17}$$

is identically zero. If we use (2.23) for $n=k+2$ together with (3.13)–(3.15), we can see that (3.17) becomes

$$\{[(k+3)\boldsymbol{A} + \boldsymbol{C}]\bar{\bar{\boldsymbol{b}}}_{k,1}\}_{k+2} + (k+1)B_{k+2,k}, \tag{3.18}$$

where the subscript $k+2$ denotes $(k+2)$th row of vector in $\{\quad\}$. Analytical and numerical inspections show that (3.18) is never zero, i.e., the vector coefficient $\boldsymbol{b}_{k,2}$ is undetermined. Therefore, the $\boldsymbol{b}_{k,i}$ cannot be determined for $i \geqslant 2$ unless $k = N-1$ or $k = N-2$. Actually, for the two roots, $r_{N-1} = N$ and $r_{N-2} = N-1$, $\boldsymbol{b}_{N-1,i}$ and $\boldsymbol{b}_{N-2,i}$ are solvable from (3.6) so that the corresponding series yield the first two linearly independent solutions of the homogeneous system.

On the other hand, for the set of roots $r_k = -(k+2)$, it is not difficult to see that corresponding to the root $r_0 = -2$, it is always possible to obtain a solution which is a polynomial of degree 2. Also for the root $r_1 = -3$, a polynomial of degree 3 may be obtained. But for the other roots we have the same problem of singularities as in the set of roots $r_k = k+1$.

Further inspections indicate that for each of the two set of roots $r_k = k+1$ and $r_k = -(k+2)$, the solutions may be separated into two groups, namely, the groups of even and odd solutions, respectively. To see these solutions in detail, we first consider the set of roots $r_k = k+1$.

3.1. The first set of exponents of the singularity, $r_k = k + 1$

Above considerations clarify that there exists only two solutions in the form of pure power series. By defining the vector function

$$\boldsymbol{F}_m(x) = \sum_{i=0}^{\infty} \boldsymbol{h}_{m,i} x^{-i-N+m}, \quad m = 0, 1, \ldots, N-1, \tag{3.19}$$

the correct form of the other solutions containing logarithmic terms can be expressed into two groups which are [1]

$$\boldsymbol{u}_{2n}(x) = \sum_{k=0}^{n} \gamma_{2n,2k} \boldsymbol{F}_{2k}(x) \ln^{n-k}(x) \tag{3.20}$$

for $n = 0, 1, \ldots, \frac{1}{2}(N-2)$ if N is even, or for $n = 0, 1, \ldots, \frac{1}{2}(N-1)$ if N is odd, and

$$\boldsymbol{u}_{2n+1}(x) = \sum_{k=0}^{n} \gamma_{2n+1,2k+1} \boldsymbol{F}_{2k+1}(x) \ln^{n-k}(x) \tag{3.21}$$

for $n=0, 1, \ldots, \frac{1}{2}(N-2)$ if N is even, or for $n=0, 1, \ldots, \frac{1}{2}(N-3)$ if N is odd. Obviously, $\boldsymbol{u}_0(x) = \boldsymbol{F}_0(x)$ corresponds to pure series solution $\boldsymbol{X}_{N-1}(x)$ with $\boldsymbol{h}_{0,i} \equiv \boldsymbol{b}_{N-1,i}$. Similarly, the first term of the odd group solutions $\boldsymbol{u}_1(x) = \boldsymbol{F}_1(x)$ stands for the other series solution $\boldsymbol{X}_{N-2}(x)$ with $\boldsymbol{h}_{1,i} \equiv \boldsymbol{b}_{N-2,i}$. Hence, we define N solutions by these two groups of functions.

The coefficients $\gamma_{n,k}$ to be determined, are introduced to simplify the calculations in the recursion formulas. In order to show that (3.20) and (3.21) are solutions of the homogeneous system of equations, it is enough to show that (3.20) is a solution. Then, in a similar fashion, by replacing every $2n$ by $2n+1$ and every $2k$ by $2k+1$, it is seen that (3.21) will also be a solution. Now substitution of $\boldsymbol{u}_{2n}(x)$, $\boldsymbol{u}'_{2n}(x)$ and $\boldsymbol{u}''_{2n}(x)$ into (1.1), we have

$$\begin{aligned}\sum_{k=0}^{n} \Bigg\{ & \gamma_{2n,2k} \boldsymbol{T}\boldsymbol{F}_{2k}(x) + (n-k+1) \Bigg[\gamma_{2n,2k-2} \boldsymbol{L}(x) \left(\frac{2\boldsymbol{F}'_{2k-2}(x)}{x} - \frac{\boldsymbol{F}_{2k-2}(x)}{x^2} \right) \\ & - \gamma_{2n,2k-2} \boldsymbol{C} \frac{\boldsymbol{F}_{2k-2}(x)}{x} + (n-k+2) \gamma_{2n,2k-4} \boldsymbol{L}(x) \frac{\boldsymbol{F}_{2k-4}(x)}{x^2} \Bigg] \Bigg\} \ln^{n-k}(x) = \boldsymbol{0}. \end{aligned} \tag{3.22}$$

Inserting the $\boldsymbol{F}_m(x)$ and their derivatives into (3.22), and collecting coefficients of $x^{-i-N+2k} \ln^{n-k}(x)$, we obtain

$$\begin{aligned} \sum_{k=0}^{n} \sum_{i=0}^{\infty} \{ & \gamma_{2n,2k} [(m_0(m_0+1)\boldsymbol{I} - \boldsymbol{D}) \boldsymbol{h}_{2k,i} \\ & + (m_0 - 1)((m_0 \boldsymbol{A} + \boldsymbol{C}) \boldsymbol{h}_{2k,i-1} + (m_0 - 2) \boldsymbol{B} \boldsymbol{h}_{2k,i-2})] \\ & - n_0 [\gamma_{2n,2k-2} ((2m_0 + 1) \boldsymbol{h}_{2k-2,i-2} + ((2m_0 - 1)\boldsymbol{A} + \boldsymbol{C}) \boldsymbol{h}_{2k-2,i-3} \\ & + (2m_0 - 3) \boldsymbol{B} \boldsymbol{h}_{2k-2,i-4}) - (n_0 + 1) \gamma_{2n,2k-4} (\boldsymbol{h}_{2k-4,i-4} \\ & + \boldsymbol{A} \boldsymbol{h}_{2k-4,i-5} + \boldsymbol{B} \boldsymbol{h}_{2k-4,i-6})]\} x^{-i-N+2k} \ln^{n-k}(x) = \boldsymbol{0} \end{aligned} \tag{3.23}$$

in which the integers m_0 and n_0 are

$$m_0 = i + N - 2k, \qquad n_0 = n - k + 1. \tag{3.24}$$

Since the $x^{-i-N+2k} \ln^{n-k}(x)$ are linearly independent, the coefficient term in Eq. (3.23) must vanish so that the $\boldsymbol{h}_{2k,i}$ satisfy the recurrence equations

$$\begin{aligned}
&[m_0(m_0+1)\boldsymbol{I} - \boldsymbol{D}]\boldsymbol{h}_{2k,i} + (m_0-1)[(m_0\boldsymbol{A} + \boldsymbol{C})\boldsymbol{h}_{2k,i-1} + (m_0-2)\boldsymbol{B}\boldsymbol{h}_{2k,i-2}] \\
&\quad = d_{2n,2k-2}\{(2m_0+1)\boldsymbol{h}_{2k-2,i-2} + [(2m_0-1)\boldsymbol{A} + \boldsymbol{C}]\boldsymbol{h}_{2k-2,i-3} \\
&\qquad + (2m_0-3)\boldsymbol{B}\boldsymbol{h}_{2k-2,i-4} - d_{2n,2k-4}(\boldsymbol{h}_{2k-4,i-4} + \boldsymbol{A}\boldsymbol{h}_{2k-4,i-5} + \boldsymbol{B}\boldsymbol{h}_{2k-4,i-6})\}
\end{aligned} \tag{3.25}$$

in which

$$\boldsymbol{h}_{-j,-l} \equiv \boldsymbol{h}_{-j,l} \equiv \boldsymbol{h}_{j,-l} \equiv \boldsymbol{0} \tag{3.26}$$

for any positive integers j and l, where we have defined a new parameter $d_{2n,2k}$ such that

$$d_{2n,2k-2} = \frac{(n-k+1)}{\gamma_{2n,2k}}\gamma_{2n,2k-2}, \quad \gamma_{n,n} = 1. \tag{3.27}$$

Even though the recurrence equations (3.25) seems to make sense only for $k \geqslant 2$, it is in fact valid for all $k \geqslant 0$ since all the vectors $\boldsymbol{h}_{j,l}$ of at least one negative subscript are zero by (3.26). When $k = 0$ the right-hand side of (3.25) becomes zero making the equation a homogeneous one

$$[(i+N)(i+N+1)\boldsymbol{I} - \boldsymbol{D}]\boldsymbol{h}_{0,i} + (i+N-1)\{[(i+N)\boldsymbol{A} + \boldsymbol{C}]\boldsymbol{h}_{0,i-1} + (i+N-2)\boldsymbol{B}\boldsymbol{h}_{0,i-2}\} = \boldsymbol{0} \tag{3.28}$$

which is completely equivalent to solving (3.6) for $\boldsymbol{b}_{N-1,i}$. For $i = 0$

$$[N(N+1)\boldsymbol{I} - \boldsymbol{D}]\boldsymbol{h}_{0,0} = \boldsymbol{0} \tag{3.29}$$

is trivially solvable by

$$\boldsymbol{h}_{0,0} = \boldsymbol{e}_{N-1}. \tag{3.30}$$

Note that Eq. (3.30) does not contain an arbitrary constant because such a constant is considered implicitly in $d_{2n,2k-2}$. On the other hand, for $i \geqslant 1$, the matrix $[(i+N)(i+N+1)\boldsymbol{I} - \boldsymbol{D}]$ becomes nonsingular. Thus, by the formula

$$\boldsymbol{h}_{0,i} = -(i+N-1)[(i+N)(i+N+1)\boldsymbol{I} - \boldsymbol{D}]^{-1}\{[(i+N)\boldsymbol{A} + \boldsymbol{C}]\boldsymbol{h}_{0,i-1} + (i+N-2)\boldsymbol{B}\boldsymbol{h}_{0,i-2}\} \tag{3.31}$$

the vector coefficients $\boldsymbol{h}_{0,i}$ have been completely determined. It is not a problem to find the inverse of the matrix appearing in (3.31) since $[(i+N)(i+N+1)\boldsymbol{I} - \boldsymbol{D}]$ is diagonal.

The vectors $\boldsymbol{h}_{2k,i}$ are obtained for all k and i by solving (3.25) recursively for $k \geqslant 1$ with initial vectors $\boldsymbol{h}_{0,i}$. It is significant to indicate that singularities appearing in each step have been removed by means of an appropriate selection of the parameter $d_{2n,2k}$ in a systematic way. The details of the calculations can be found in [1]. Therefore N linearly independent solutions $\boldsymbol{u}_{2n}(x)$ and $\boldsymbol{u}_{2n+1}(x)$ of the homogeneous system of differential equations have been determined.

3.2. The second set of exponents of the singularity, $r_k = -(k+2)$

Corresponding to the roots $r_k = -(k+2)$, we have the recurrence relation

$$[(i-k-2)(i-k-1)\boldsymbol{I} - \boldsymbol{D}]\boldsymbol{b}_{k,i} + (i-k-3)\{[(i-k-2)\boldsymbol{A} + \boldsymbol{C}]\boldsymbol{b}_{k,i-1} + (i-k-4)\boldsymbol{B}\boldsymbol{b}_{k,i-2}\} = \boldsymbol{0} \tag{3.32}$$

which is solvable for $i=0$ and 1. For $i \geqslant 2$, however, the coefficient vectors cannot be determined unless $k=0$ and 1. Actually, for $k=0$ or $r_0 = -2$, a polynomial of the second degree $\boldsymbol{b}_{0,0}x^2 + \boldsymbol{b}_{0,1}x + \boldsymbol{b}_{0,2}$ may be seen to be a solution of the homogeneous system of equations. Similarly, a polynomial solution of the third degree can be obtained for $r_1 = -3$. The construction of the other solutions, however, is much more complicated comparing with that determined by the first set of exponents of the singularity. This is due to the fact that the matrix $[(i-k-2)(i-k-1)\boldsymbol{I} - \boldsymbol{D}]$ in (3.32) becomes singular in each step of $i=0,1,\ldots,k$; and nonsingular for $i=k+1$ and $i=k+2$, and then singular again in each step of $i=k+3, k+4, \ldots, N+k+2$. To remove the singularities in steps of $i=0,1,\ldots,k$, a logarithmic structure in the solution vectors similar to the previous ones is necessary. But for $i=k+3, k+4, \ldots, N+k+2$ additional logarithmic terms should be introduced. Therefore, we again split these N linearly independent solutions into two groups, define a vector-valued function

$$\boldsymbol{G}_m(x) = \sum_{i=0}^{\infty} \boldsymbol{g}_{m,i} x^{-i+m+2}, \quad m = 0,1,\ldots,N-1 \tag{3.33}$$

and express the solutions in the forms

$$\boldsymbol{v}_{2n}(x) = \sum_{k=0}^{n} \beta_{2n,2k} \boldsymbol{G}_{2k}(x) \ln^{n-k}(x) + \boldsymbol{z}_0(x) + \boldsymbol{z}_2(x) \tag{3.34}$$

for $n=0,1,\ldots,\frac{1}{2}(N-2)$ if N is even, or for $n=0,1,\ldots,\frac{1}{2}(N-1)$ if N is odd, and

$$\boldsymbol{v}_{2n+1}(x) = \sum_{k=0}^{n} \beta_{2n+1,2k+1} \boldsymbol{G}_{2k+1}(x) \ln^{n-k}(x) + \boldsymbol{z}_1(x) + \boldsymbol{z}_3(x) \tag{3.35}$$

for $n=0,1,\ldots,\frac{1}{2}(N-2)$ if N is even, or for $n=0,1,\ldots,\frac{1}{2}(N-3)$ if N is odd. The additional logarithmic structures in the solutions have been presented by means of the functions $\boldsymbol{z}_n(x)$, $n=0,1,2,3$, the definitions of which are

$$\boldsymbol{z}_0(x) = \sum_{k=0}^{n} \sum_{m=0}^{l_0} \mu_{2n,2k,2m+j_0} \boldsymbol{F}_{2m+j_0}(x) \ln^{l_0-m+n-k}(x), \tag{3.36}$$

$$\boldsymbol{z}_1(x) = \sum_{k=0}^{n} \sum_{m=0}^{l_0} \mu_{2n+1,2k+1,2m+j_0} \boldsymbol{F}_{2m+j_0}(x) \ln^{l_0-m+n-k}(x), \tag{3.37}$$

$$\boldsymbol{z}_2(x) = \sum_{k=0}^{n} \sum_{m=0}^{l_1} \mu_{2n,2k,2m+j_1} \boldsymbol{F}_{2m+j_1}(x) \ln^{l_1-m+n-k}(x) \tag{3.38}$$

and

$$\boldsymbol{z}_3(x) = \sum_{k=0}^{n} \sum_{m=0}^{l_1} \mu_{2n+1,2k+1,2m+j_1} \boldsymbol{F}_{2m+j_1}(x) \ln^{l_1-m+n-k}(x), \tag{3.39}$$

respectively. It should be noted that the integers l_0, l_1, j_0 and j_1 are chosen in such a way that $l_0 = l_1 = \frac{1}{2}(N-2)$, $j_0 = 1$ and $j_1 = 0$ if N is even; otherwise, $l_0 = \frac{1}{2}(N-1)$, $l_1 = \frac{1}{2}(N-3)$, $j_0 = 0$ and $j_1 = 1$. Here we have the relations $2l_0 + j_0 = N - 1$ and $2l_1 + j_1 = N - 2$. The parameters $\beta_{n,k}$ and $\mu_{n,k,m}$ will be determined so as to remove the singularities appearing in the recurrence relations of the vector coefficients $\boldsymbol{g}_{m,i}$ in (3.33).

We are not going to give further details about the calculations of those solutions in order not to overfill the content of the paper with a material of lengthy mathematical formulas anymore. The readers may refer [1] for details.

Now we have the occasion to define two $N \times N$ fundamental matrices $\boldsymbol{\Omega}_1(x)$ and $\boldsymbol{\Omega}_2(x)$,

$$\boldsymbol{\Omega}_1(x) = [\boldsymbol{u}_0(x), \boldsymbol{u}_1(x), \ldots, \boldsymbol{u}_{N-1}(x)] \tag{3.40}$$

and

$$\boldsymbol{\Omega}_2(x) = [\boldsymbol{v}_0(x), \boldsymbol{v}_1(x), \ldots, \boldsymbol{v}_{N-1}(x)] \tag{3.41}$$

having the vector solutions $\boldsymbol{u}_n(x)$ and $\boldsymbol{v}_n(x)$ as their columns, respectively. Therefore, the complementary solution $\boldsymbol{X}_c(x)$ of the homogeneous system of differential equations (1.1) can be written in a more neatly form

$$\boldsymbol{X}_c(x) = \boldsymbol{\Omega}_1(x)\boldsymbol{c}_1 + \boldsymbol{\Omega}_2(x)\boldsymbol{c}_2 \tag{3.42}$$

where $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are some arbitrary constant vectors of order N.

4. Solution of the inhomogeneous system of equations

In this section, we consider the inhomogeneous equation (1.2) in vector form

$$\boldsymbol{L}(x)\boldsymbol{Y}''(x) - \boldsymbol{C}\boldsymbol{Y}'(x) - \boldsymbol{D}\boldsymbol{Y}(x) = \boldsymbol{S}(x)\boldsymbol{X}_c(x), \tag{4.1}$$

where

$$\boldsymbol{S}(x) = \alpha_0^2(x^2\boldsymbol{I} + x\boldsymbol{A} + \boldsymbol{E}). \tag{4.2}$$

It is obvious that the complementary solution corresponding to (4.1) is given by

$$\boldsymbol{Y}_c(x) = \boldsymbol{\Omega}_1(x)\boldsymbol{c}_3 + \boldsymbol{\Omega}_2(x)\boldsymbol{c}_4, \tag{4.3}$$

where $\boldsymbol{\Omega}_1(x)$ and $\boldsymbol{\Omega}_2(x)$ are fundamental matrices defined in the previous section while $\boldsymbol{c}_3$ and $\boldsymbol{c}_4$ are additional arbitrary constant vectors of order N. In order to obtain the general solution it remains only to find a particular solution of (4.1). To this end, it is preferable to reduce its order from second to first order. Introducing the $2N \times 1$ vector-valued function $\boldsymbol{W}(x)$ of the form

$$\boldsymbol{W}(x) = \begin{bmatrix} \boldsymbol{Y}(x) \\ \boldsymbol{Y}'(x) \end{bmatrix}, \tag{4.4}$$

the inhomogeneous equation (4.1) can be converted to a $2N \times 2N$ system of first-order ordinary differential equations

$$\boldsymbol{W}'(x) = \boldsymbol{M}(x)\boldsymbol{W}(x) + \boldsymbol{R}(x), \tag{4.5}$$

where the $2N \times 2N$ matrix $\boldsymbol{M}(x)$ and the $2N \times 1$ vector $\boldsymbol{R}(x)$ are defined, respectively, as

$$\boldsymbol{M}(x) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{L}^{-1}(x)\boldsymbol{D} & \boldsymbol{L}^{-1}(x)\boldsymbol{C} \end{bmatrix}, \qquad \boldsymbol{R}(x) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{L}^{-1}(x)\boldsymbol{S}(x)\boldsymbol{X}_c(x) \end{bmatrix}. \tag{4.6}$$

It is clear that the $2N \times 2N$ matrix

$$\boldsymbol{U}(x) = \begin{bmatrix} \boldsymbol{\Omega}_1(x) & \boldsymbol{\Omega}_2(x) \\ \boldsymbol{\Omega}_1'(x) & \boldsymbol{\Omega}_2'(x) \end{bmatrix} = [\boldsymbol{U}_0(x), \boldsymbol{U}_1(x), \ldots, \boldsymbol{U}_{2N-1}(x)] \tag{4.7}$$

is the fundamental matrix solution of the homogeneous equation

$$\boldsymbol{W}'(x) = \boldsymbol{M}(x)\boldsymbol{W}(x) \tag{4.8}$$

corresponding to (4.5). Moreover, we shall use the notion of the homogeneous system adjoint to (4.8) which is given by

$$\boldsymbol{Z}'(x) = -\boldsymbol{M}^{t}\boldsymbol{Z}(x). \tag{4.9}$$

It is well known that if $\boldsymbol{Z}_k(x)$ is a solution of the adjoint system, the relation between $\boldsymbol{Z}_k(x)$ and $\boldsymbol{U}_j(x)$ is

$$\boldsymbol{Z}_k^{t}(x)\boldsymbol{U}_j(x) = \boldsymbol{U}_j^{t}(x)\boldsymbol{Z}_k(x) = \text{constant} \tag{4.10}$$

for all x. Let $\boldsymbol{Q}(x)$ be the fundamental matrix solution of the adjoint system (4.9) defined by

$$\boldsymbol{Q}(x) = [\boldsymbol{Z}_0(x), \boldsymbol{Z}_1(x), \ldots, \boldsymbol{Z}_{2N-1}(x)], \tag{4.11}$$

then

$$\boldsymbol{Q}^{t}(x)\boldsymbol{U}(x) = \mathscr{C} \tag{4.12}$$

for all x, where $\mathscr{C}$ is a $2N \times 2N$ constant matrix.

By the variation of parameters a particular solution of (4.5) can be written in the form

$$\boldsymbol{W}_p(x) = \boldsymbol{U}(x) \int_{x_0}^{x} \boldsymbol{U}^{-1}(t)\boldsymbol{R}(t)\,dt \tag{4.13}$$

where $\boldsymbol{U}^{-1}(x)$ is the inverse of (4.7). To find the inverse of a matrix with variable entries by conventional techniques is not practical which is rather difficult even for small orders. Therefore, in order to evaluate the inverse of the fundamental matrix $\boldsymbol{U}(x)$ we employ the relation given by (4.12). More specifically,

$$\boldsymbol{U}^{-1}(x) = \mathscr{C}^{-1}\boldsymbol{Q}^{t}(x) \tag{4.14}$$

so that the particular solution (4.13) takes the form

$$\boldsymbol{W}_p(x) = \boldsymbol{U}(x)\mathscr{C}^{-1} \int_{x_0}^{x} \boldsymbol{Q}^{t}(t)\boldsymbol{R}(t)\,dt. \tag{4.15}$$

Now the problem is to determine the fundamental matrix $\boldsymbol{Q}(x)$ of the adjoint system. A careful inspection shows that the $2N$ vector solutions in (4.11) are of the form

$$\boldsymbol{Z}_k(x) = \begin{bmatrix} \boldsymbol{D}^{-1}\boldsymbol{H}_k(x) \\ -\boldsymbol{L}(x)\boldsymbol{H}_k'(x) \end{bmatrix}, \tag{4.16}$$

where the vector-valued functions $\boldsymbol{H}_k(x)$ have to be the $2N$ solutions of the homogeneous equation

$$\boldsymbol{L}(x)\boldsymbol{H}''(x) + (2x\boldsymbol{I} + \boldsymbol{A} - \boldsymbol{C})\boldsymbol{H}'(x) - \boldsymbol{D}\boldsymbol{H}(x) = \boldsymbol{0}. \tag{4.17}$$

The method of the previous section is also applicable to solve this equation [1]. Therefore, we may define the two fundamental matrices $\boldsymbol{\Omega}_3(x)$ and $\boldsymbol{\Omega}_4(x)$ of order N, and express the fundamental matrix $\boldsymbol{Q}(x)$ of order $2N$ of the adjoint system (4.9) in the form

$$\boldsymbol{Q}(x) = \begin{bmatrix} \boldsymbol{D}^{-1}\boldsymbol{\Omega}_3(x) & \boldsymbol{D}^{-1}\boldsymbol{\Omega}_4(x) \\ \boldsymbol{L}(x)\boldsymbol{\Omega}_3'(x) & \boldsymbol{L}(x)\boldsymbol{\Omega}_4'(x) \end{bmatrix}. \tag{4.18}$$

The constant matrix $\mathscr{C}$ in (4.12) may be taken as

$$\mathscr{C} = \begin{bmatrix} \mathscr{C}_1 & \mathscr{C}_2 \\ \mathscr{C}_3 & \mathscr{C}_4 \end{bmatrix}, \tag{4.19}$$

where the matrices $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_4$ are expressible as

$$\mathscr{C}_1 = \boldsymbol{\Omega}_3^{\mathrm{t}}(x)\boldsymbol{D}^{-1}\boldsymbol{\Omega}_1(x) - [\boldsymbol{\Omega}_3^{\mathrm{t}}(x)]'\boldsymbol{L}(x)\boldsymbol{\Omega}_1'(x), \tag{4.20a}$$

$$\mathscr{C}_2 = \boldsymbol{\Omega}_3^{\mathrm{t}}(x)\boldsymbol{D}^{-1}\boldsymbol{\Omega}_2(x) - [\boldsymbol{\Omega}_3^{\mathrm{t}}(x)]'\boldsymbol{L}(x)\boldsymbol{\Omega}_2'(x), \tag{4.20b}$$

$$\mathscr{C}_3 = \boldsymbol{\Omega}_4^{\mathrm{t}}(x)\boldsymbol{D}^{-1}\boldsymbol{\Omega}_1(x) - [\boldsymbol{\Omega}_4^{\mathrm{t}}(x)]'\boldsymbol{L}(x)\boldsymbol{\Omega}_1'(x), \tag{4.20c}$$

and

$$\mathscr{C}_4 = \boldsymbol{\Omega}_4^{\mathrm{t}}(x)\boldsymbol{D}^{-1}\boldsymbol{\Omega}_2(x) - [\boldsymbol{\Omega}_4^{\mathrm{t}}(x)]'\boldsymbol{L}(x)\boldsymbol{\Omega}_2'(x) \tag{4.20d}$$

respectively. The inner product in (4.12) is valid for all x so that letting x tends to infinity implies that

$$\mathscr{C}_1 = \boldsymbol{0} \tag{4.21}$$

since the terms which have negative powers of x vanish. Therefore, the constant matrix in (4.19) simplifies into

$$\mathscr{C} = \begin{bmatrix} \boldsymbol{0} & \mathscr{C}_2 \\ \mathscr{C}_3 & \mathscr{C}_4 \end{bmatrix} \tag{4.22}$$

whose inverse is of the form

$$\mathscr{C}^{-1} = \begin{bmatrix} \mathscr{C}_5 & \mathscr{C}_6 \\ \mathscr{C}_7 & \boldsymbol{0} \end{bmatrix} \tag{4.23}$$

in which the matrices $\mathscr{C}_5$, $\mathscr{C}_6$ and $\mathscr{C}_7$ are new matrices related to the matrices $\mathscr{C}_2$, $\mathscr{C}_3$ and $\mathscr{C}_4$.

Now the complete solution of Eq. (4.5) is given by

$$\begin{bmatrix} \boldsymbol{Y}(x) \\ \boldsymbol{Y}'(x) \end{bmatrix} = \boldsymbol{U}(x)\left\{ \begin{bmatrix} \boldsymbol{c}_3 \\ \boldsymbol{c}_4 \end{bmatrix} - \mathscr{C}^{-1}\int_{x_0}^{x} \begin{bmatrix} [\boldsymbol{\Omega}_3^{\mathrm{t}}(t)]'\,\boldsymbol{S}(t)\boldsymbol{X}_{\mathrm{c}}(t) \\ [\boldsymbol{\Omega}_4^{\mathrm{t}}(t)]'\,\boldsymbol{S}(t)\boldsymbol{X}_{\mathrm{c}}(t) \end{bmatrix} \mathrm{d}t \right\}. \tag{4.24}$$

Using the boundary conditions $\boldsymbol{Y}(x_0) = \boldsymbol{Y}'(x_0) = \boldsymbol{0}$, we obtain that

$$\boldsymbol{c}_3 \equiv \boldsymbol{c}_4 \equiv \boldsymbol{0}. \tag{4.25}$$

Hence, the general solution $\boldsymbol{Y}_{\mathrm{g}}(x)$ of (4.1) takes the form

$$\begin{aligned} \boldsymbol{Y}_{\mathrm{g}}(x) = &-\{\boldsymbol{\Omega}_1(x)\mathscr{C}_5 + \boldsymbol{\Omega}_2(x)\mathscr{C}_7\}\int_{x_0}^{x}[\boldsymbol{\Omega}_3^{\mathrm{t}}(t)]'\boldsymbol{S}(t)\boldsymbol{X}_{\mathrm{c}}(t)\,\mathrm{d}t \\ &-\boldsymbol{\Omega}_1(x)\mathscr{C}_6\int_{x_0}^{x}[\boldsymbol{\Omega}_4^{\mathrm{t}}(t)]'\boldsymbol{S}(t)\boldsymbol{X}_{\mathrm{c}}(t)\,\mathrm{d}t. \end{aligned} \tag{4.26}$$

It is clear that $\boldsymbol{X}_{\mathrm{c}}(x)$ in (4.26) contains the arbitrary vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ which are to be determined according to the boundary conditions in (1.5). If we carefully study the leading powers of x dividing by x^2 as $x \to \infty$, the existence of the limit in (1.5a) then implies that $\boldsymbol{c}_2$ must vanish. Therefore, taking $\boldsymbol{c}_2$ as the zero vector reduces the general solution to

$$\begin{aligned} \boldsymbol{Y}_{\mathrm{g}}(x) = -\Big\{&[\boldsymbol{\Omega}_1(x)\mathscr{C}_5 + \boldsymbol{\Omega}_2(x)\mathscr{C}_7]\int_{x_0}^{x}[\boldsymbol{\Omega}_3^{\mathrm{t}}(t)]'\boldsymbol{S}(t)\boldsymbol{\Omega}_1(t)\,\mathrm{d}t \\ &- \boldsymbol{\Omega}_1(x)\mathscr{C}_6\int_{x_0}^{x}[\boldsymbol{\Omega}_4^{\mathrm{t}}(t)]'\boldsymbol{S}(t)\boldsymbol{\Omega}_1(t)\,\mathrm{d}t\Big\}\,\boldsymbol{c}_1 \end{aligned} \tag{4.27}$$

in which the only unkown is the arbitrary constant vector $\boldsymbol{c}_1$. Now employing the boundary condition (1.5b) yields an algebraic system of equations of order N for the determination of $\boldsymbol{c}_1$. Hence, we obtain the required solution of the inhomogeneous system of differential equations (1.2) or (4.1) satisfying the boundary conditions.

5. Modification of the method for centrally symmetric bodies

In this section we provide certain remarks about the modification of the method presented in PI for bodies which are centrally symmetric. For this particular case, the odd indexed terms in the function $f(\eta)$ are identically zero so that we may redefine the shape function, $r = F(\eta)$,

$$F(\eta) = \alpha_0\,[1 + f(\eta)], \quad f(\eta) = \sum_{i=1}^{\infty}\alpha_i\eta^{2i} \tag{5.1}$$

to contain only the even powers of η. Such a symmetry of the shape function implies that $\Phi(\xi,\eta)$ and $\Psi(\xi,\eta)$ cannot include Gegenbauer polynomials with odd indices in their series expansions. Thus,

$$\Phi(\xi,\eta) = (1-\eta^2)\sum_{k=0}^{\infty}X_k(\xi)\mathscr{G}_{2k}(\eta) \tag{5.2}$$

and

$$\Psi(\xi,\eta) = (1-\eta^2)\sum_{k=0}^{\infty} Y_k(\xi)\mathscr{G}_{2k}(\eta). \tag{5.3}$$

Consequently, we obtain the truncated systems of equations similar to (1.1) and (1.2),

$$\sum_{j=0}^{N_e-1} \tilde{T}_{k,j}(\xi)X_j(\xi) = 0 \tag{5.4}$$

and

$$\sum_{j=0}^{N_e-1} \tilde{T}_{k,j}(\xi)Y_j(\xi) = \alpha_0^2 \sum_{j=0}^{N_e-1} (\xi^2\delta_{k,j} + \xi\tilde{A}_{k,j} + \tilde{E}_{k,j})X_j(\xi). \tag{5.5}$$

The matrix operator $\tilde{T}_{k,j}$ is formally the same as the original one,

$$\tilde{T}_{k,j} = (\xi^2\delta_{k,j} + \xi\tilde{A}_{k,j} + \tilde{B}_{k,j})\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - \tilde{C}_{k,j}\frac{\mathrm{d}}{\mathrm{d}\xi} - \tilde{D}_{k,j}, \tag{5.6}$$

however, symmetric matrices $\tilde{\boldsymbol{A}}$, $\tilde{\boldsymbol{B}}$ and $\tilde{\boldsymbol{E}}$ have the new definitions

$$\tilde{A}_{j,k} = 2\sum_{i=1}^{\infty}\alpha_i f_{2i,2j,2k}, \tag{5.7}$$

$$\tilde{B}_{j,k} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_m\alpha_n \left\{(1-4nm)\, f_{2n+2m,2j,2k} + 4nm\, f_{2n+2m-2,2j,2k}\right\} \tag{5.8}$$

and

$$\tilde{E}_{j,k} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_m\alpha_n f_{2n+2m,2j,2k}, \tag{5.9}$$

respectively; the skew-symmetric matrix $\tilde{\boldsymbol{C}}$ is

$$\begin{aligned}\tilde{C}_{j,k} = \sum_{i=1}^{\infty} 4i\alpha_i \Bigg\{ &(j-k) f_{2i,2j,2k} - (j+1)\sqrt{\frac{\mathscr{N}_{2j-1}}{\mathscr{N}_{2j}}}\, f_{2i-1,2j-1,2k} \\ &+(k+1)\sqrt{\frac{\mathscr{N}_{2k-1}}{\mathscr{N}_{2k}}}\, f_{2i-1,2j,2k-1} \Bigg\}\end{aligned} \tag{5.10}$$

and the diagonal matrix $\tilde{\boldsymbol{D}}$ takes the form

$$\tilde{D}_{j,k} = (2k+1)(2k+2)\delta_{j,k}, \tag{5.11}$$

where indices of the above matrices differ from 0 to $N_e - 1$.

The boundary conditions, on the other hand, remain unchanged. Complementary solution of (5.4) can be obtained by a procedure which is slightly different from that of (1.1). Another remark is that we find also the series representation of the particular solution in this case of centrally symmetric bodies [1].

6. A specimen numerical application

In this section, we apply our method to a particular body, namely the prolate spheroid, in order to compute hydrodynamic drag on the body. Of course, the determination of the stream function Ψ allows us to obtain any of hydrodynamic quantities such as velocity or vorticity distribution. However, drag on a body is a much more typical quantity which gives a better idea about the precision of the present method. To this end, we use our N-truncated solutions to calculate the drag on prolate spheroids with various b/a ratios, in a systematic way. Numerical results are reported in Tables 1–7.

From a computational point of view, our method mainly requires the shape parameters of the body as its input data. Obviously, the equation of the body under consideration should be expressed in such a way that it is compatible with our proposed shape function. Therefore, for a prolate spheroid the formulas (4.53) and (4.54) in PI are sufficient to this end.

Furthermore, the order N of the simultaneous vector differential equation (1.1) and (1.2), or, in other words, the dimension N of the truncated vector subspace is an important parameter of the method. The series parts in the solution of the simultaneous system of differential equations and the shape function in the form of a power series are also truncated, and the truncation orders are denoted by L and M, respectively. On the other hand, in the particular case of a centrally symmetric body like a prolate spheroid, the aforementioned truncation orders are denoted by N_e, L_e and M_e, respectively,

Table 1
Drag calculations for the spheroid with $b/a = 0.95$, as a function of the truncation order N, where $\xi_0 = 0$

N	M	L	Drag
1	24	12	0.993 898 819 162 897
3	24	14	0.993 425 823 878 110
5	24	20	0.993 424 774 381 214
7	24	25	0.993 424 772 612 751
9	24	40	0.993 424 772 609 707
11	26	60	0.993 424 772

Table 2
Drag calculations for the spheroid with $b/a = 0.95$, as a function of the truncation order N_e, where $\xi_0 = 0$

N_e	M_e	L_e	Drag
1	12	13	0.993 898 819 162 897
2	12	17	0.993 425 823 878 110
3	12	20	0.993 424 774 381 214
4	12	25	0.993 424 772 612 751
5	12	26	0.993 424 772 609 702
6	13	29	0.993 424 772 609 696
			$D_{\text{exact}} = 0.993\ 424\ 772\ 609\ 696$

Table 3
Accurate drag calculations for the spheroid with $b/a = 0.95$, as a function of the truncation order N_e, where $\xi_0 = 0$

N_e	L_e	Drag
1	13	0.993 9
2	18	0.993 425 8
3	20	0.993 424 774 4
4	24	0.993 424 772 613
5	26	0.993 424 772 609 701
6	28	0.993 424 772 609 696 101
7	35	0.993 424 772 609 696 091 706
8	40	0.993 424 772 609 696 091 680 361
9	43	0.993 424 772 609 696 091 680 301 916
10	50	0.993 424 772 609 696 091 680 301 777 1
11	60	0.993 424 772 609 696 091 680 301 776 818
		D_{exact} = 0.993 424 772 609 696 091 680 301 776 817

Table 4
Accurate drag calculations for the spheroid with $b/a = 0.90$, as a function of the truncation order N_e, where $\xi_0 = 0$

N_e	L_e	Drag
1	20	0.989
2	26	0.987 08
3	35	0.987 065 14
4	43	0.987 065 015
5	51	0.987 065 014 41
6	55	0.987 065 014 404 3
7	60	0.987 065 014 404 223 93
8	64	0.987 065 014 404 223 923 4
9	70	0.987 065 014 404 223 917 895
10	75	0.987 065 014 404 223 917 841 03
11	80	0.987 065 014 404 223 917 840 488
12	85	0.987 065 014 404 223 917 840 482 83
13	90	0.987 065 014 404 223 917 840 482 768 2
14	130	0.987 065 014 404 223 917 840 482 767 620
		D_{exact} = 0.987 065 014 404 223 917 840 482 767 614

in the numerical tables. It is significant to note that $N_e = \frac{1}{2}(N+1)$ and $M_e = \frac{1}{2}M$, which clarifies the computational advantage of introducing the modified method for centrally symmetric bodies.

The software is written in a machine-independent manner. The Fortran program is executed in quadruple precision arithmetic on the IBM-3070 computer system for large values of N and N_e. However, we employed a PC for relatively small truncation orders.

In Tables 1 and 2, the first six successive approximations to the drag of a prolate spheroid, having the ratio $b/a = 0.95$, are given. In the former, the method which is valid for an arbitrary body, centrally symmetric or not, is used. In the latter, however, we consider the modified method which is introduced for a centrally symmetric body, where both homogeneous and particular solutions

Table 5
Accurate drag calculations for the spheroid with $b/a = 0.75$, as a function of the truncation order N_e, where $\xi_0 = 0$

N_e	L_e	Drag
1	30	0.982
2	42	0.970 8
3	45	0.969 977
4	55	0.969 934 5
5	62	0.969 932 31
6	70	0.969 932 185
7	75	0.969 932 177 2
8	85	0.969 932 176 71
9	102	0.969 932 176 678
10	120	0.969 932 176 675 31
11	135	0.969 932 176 675 139
12	150	0.969 932 176 675 126 7
13	164	0.969 932 176 675 125 86
14	170	0.969 932 176 675 125 797
		$D_{\text{exact}} = 0.969\ 932\ 176\ 675\ 125\ 793$

Table 6
Drag calculations for the spheroid with $b/a = 0.60$, as a function of the truncation order N_e

N_e	ξ_0	L_e	Drag
1	0.1	10	0.988
2	0.3	15	0.964 7
3	0.3	26	0.959 04
4	0.4	29	0.958 15
5	0.5	33	0.958 01
6	0.6	43	0.957 992
7	0.7	58	0.957 988 1
8	0.7	68	0.957 987 39
9	0.7	86	0.957 987 258
10	0.8	99	0.957 987 233
11	0.8	110	0.957 987 227 9
12	0.8	125	0.957 987 226 9
13	1.0	180	0.957 987 226 71
			$D_{\text{exact}} = 0.957\ 987\ 226\ 68$

are in the form of infinite power series. In these tables the convergence parameter ξ_0, the number of terms taken from the series appearing in the solution of the system and the number of shape parameters are also included. We observe by means of numerical experiments that the parameter ξ_0 may be taken as zero for prolate spheroids having b/a ratios grater than 0.60.

In Tables 3–7, by using the modified method, highly accurate drag calculations for various prolate spheroids have been presented as a function of N_e. The tables also include L_e and ξ_0 values corresponding to each N_e. Particular comments on the numerical tables are given in the next section.

Table 7
Drag calculations for the spheroid with $b/a = 0.50$, as a function of the truncation order N_e

N_e	ξ_0	L_e	Drag
1	0.1	12	0.999 6
2	0.5	18	0.974 1
3	0.9	24	0.960 59
4	1.0	35	0.956 87
5	1.1	47	0.955 915
6	1.1	59	0.955 666
7	1.3	77	0.955 597
8	1.3	95	0.955 577 4
9	1.3	113	0.955 571 4
10	1.3	135	0.955 569 5
11	1.3	151	0.955 568 89
12	1.3	163	0.955 568 69
13	1.3	187	0.955 568 63
			$D_{exact} = 0.955\ 568\ 60$

7. Discussion and conclusion

In this paper the general solution of a simultaneous system of vector differential equations, driven by Taşeli and Demiralp [5] in their methodological work concerning the hydrodynamical problem of determining Stokes flow past an arbitrary axisymmetric body, has been studied. This linear double infinite system of the second order ordinary differential equations with variable coefficients has been truncated to a finite system of order N. The first system is homogeneous while the second is inhomogeneous with the same homogeneous part as the first system. The solution of the first system appears as a factor on the right-hand side of the second system.

Since the truncated system being considered does not admit solutions in closed form, except for $N = 1$ [5], we obtain its series solutions. For this purpose, the classical Frobenius method used frequently for a single differential equation has been extended to, determining series solutions of systems of equations which have been confronted in our study. Such an extension, in fact, is not very trivial, especially in the case of expanding the solution around a regular singular point. As is well-known, in such a situation solutions may involve certain logarithmic terms. Therefore, the determination of the correct form of the combination of logarithms and infinite power series becomes a difficult and a more elaborate task as the order N of the system increases. From this point of view, the systematic procedure given in Section 3 may be considered as a significant contribution to problems of constructing Frobenius type series solutions for systems of differential equations.

In PI by Taşeli and Demiralp, the methodology has been developed for an arbitrary axisymmetrical body which is not necessarily centrally symmetric. In the case of a centrally symmetric body, however, the shape function does not contain the odd powers of the transformed variable η. As a result of this fact the Gegenbauer polynomials with odd indices in the expansion of the stream function do not make any contributions to the hydrodynamical quantities. For this reason, a modified method for such bodies is also introduced. More specifically, we consider the expansion of the stream function in terms of the $\mathscr{G}_{2k}(\eta)$ polynomials corresponding to the problem of a centrally symmetric body, the boundary shape of which has a power series representation in terms of η^2.

A particular solution to the original system of equations has been obtained by the use of the variation of parameters. In the modified method, alternatively, the series expansion of a particular solution of the resulting vector differential equations is constructed which seems to be computationally more useful. Indeed, numerical results presented in Table 1 and 2 support this argument.

The comparison of the two methods in connection with Tables 1 and 2, including drag calculations for the spheroid with $b/a = 0.95$, shows that the first five consecutive approximations are in a very good agreement. For $N = 11$ (Table 1), however, the general method starts to lose its efficiency due to the numerical difficulties in the calculation of the inverse of the fundamental matrix which appears in the particular solution. Note that the corresponding truncation order to $N = 11$ is $N_e = 6$ in the modified method, and successive approximations are still converging to the exact drag value (Table 2). Furthermore, it is shown from Table 3 that the drag of the same body is calculated accurate to 30 digits by systematically increasing N_e and L_e values. It is seen that a truncated system of equations of order 11, $N_e = 11$, and taking 60 terms, $L_e = 60$, from the series solutions are sufficient to obtain such an extreme accuracy. The general method fails to yield the same accuracy since the inverse of the fundamental matrix becomes an ill-conditioned one as N increases. Actually, we observe that the algorithm is very sensitive to error accumulations when $N = 11$ which explains the numerical problems.

We have seen from Tables 3–7 that the rate of convergence of the modified method falls off rapidly as b/a ratio of the prolate spheroid decreases. More specifically, the required truncation order N_e and the number of terms L_e taken from infinite series solutions are 11 and 60, respectively, in order to reach 30 digits accuracy for the spheroid with $b/a = 0.95$, whereas nearly the same accuracy requires $N_e = 14$ truncation order and $L_e = 130$ terms for the spheroid with $b/a = 0.90$. Furthermore, as b/a ratio decreases the number of terms L_e in the series solutions rapidly increases. To deal with a problem in which the series are beyond a certain size results in an accuracy loss in drag calculations owing to the error accumulations. Therefore, we can present approximately 18, 11 and eight significant figures for the drag values of the prolate spheroids with b/a ratios 0.75, 0.60 and 0.50, respectively.

On the other hand, the very important parameter of the method is the convergence parameter ξ_0. This parameter stands for the center of the circle in which all the singularities of the differential operator $T_{i,j}$ are located. By numerical experiments we have shown that ξ_0 is zero for a spheroid having the ratio greater than 0.60. However, for $b/a \leqslant 0.60$, it is given as a function of N_e in Tables 6 and 7. This is normally the case since the singularities of the system of equations completely depend on the body in question and the truncation order N_e. Moreover, the convergence parameter should have an optimum value depending on the body shape and N_e. As a matter of fact, if ξ_0 is not large enough the circle centered at ξ_0 may include a part of the physical interval $\xi \in [1, \infty)$ to cover all of the singularities of the system. On the contrary, if ξ_0 is larger than an optimum value we consider a very large circle unnecessarily. In such cases, the series solutions in terms of the inverse powers of $(\xi + \xi_0)$ are not guaranteed to converge, and we may face some divergence problems, especially, at the point $\xi = 1$.

A specimen calculation of the optimum ξ_0 value is reported in Table 8. Here, to obtain the same drag values of the prolate spheroid with $b/a = 0.50$ in the truncated subspace $N_e = 3$, the numbers of terms L_e should be taken from the series solutions are presented for various ξ_0 values. It is shown

Table 8
Convergence rate of the method as a function of the parameter ξ_0 for the spheriod with $b/a = 0.50$, where $N_e = 3$

ξ_0	L_e
0.0	divergent
0.3	244
0.4	143
0.5	112
0.6	90
0.7	84
0.9	71
1.0	69
1.1	65
1.2	68
1.3	71
1.6	84
2.0	103
2.5	126
4.0	196
5.0	243

that the series are divergent at least numerically when $\xi_0 = 0$. For $\xi_0 = 0.3$ we find the target by taking $L_e = 244$. L_e values then decrease, and we attain the optimum value of the convergence parameter around $\xi_0 = 1.1$. As a consequence Table 8 supports exactly our intuition about ξ_0.

It should be noted that the high accuracy of the drag presented in the numerical tables is unnecessary in practice. However, such a numerical implementation of the method is of importance in understanding the capability and the limitations of the present method. Furthermore, we have shown numerically the convergence of the method which was discussed in the last section of PI. The method will be more generally applicable to arbitrary bodies if we get rid of some numerical difficulties mentioned above.

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