ELSEVIER

Journal of Computational and Applied Mathematics 78 (1997) 213–232

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

# A new approach to the classical Stokes flow problem: Part I Methodology and first-order analytical results

H. Taşeli [a,*], M. Demiralp [b]

[a] *Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey*

[b] *Department of Engineering Sciences, Istanbul Technical University, 80626 Maslak Istanbul, Turkey*

Received 25 March 1996; revised 25 September 1996

**Abstract**

The problem of determining the axisymmetric Stokes flow past an arbitrary body, the boundary shape of which can be represented by an analytic function, is examined by developing an exact method. An appropriate nonorthogonal coordinate system is introduced, and it is shown that the Hilbert space to which the stream function belongs is spanned by the set of Gegenbauer polynomials based on the physical argument that the drag on a body should be finite. The partial differential equation of the original problem is then reduced to two simultaneous vector differential equations. By the truncation of this infinite-dimensional system to the one-dimensional subspace, an explicit analytic solution to the Stokes equation valid for all bodies in question is obtained as a first approximation.

*Keywords:* Stokes flow; Eigenfunction expansion; Vector differential equations; Drag

*AMS classification:* 76D07, 76M99, 41A30, 65P05

## 1. Introduction

The classical Stokes flow problem describing the creeping motion of a single body without rotation has been studied for more than a century. The determination of the drag on a body is of special importance in many areas of applied sciences. As is well known, the first explicit analytic solutions are due to Stokes [26] and Oberbeck [17] for a sphere and an ellipsoid, respectively. For axisymmetrical flow fields, general solution of Stokes equation in spherical coordinates is given in [23] in terms of the stream function as an infinite series of Legendre polynomials by separation

* Corresponding author. E-mail: taseli@rorqual.cc.metu.edu.tr.


*PII* S0377-0427(96)00141-0

of variables. However, both analytical and numerical implementations of the exact serial solution is a very difficult task for a treatment of the problem of an arbitrary axisymmetrical body except for some special geometries. For instance, Happel and Brenner [13] and Ramkissoon [22] obtained perturbative solutions for flow past slightly deformed spheres. Furthermore, the classical theory has been developed for a few body shapes based mainly upon the use of the separability of a special orthogonal coordinate system. Such solutions were derived for lens-shaped bodies [21] and spherical caps [8].

An alternative approach, which has been especially used in the case of flows in three-dimensional domains to obtain the integral representation of the solution, is the Green's function technique. We may recall several works to study the main concepts of Green's function method and to find detailed discussions on the advantages of the solution in integral form [12, 19, 14, 20, 24, 11]. Since no analytic solutions are at present available, numerical analysis is required for the determination of the problem describing Stokes flow past general bodies. Analytical results can only be obtained when the boundary shapes are simple enough [5, 3, 9, 10].

Various numerical methods using the general integral form of the solution or the truncated series expansion of the stream function have been proposed to approximately solve the problem. A few of them may be referred here. Ladyzhenskaya's [14] general solution was applied in [29] to formulate Stokes flow problem as a system of linear integral equations. They evaluated the unknown density of point forces numerically by reducing the integral equations to a system of linear algebraic equations. Numerical results obtained for general bodies, however, were not always in agreement with experimental data. Lee and Leal [16] proposed a numerical method to analyze the two-dimensional flow by asymptotic matching of the general solutions of Stokes and Oseen equations in integral form. On the other hand, O'brien [18] truncated Sampson's series expansion of the stream function written in spherical coordinates and satisfied approximately the no-slip conditions by employing a boundary collocation method. Bowen and Masliyah [4] also expressed the stream function in terms of Sampson's separable solutions, but they performed a least-squares fitting in order to satisfy the boundary conditions. Although the numerical results were seen to be consistent with the well-known exact analytical ones, i.e., sphere and ellipsoid, the validity of these two works is not clear in mathematical sense.

In this paper, we show how an exact theory can be constructed to determine Stokes flow past an arbitrary axisymmetrical body. The statements of the problem are summarized in Section 2 to clarify the starting point of the present method. The mathematical presentation is given in detail in Section 3 which will be helpful in understanding the method. The formalism is developed by recalling some of the basic properties of the Hilbert space of the problem and the theory of differential equations. With the representation of the body shape by an analytic function including arbitrary number of design parameters, the Stokes equation written for such an arbitrary body is reduced to a system of ordinary differential equations by utilizing the basis function expansion in the Hilbert space of the problem. Sections 3.1 and 3.2 cover the reconstruction of the boundary conditions compatible with the present theory and the sphere problem to show whether the method yields the well-known exact solution of this particular case, respectively. In Section 4, a first approximate solution of the resulting simultaneous vector differential equations is derived in closed form. An application is realized to calculate the drags of various spheroids in Section 4.1 for illustrative purposes. The last section includes a discussion of the method and some concluding remarks.

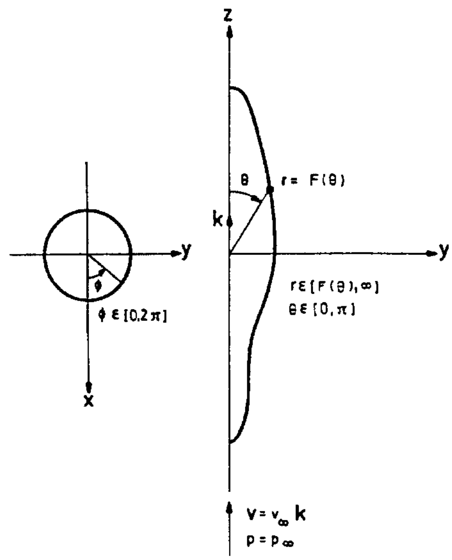


Fig. 1. Axisymmetric Stokes flow past an arbitrary body.

## 2. The basic formulae

Stokes flow problem of the uniform motion of an inertialess unbounded viscous, incompressible fluid past an arbitrary body defined by $r = F(\theta)$ in a domain $\mathscr{D}$ (Fig. 1) is described by

$$\nabla^2 \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\nabla} p(\boldsymbol{x}), \quad \boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathscr{D}. \tag{2.1}$$

The boundary condition on the surface, $r = F(\theta)$, of the body

$$v_i(\boldsymbol{x}) = 0 \tag{2.2}$$

is known as the no-slip condition, where the $v_i$ are velocity components, and the conditions at infinity as $\|\boldsymbol{x}\| \to \infty$ are given by

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{k}, \qquad p(\boldsymbol{x}) = p_\infty \tag{2.3}$$

where $\|\boldsymbol{x}\|$, $p_\infty$ and $\boldsymbol{k}$ denote the norm of the position vector, the pressure at infinity and the unit vector in the $z$-direction, respectively. It is noteworthy that the lengths, velocities and pressures have been nondimensionalized by a characteristic length $l$, $l = (3V/4\pi)^{1/3}$, uniform fluid speed $v_\infty$ and $\mu v_\infty / l$, respectively, where $V$ is the volume of the body and $\mu$ is the dynamic viscosity of fluid.

It is well known that the azimuthal component of the velocity vector is everywhere zero in the case of flow in axisymmetrical domains, and the following velocity field [13]

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\Psi(\boldsymbol{x}) \tag{2.4}$$

satisfies the continuity equation $\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x}) = 0$. Hence, the problem is reduced to a search for a so-called stream function $\Psi(\boldsymbol{x})$, and all flow characteristics can be determined completely by evaluating this scalar function. In spherical coordinates $(r, \theta, \phi)$, the expressions of the velocity components in terms of the stream function are

$$v_r(r,\theta) = -\frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}, \qquad v_\theta(r,\theta) = \frac{1}{r \sin\theta}\frac{\partial \Psi}{\partial r} \tag{2.5}$$

from which the sole component of the vorticity vector is derived as

$$w_\phi(r,\theta) = \frac{1}{r\sin\theta} E^2\Psi, \tag{2.6}$$

where the second-order partial differential operator $E^2$ is defined by [15]

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{2.7}$$

The governing equation (2.1) can now be altered to

$$\nabla^2 w_\phi(r,\theta) = E^4\Psi(r,\theta) = 0. \tag{2.8}$$

Dividing this fourth-order equation into two parts, one obtains the simultaneous partial differential equations of the second order as follows:

$$E^2\Phi(r,\theta) = 0, \qquad E^2\Psi(r,\theta) = \Phi(r,\theta). \tag{2.9}$$

In the case where the body is assumed to be fixed and there is no slip between fluid and boundary, the drag on a body which is the most important flow quantity may be calculated by [6]

$$D = \frac{1}{6\pi}\int_{\mathscr{D}} w_\phi^2 \,\mathrm{d}V = \frac{1}{3}\int_0^\pi \int_{F(\theta)}^\infty \frac{\Phi^2(r,\theta)}{r\sin\theta}\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.10}$$

The drag is nondimensionalized by $6\pi\mu v_\infty l$, so it is obvious that if a sphere with unit radius is under consideration then the characteristic length and drag become unity due to the Stokes' low, according to the aforementioned dimensional analysis. Lastly, it should be noted that the boundary conditions given on the velocity components are

$$v_r(r,\theta) \equiv v_\theta(r,\theta) \equiv 0 \quad \text{at } r = F(\theta), \tag{2.11}$$

$$\lim_{r\to\infty} v_r(r,\theta) = \cos\theta, \qquad \lim_{r\to\infty} v_\theta(r,\theta) = -\sin\theta \tag{2.12}$$

in spherical coordinates.

## 3. Methodology

Let us consider the definite integral expression of the drag. By the introduction of the coordinate transformations,

$$\eta = \cos\theta, \quad \tau = r - F(\eta), \tag{3.1}$$

$$\eta \in [-1, 1], \quad \tau \in [0, \infty), \tag{3.2}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial\theta} = \sqrt{1-\eta^2}\left[F'(\eta)\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\eta}\right], \tag{3.3}$$

the drag can be rewritten in the form

$$D = \frac{1}{3}\int_{-1}^{1}\int_{0}^{\infty} \frac{\Phi^2(\tau,\eta)}{1-\eta^2}\,\mathrm{d}\tau\,\mathrm{d}\eta. \tag{3.4}$$

Now we get rid of the undetermined shape function at the lower limit of the integral over $r$. By (3.1), the operator $E^2$ is changed into $\mathscr{L}^2$ such that

$$\mathscr{L}^2 = \frac{\partial^2}{\partial\tau^2} + \frac{1-\eta^2}{[\tau + F(\eta)]^2}\left[F'(\eta)\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\eta}\right]\left[F'(\eta)\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\eta}\right], \tag{3.5}$$

where the prime stands for the derivative with respect to the independent variable. Hence, the shape function $F(\eta)$ is inserted into the operator, and the mathematical problem is converted to

$$\mathscr{L}^2\Phi(\tau,\eta) = 0, \quad \mathscr{L}^2\Psi(\tau,\eta) = \Phi(\tau,\eta). \tag{3.6}$$

These equations cannot be solved by separation of variables since the pair $(\tau,\eta)$ does not form an orthogonal coordinate system. However, the physically acceptable values of drag have to be finite, so that

$$D < \infty. \tag{3.7}$$

This is an interesting point of view of the treatment of the problem. In what follows, it may be stated that $\Phi$ is in the Hilbert space of the square integrable functions denoted by $L_2(\mathscr{D})$, which is defined by the integral operation under the weight $(1-\eta^2)^{-1}$ over the domain of $\tau$ and $\eta$, according to (3.4). The stream function is also in $L_2$ with respect to the same inner product due to the differential relation (3.6) between $\Phi$ and $\Psi$. Further investigation on this space is, however, outside the scope of the work.

With the general considerations in perspective, both $\Phi$ and $\Psi$ can be written as a linear combination of Gegenbauer or Legendre polynomials, since such functions generate orthonormal basis sets of the $L_2$ space on the interval $-1 \leqslant \eta \leqslant 1$ [2]. The consideration of the eigenvalue problem

$$(1-\eta^2)G''(\eta) = \lambda G(\eta), \qquad G(-1) = G(1) = 0, \tag{3.8}$$

which is related to the last term of the operator $\mathscr{L}^2$, guides us towards the more convenient expansion to choose. Putting

$$2x = 1-\eta, \qquad 1-\eta^2 = 4x(1-x), \quad x \in [0,1] \tag{3.9}$$

and proposing a trial solution of the form

$$G(x) = x(1-x)y(x) \tag{3.10}$$

satisfying the boundary conditions, the problem is reduced to the hypergeometric equation [25]

$$x(1-x)y''(x) + 2(1-2x)y'(x) - (\lambda+2)y(x) = 0. \tag{3.11}$$

A regular solution of (3.11) is the so-called hypergeometric function

$$y(x) = {}_2F_1(-n, n+3, 2; x), \quad n \in \mathbb{Z}^+ \tag{3.12}$$

which can be expressed in terms of Gegenbauer polynomials

$${}_2F_1(-n, n+3, 2; x) = \frac{n!}{(3)_n}\mathscr{C}_n^{(3/2)}(1-2x), \tag{3.13}$$

where $\mathbb{Z}^+$ is the set of natural numbers, $(p)_n$ is Pochhammer symbol and $\mathscr{C}_n^{(\alpha)}$ is the Gegenbauer polynomials of order $n$ and degree $\alpha$ [1]. From now on, we are going to employ $\mathscr{G}_n$ instead of $\mathscr{C}_n^{(3/2)}$ to simplify the notation. If constants are not taken into account, we consequently obtain a general solution

$$G_n(\eta) = (1 - \eta^2)\mathscr{G}_n(\eta) \tag{3.14}$$

for the eigenfunctions and

$$\lambda_n = -(n+1)(n+2) \tag{3.15}$$

for the eigenvalues of the problem (3.8). It is now preferable to expand $\Phi$ and $\Psi$ in a series of Gegenbauer polynomials so as to use the results of the above eigenvalue problem. Therefore, the solutions of (3.6) are most easily found by assuming the following expansions of the form

$$\Phi(\tau,\eta) = (1-\eta^2)\sum_{k=0}^{\infty} X_k(\tau)\mathscr{G}_k(\eta), \tag{3.16}$$

$$\Psi(\tau,\eta) = (1-\eta^2)\sum_{k=0}^{\infty} Y_k(\tau)\mathscr{G}_k(\eta), \tag{3.17}$$

where the $X_k$ and $Y_k$ are solely $\tau$-dependent linear combination coefficients to be determined. Thus, if the homogeneous equation $\mathscr{L}^2\Phi = 0$ is considered in conjunction with (3.16), we have

$$\begin{aligned} \frac{[\tau+F(\eta)]^2}{1-\eta^2}\mathscr{L}^2\Phi(\tau,\eta) = \{[\tau+F(\eta)]^2 + (1-\eta^2)[F'(\eta)]^2\}\sum_{k=0}^{\infty} X_k''(\tau)\mathscr{G}_k(\eta) \\ - \sum_{k=0}^{\infty} X_k'(\tau)\{2F'(\eta)[(1-\eta^2)\mathscr{G}_k(\eta)]' + (1-\eta^2)F''(\eta)\mathscr{G}_k(\eta)\} \\ - \sum_{k=0}^{\infty}(k+1)(k+2)X_k(\tau)\mathscr{G}_k(\eta) = 0 \end{aligned} \tag{3.18}$$

in which the solutions of (3.8) are employed to obtain the last term. The linearly independency of Gegenbauer polynomials implies that it is necessary to express all $\eta$-dependent functions in terms of the consecutive $\mathscr{G}_k$, and resulting $\tau$-dependent coefficients should be equated to zero. To achieve this we need only to propose an explicit form for the equation of surface. The following structure is therefore taken:

$$r = F(\eta) = \alpha_0[1 + f(\eta)], \qquad f(\eta) = \sum_{i=1}^{\infty} \alpha_i \eta^i, \tag{3.19}$$

which is almost equivalent to the Fourier series representation, providing that $f(\eta)$ is analytic. The $\alpha_i$ may be called shape or design parameters. It is obvious that a sphere of radius $\alpha_0$ is under consideration if all $\alpha_i$ with $i \geqslant 1$ vanish. For an arbitrary body, the geometrical conditions imposed on the shape function $f(\eta)$ are those that it should be positive and finite-valued:

$$0 < f(\eta) < \infty \quad \text{for all } \eta \in [-1,1]. \tag{3.20}$$

There are no further restrictions on the shape function. Now, the $\eta^i$ may be transformed into an expansion in terms of Gegenbauer polynomials making use of the expansion of the form

$$\eta^i \mathscr{G}_j(\eta) = \sum_{k=0}^{\infty} f_{i,j,k} \mathscr{G}_k(\eta). \tag{3.21}$$

Multiplying both sides of (3.21) by $(1-\eta^2)\mathscr{G}_m(\eta)$ and integrating from $-1$ to 1, the $f_{i,j,k}$ can be defined as

$$f_{i,j,k} = \int_{-1}^{1} \eta^i (1-\eta^2) \mathscr{G}_j(\eta) \mathscr{G}_k(\eta)\, \mathrm{d}\eta \tag{3.22}$$

due to the orthonormality of the $\mathscr{G}_k(\eta)$, i.e.,

$$\int_{-1}^{1} (1-\eta^2) \mathscr{G}_m(\eta) \mathscr{G}_n(\eta)\, \mathrm{d}\eta = \delta_{m,n}, \tag{3.23}$$

where $\delta_{m,n}$ is the Kronecker delta. Hence, substituting (3.21) into (3.18) and employing the recurrence

$$\frac{2\eta}{\sqrt{\mathscr{N}_k}} \mathscr{G}_k(\eta) = \frac{\sqrt{\mathscr{N}_{k+1}}}{k+2} \mathscr{G}_{k+1}(\eta) + \frac{\sqrt{\mathscr{N}_{k-1}}}{k+1} \mathscr{G}_{k-1}(\eta) \tag{3.24}$$

and differential relations [1]

$$(1-\eta^2)\mathscr{G}_k'(\eta) = (k+2)\sqrt{\frac{\mathscr{N}_{k-1}}{\mathscr{N}_k}}\, \mathscr{G}_{k-1}(\eta) - k\eta \mathscr{G}_k(\eta) \tag{3.25}$$

of the $\mathscr{G}_k(\eta)$, we may define the following matrices:

$$A_{j,k} = 2\sum_{i=1}^{\infty} \alpha_i f_{i,j,k}, \tag{3.26}$$

$$B_{j,k} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha_m \alpha_n [(1-nm) f_{n+m,j,k} + nm f_{n+m-2,j,k}], \tag{3.27}$$

$$C_{j,k} = \sum_{i=1}^{\infty} i\alpha_i \left[ (k-j) f_{i,j,k} - (k+2)\sqrt{\frac{\mathscr{N}_{k-1}}{\mathscr{N}_k}} f_{i-1,j,k-1} + (j+2)\sqrt{\frac{\mathscr{N}_{j-1}}{\mathscr{N}_j}} f_{i-1,k,j-1} \right], \tag{3.28}$$

$$D_{j,k} = (k+1)(k+2)\delta_{j,k}, \tag{3.29}$$

where the normalization factor $\mathscr{N}_k$ is given by

$$\mathscr{N}_k = \frac{2(k+1)(k+2)}{(2k+3)}. \tag{3.30}$$

The homogeneous equation (3.18) can then be written in the form

$$\frac{[\tau+F(\eta)]^2}{1-\eta^2} \mathscr{L}^2 \Phi(\tau,\eta) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \mathscr{G}_j(\eta)\{\alpha_0^2[(\tau/\alpha_0+1)^2 \delta_{j,k} + (\tau/\alpha_0+1)A_{j,k} + B_{j,k}]X_k''(\tau) - \alpha_0 C_{j,k} X_k'(\tau) - D_{j,k} X_k(\tau)\}. \tag{3.31}$$

Now, the linearly independency of the $\mathscr{G}_k(\eta)$ evidently implies that

$$\sum_{k=0}^{\infty}\left[(\xi^2\delta_{j,k}+\xi A_{j,k}+B_{j,k})\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}-C_{j,k}\frac{\mathrm{d}}{\mathrm{d}\xi}-D_{j,k}\right]X_k(\xi)=0,\quad j=0,1,\ldots, \tag{3.32}$$

where, for conciseness, we have changed the variable from $\tau$ to $\xi$,

$$\xi=\frac{\tau}{\alpha_0}+1,\quad \xi\in[1,\infty). \tag{3.33}$$

Finally, we obtain the equation

$$\boldsymbol{TX}(\xi)=\boldsymbol{0} \tag{3.34}$$

in vector-matrix notation for the determination of the vector-valued function $\boldsymbol{X}(\xi)$ whose transpose is formed by means of the functions $X_k(\xi)$, i.e.,

$$\boldsymbol{X}^{\mathrm{t}}(\xi)=[X_0(\xi),X_1(\xi)\ldots,X_k(\xi),\ldots]. \tag{3.35}$$

In (3.34), $\boldsymbol{T}$ is a differential operator,

$$\boldsymbol{T}=(\xi^2\boldsymbol{I}+\xi\boldsymbol{A}+\boldsymbol{B})\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}-\boldsymbol{C}\frac{\mathrm{d}}{\mathrm{d}\xi}-\boldsymbol{D}, \tag{3.36}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ are matrices defined in (3.26)–(3.29), and $\boldsymbol{I}$ stands for the identity matrix.

On the other hand, the inhomogeneous equation $\mathscr{L}^2\Psi=\Phi$ considered in the form

$$\alpha_0^2\frac{[\xi+F(\eta)]^2}{1-\eta^2}\mathscr{L}^2\Psi(\xi,\eta)=\alpha_0^2[\xi+f(\eta)]^2\sum_{k=0}^{\infty}X_k(\xi)\mathscr{G}_k(\eta) \tag{3.37}$$

may be worked out in a similar fashion to find the relation

$$\sum_{j=0}^{\infty}\mathscr{G}_j(\eta)\sum_{k=0}^{\infty}T_{j,k}Y_k(\xi)=\alpha_0^2\sum_{j=0}^{\infty}\mathscr{G}_j(\eta)\sum_{k=0}^{\infty}(\xi^2\delta_{j,k}+\xi A_{j,k}+E_{j,k})X_k(\xi). \tag{3.38}$$

It is readily seen that the left-hand side has been derived analogous to (3.31) on replacing $X_k(\xi)$ by $Y_k(\xi)$, and an additional matrix $E_{j,k}$ on the right-hand side,

$$E_{j,k}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_m\alpha_n f_{n+m,j,k}, \tag{3.39}$$

has been defined for the transformation of $f^2(\eta)$ into an expansion in terms of Gegenbauer polynomials. Now (3.38) leads to

$$\boldsymbol{TY}(\xi)=\alpha_0^2(\xi^2\boldsymbol{I}+\xi\boldsymbol{A}+\boldsymbol{E})\boldsymbol{X}(\xi) \tag{3.40}$$

in vector-matrix form, where the transpose of $\boldsymbol{Y}(\xi)$ is

$$\boldsymbol{Y}^{\mathrm{t}}(\xi)=[Y_0(\xi),Y_1(\xi),\ldots,Y_k(\xi),\ldots]. \tag{3.41}$$

The systems (3.34) and (3.40) are referred to as the vector differential equations which have to be solved simultaneously. The investigation of the general solutions of the systems is left to a future study. We may, however, state that the complete solution will be valid for all bodies provided that their surface equations are given by (3.19). As a matter of fact, changing the variable from $\tau$ to $\xi$, an arbitrary body in question has been transformed to a fixed region in the new fluid domain, and the shape effects are now characterized by the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{E}$ appearing in the operator $\boldsymbol{T}$ and on the right-hand side of (3.40) as matrix-valued coefficients. The determination of these matrices depends on the shape parameters $\alpha_i$. Since $f_{i,j,k}$ is symmetric in the second and third indices, it should be noted that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{E}$ are symmetric; $\boldsymbol{C}$ is skew-symmetric. Furthermore, the last matrix $\boldsymbol{D}$ in $\boldsymbol{T}$ is diagonal. Another remark is that the $f_{i,j,k}$ can be evaluated recursively by utilizing the recurrence relation (3.24) of the $\mathscr{G}_k(\eta)$.

Finally, we may rewrite some of the basic flow quantities by taking into consideration the foregoing new representation of the problem. According to (2.5), the velocity components are

$$v_\xi(\xi,\eta) = \frac{1}{\alpha_0^2[\xi + f(\eta)]^2}\left[\frac{\partial}{\partial\eta} - f'(\eta)\frac{\partial}{\partial\xi}\right]\left[(1-\eta^2)\sum_{k=0}^{\infty} Y_k(\xi)\mathscr{G}_k(\eta)\right], \tag{3.42}$$

$$v_\eta(\xi,\eta) = \frac{\sqrt{1-\eta^2}}{\alpha_0^2[\xi + f(\eta)]}\sum_{k=0}^{\infty} Y_k'(\xi)\mathscr{G}_k(\eta) \tag{3.43}$$

in the $(\xi,\eta)$ coordinates. From (3.4), the drag is expressible as

$$D = \frac{1}{3}\alpha_0\int_1^\infty\int_{-1}^{1}(1-\eta^2)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} X_j(\xi)X_k(\xi)\mathscr{G}_j(\eta)\mathscr{G}_k(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi \tag{3.44}$$

which can be reduced to

$$D = \frac{1}{3}\alpha_0\int_1^\infty\left[\sum_{k=0}^{\infty} X_k^2(\xi)\right]\mathrm{d}\xi \tag{3.45}$$

on recalling again the orthonormality of the $\mathscr{G}_k(\eta)$.

### 3.1. Reconstruction of the boundary conditions

The boundary conditions (2.11) and (2.12) given on the velocity components have to be modified so as to correspond to the vector differential equation (3.40) for the completion of the methodology. On the body surface, or equivalently when $\xi = 1$, the no-slip condition requires that

$$v_\xi(1,\eta) = 0, \qquad v_\eta(1,\eta) = 0. \tag{3.46}$$

It is evident from (3.43) that the condition $v_\eta(1,\eta) = 0$ is fulfilled if the $Y_k'(1)$ are equated to zero for $k \geqslant 0$, i.e.,

$$\boldsymbol{Y}'(1) = \boldsymbol{0} \tag{3.47}$$

since Gegenbauer polynomials are linearly independent. Substitution of (3.47) into (3.42) gives

$$\sum_{k=0}^{\infty} Y_k(1)[(1-\eta^2)\mathscr{G}_k(\eta)]' = 0 \tag{3.48}$$

to satisfy the other condition $v_\xi(1,\eta)=0$. Since (3.48) is identically valid for all $\eta$, $\eta\in[-1,1]$, one can take the derivative with respect to $\eta$ of both sides of the equality and arrive at

$$\sum_{k=0}^{\infty}(k+1)(k+2)Y_k(1)\mathscr{G}_k(\eta)=0 \tag{3.49}$$

by recalling the results, (3.14) and (3.15), of the eigenvalue problem (3.8). Again taking into account the linearly independency of the $\mathscr{G}_k(\eta)$, it may be seen that

$$Y_k(1)=0,\quad k=0,1,2,\ldots \tag{3.50}$$

or

$$\boldsymbol{Y}(1)=\boldsymbol{0}. \tag{3.51}$$

The condition of uniform flow at infinity, on the other hand, requires that

$$\lim_{\xi\to\infty} v_\xi(\xi,\eta)=\eta,\qquad \lim_{\xi\to\infty} v_\eta(\xi,\eta)=-\sqrt{1-\eta^2}. \tag{3.52}$$

According to (3.43), the second condition in (3.52) may be written as

$$\frac{1}{\xi}\sum_{k=0}^{\infty} Y_k'(\xi)\mathscr{G}_k(\eta)\approx-\alpha_0^2\left[1+\frac{1}{\xi}f(\eta)\right] \tag{3.53}$$

for sufficiently large values of $\xi$. Multiplying the left-hand side by $\sqrt{\mathscr{N}_0}\mathscr{G}_0(\eta)$ $(=1)$ and using the relations (3.19), (3.21) and (3.26), the equation takes the form

$$\frac{1}{\xi}\sum_{k=0}^{\infty} Y_k'(\xi)\mathscr{G}_k(\eta)\approx-\frac{1}{2}\alpha_0^2\sqrt{\mathscr{N}_0}\left[2\mathscr{G}_0(\eta)+\frac{1}{\xi}\sum_{k=0}^{\infty}A_{0,k}\mathscr{G}_k(\eta)\right] \tag{3.54}$$

which implies, for $k=0,1,\ldots$, that

$$\frac{1}{\xi}Y_k'(\xi)\approx-\frac{1}{2}\alpha_0^2\sqrt{\mathscr{N}_0}\left(2\delta_{0,k}+\frac{1}{\xi}A_{0,k}\right) \tag{3.55a}$$

or in vector–matrix notation

$$\frac{1}{\xi}\boldsymbol{Y}'(\xi)\approx-\frac{1}{2}\alpha_0^2\sqrt{\mathscr{N}_0}\left(2\boldsymbol{I}+\frac{1}{\xi}\boldsymbol{A}\right)\boldsymbol{e}_1 \tag{3.55b}$$

and that, in the limiting case of $\xi\to\infty$, we must have

$$\lim_{\xi\to\infty}\frac{1}{\xi}Y_k'(\xi)=-\alpha_0^2\sqrt{\mathscr{N}_0}\,\delta_{0,k},\quad k=0,1,\ldots, \tag{3.56}$$

where $\boldsymbol{e}_1$ denotes the unit vector

$$\boldsymbol{e}_1^{\mathrm{t}}=[1,0,\ldots,0,\ldots]. \tag{3.57}$$

Similarly, the first condition in (3.52) may be written from (3.42) as

$$\frac{1}{\xi^2}\left\{\sum_{k=0}^{\infty}Y_k(\xi)[(1-\eta^2)\mathscr{G}_k(\eta)]'-(1-\eta^2)f'(\eta)\sum_{k=0}^{\infty}Y_k'(\xi)\mathscr{G}_k(\eta)\right\}\approx\alpha_0^2\eta\left[1+\frac{1}{\xi}f(\eta)\right]^2 \tag{3.58}$$

for large enough values of $\xi$. Taking again the derivative of both sides with respect to $\eta$ and neglecting the terms which decay proportional to $1/\xi^2$ at infinity, we obtain

$$-(k+1)(k+2)\frac{1}{\xi^2}Y_k(\xi) \approx -\alpha_0^2\sqrt{\mathcal{N}_0}\left[\delta_{0,k} + \frac{1}{\xi}(A_{0,k} - C_{0,k})\right], \quad k = 0, 1, \ldots \tag{3.59a}$$

or

$$\frac{1}{\xi^2}\boldsymbol{Y}(\xi) \approx -\alpha_0^2\sqrt{\mathcal{N}_0}\,\boldsymbol{D}^{-1}\left[\boldsymbol{I} + \frac{1}{\xi}(\boldsymbol{A} - \boldsymbol{C})\right]\boldsymbol{e}_1 \tag{3.59b}$$

which leads to the condition that

$$\lim_{\xi\to\infty}\frac{1}{\xi^2}Y_k(\xi) = -\frac{1}{2}\alpha_0^2\sqrt{\mathcal{N}_0}\,\delta_{0,k}, \quad k = 0, 1, \ldots, \tag{3.60}$$

where the inverse of the diagonal matrix is immediately defined by

$$D_{j,k}^{-1} = \frac{\delta_{j,k}}{(k+1)(k+2)}. \tag{3.61}$$

It should be noted that the definitions of the related quantities and the condition (3.55) have been employed in intermediate steps of the derivation. Therefore, the conditions (3.47), (3.51), (3.56) and (3.60) can be taken as the accompanying boundary conditions of the vector differential equation (3.40).

### 3.2. A particular case: Streaming flow past a sphere

The problem of streaming flow past a stationary sphere when setting $f(\eta) = 0$ is a particular case of our presentation, which was originally treated and solved in [26]. It may be interesting to take up this well-known problem for checking purposes. In this case the matrices, whose elements depend on the shape parameters, are all identical to the zero matrix. So, the operator $\boldsymbol{T}$ is reduced to

$$T_{j,k} = \left[\xi^2\frac{d^2}{d\xi^2} - (k+1)(k+2)\right]\delta_{j,k} \tag{3.62}$$

and the homogeneous equation $\boldsymbol{TX}(\xi) = \boldsymbol{0}$ turns out to be a Cauchy–Euler equation which possesses solutions of the type

$$X_k(\xi) = a_k\xi^{k+2} + b_k\xi^{-(k+1)}, \quad k = 0, 1, 2, \ldots \tag{3.63}$$

The general solution of the inhomogeneous equation $\boldsymbol{TY}(\xi) = \alpha_0^2\xi^2\boldsymbol{X}(\xi)$ then takes the form

$$Y_k(\xi) = \frac{a_k}{2(2k+5)}\xi^{k+4} - \frac{b_k}{2(2k+1)}\xi^{-(k-1)} + c_k\xi^{k+2} + d_k\xi^{-(k+1)}. \tag{3.64}$$

Here, the $a_k$, $b_k$, $c_k$ and $d_k$ are constants, and $\alpha_0$ has been taken as one to consider a sphere with unit radius, for simplicity. The boundary conditions imply that

$$d_k = -\tfrac{1}{4}\sqrt{\mathcal{N}_0}\delta_{k,0}, \qquad c_k = 2d_k, \qquad b_k = 6d_k, \qquad a_k = 0. \tag{3.65}$$

Therefore, all constants vanish except the first ones when $k=0$. Upon substituting these constants into (3.17), the stream function can be terminated to give

$$\Psi(\xi,\eta) = -\tfrac{1}{4}(1-\eta^2)\left(2\xi^2 - 3\xi + \frac{1}{\xi}\right). \tag{3.66}$$

Returning to the original variables in spherical coordinates, it is possible to deduce that this is exactly the same as the Stokes' solution in terms of the dimensionless stream function [13]. Hence, all flow quantities such as the drag can also be obtained exactly by the present formalism.

## 4. Truncated solutions in the one-dimensional subspace

Keeping in mind that $N$ goes to infinity, it may be appropriate to express the vector differential equations (3.34) and (3.40), for $j=0,1,\ldots,N-1$, in the form

$$\sum_{k=0}^{N-1} T_{j,k}X_k(\xi) = 0, \tag{4.1}$$

$$\sum_{k=0}^{N-1} T_{j,k}Y_k(\xi) = \alpha_0^2 \sum_{k=0}^{N-1} (\xi^2\delta_{j,k} + \xi A_{j,k} + E_{j,k})X_k(\xi). \tag{4.2}$$

It is obvious that the best approximations to $\Phi(\xi,\eta)$ and $\Psi(\xi,\eta)$ in the least-squares sense are assumed for a finite $N$. We, therefore, consider the resulting $N$-truncated simultaneous systems of differential equations. Such a truncation will be justified properly in the last section. However, the investigation of the general solution in $N$-dimensional subspace is left to the second part of this work. In this section, we attempt to obtain a first approximate solution in the one-dimensional subspace by letting $N=1$. Hence, $\Phi$ and $\Psi$ can be written, from (3.16) and (3.17), as

$$\Phi(\xi,\eta) = \mathcal{N}_0^{-1/2}(1-\eta^2)X_0(\xi), \tag{4.3}$$

$$\Psi(\xi,\eta) = \mathcal{N}_0^{-1/2}(1-\eta^2)Y_0(\xi), \tag{4.4}$$

where $X_0(\xi)$ and $Y_0(\xi)$ will be determined by solving the following simultaneous equations:

$$T_{0,0}X_0(\xi) = 0, \tag{4.5}$$

$$T_{0,0}Y_0(\xi) = \alpha_0^2(\xi^2\delta_{0,0} + \xi A_{0,0} + E_{0,0})X_0(\xi). \tag{4.6}$$

In this case, the matrix-valued coefficients are reduced to scalar quantities such that

$$\boldsymbol{I}\to\delta_{0,0} = 1, \quad \boldsymbol{D}\to 2\delta_{0,0} = 2, \quad \boldsymbol{C}\to C_{0,0} = 0, \tag{4.7}$$

$$\boldsymbol{A}\to A_{0,0} = a, \quad \boldsymbol{B}\to B_{0,0} = b, \quad \boldsymbol{E}\to E_{0,0} = \tilde{b}, \tag{4.8}$$

and the operator $T_{0,0}$ is

$$T_{0,0} = (\xi^2 + a\xi + b)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - 2. \tag{4.9}$$

The coefficients $a$, $b$ and $\tilde{b}$ are evaluated from the definitions of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{E}$ respectively, as follows:

$$a = \frac{4}{\mathcal{N}_0} \sum_{i=1}^{\infty} \frac{1+(-1)^i}{(i+1)(i+3)} \alpha_i, \tag{4.10}$$

$$b = \tilde{b} + \frac{8}{\mathcal{N}_0} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} ij \frac{1+(-1)^{i+j}}{(i+j-1)(i+j+1)(i+j+3)} \alpha_i \alpha_j, \tag{4.11}$$

$$\tilde{b} = \frac{2}{\mathcal{N}_0} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1+(-1)^{i+j}}{(i+j+1)(i+j+3)} \alpha_i \alpha_j. \tag{4.12}$$

For brevity, if we now transform the variable from $\xi$ to $x$,

$$x = \frac{1}{\Delta_0}(2\xi + a), \quad x \in [x_0, \infty), \tag{4.13}$$

wherein

$$x_0 = \frac{1}{\Delta_0}(2 + a), \qquad \Delta_0 = \sqrt{4b - a^2} \tag{4.14}$$

the differential operator $T_{0,0}$ can be altered to $\mathscr{T}$,

$$\mathscr{T} = (x^2 + 1)\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2 \tag{4.15}$$

and the homogeneous equation $\mathscr{T} X_0(x) = 0$ possesses an obvious solution of the type

$$u_1(x) = x^2 + 1. \tag{4.16}$$

Then it is not difficult to obtain a second linearly independent solution,

$$u_2(x) = u_1(x) \operatorname{arc\,cot} x - x \tag{4.17}$$

by the reduction of order of the differential equation. Therefore, the complementary solution can be written as

$$X_0(x) = c_1 u_1(x) + c_2 u_2(x), \tag{4.18}$$

where $c_1$ and $c_2$ are arbitrary constants.

The inhomogeneous equation, on the other hand, becomes

$$\mathscr{T} Y_0(x) = \tfrac{1}{4} \alpha_0^2 \Delta_0^2 (x^2 + \lambda)[c_1 u_1(x) + c_2 u_2(x)] \tag{4.19}$$

in terms of $x$, where $\lambda$ parameter is

$$\lambda = \frac{4\tilde{b} - a^2}{4b - a^2}. \tag{4.20}$$

If $y_{p_1}(x)$ and $y_{p_2}(x)$ are particular integrals of (4.19), we can provide a general solution of the form

$$Y_0(x) = \tfrac{1}{4} \alpha_0^2 \Delta_0^2 [c_1 y_{p_1}(x) + c_2 y_{p_2}(x) + c_3 u_1(x) + c_4 u_2(x)], \tag{4.21}$$

where $c_3$ and $c_4$ are additional arbitrary constants. The following trial solutions

$$y_{p_1}(x) = u_1(x)y_1(x), \tag{4.22}$$

$$y_{p_2}(x) = u_2(x)y_1(x) + y_2(x), \tag{4.23}$$

may be suggested to satisfy the equations

$$\mathscr{T}y_{p_1}(x)(x) = (x^2 + \lambda)u_1(x), \tag{4.24}$$

$$\mathscr{T}y_{p_2}(x) = (x^2 + \lambda)u_2(x), \tag{4.25}$$

respectively, the integrations of which result in

$$y_1(x) = \frac{1}{30}\left\{3u_1(x) + (5\lambda - 1)\left[\ln u_1(x) - \frac{2}{u_1(x)}\right]\right\}, \tag{4.26}$$

$$y_2(x) = \frac{1}{15}\left\{-6x + (5\lambda - 1)\left[x\ln u_1(x) - \frac{x}{u_1(x)} + 2(1 - \ln 2)x - u_1(x)\mathscr{E}(x)\right]\right\}. \tag{4.27}$$

Note here that the function $\mathscr{E}(x)$ in (4.27) is a special function which cannot be expressed as a finite combination of elementary functions and is known as the Clausen integral [1] defined by

$$\mathscr{E}(x) = -\int_0^{\Theta(x)} \ln\left(2\sin\frac{t}{2}\right)\mathrm{d}t = \sum_{k=1}^{\infty}\frac{1}{k^2}\sin[k\Theta(x)] \tag{4.28}$$

whose derivative is in the elementary form

$$\mathscr{E}'(x) = -\frac{1}{u_1(x)}\ln\left[\frac{u_1(x)}{4}\right], \tag{4.29}$$

where

$$\Theta(x) = 2 \operatorname{arc\,cot} x. \tag{4.30}$$

In the case of the one-dimensional subspace, the corresponding boundary conditions worked out in Section 3.1 are

$$\lim_{x\to\infty}\frac{Y_0(x)}{x^2} = -\tfrac{1}{8}\alpha_0^2\Delta_0^2\sqrt{\mathscr{N}_0}, \tag{4.31}$$

$$\lim_{x\to\infty}\frac{Y_0'(x)}{x} = -\tfrac{1}{4}\alpha_0^2\Delta_0^2\sqrt{\mathscr{N}_0} \tag{4.32}$$

and

$$Y_0(x_0) = Y_0'(x_0) = 0. \tag{4.33}$$

Since the first term of $Y_0(x)$ in (4.21), i.e., $y_{p_1}(x)$, goes to infinity as $x^4$, the first condition requires that $c_1$ must vanish. Without giving the details of the limit operations, the second condition at infinity implies that

$$c_3\lim_{x\to\infty}\frac{u_1'(x)}{x} = -\sqrt{\mathscr{N}_0} \tag{4.34}$$

from which we obtain

$$c_3 = -\tfrac{1}{2}\sqrt{\mathcal{N}_0}. \tag{4.35}$$

The conditions (4.33) yield the following system of algebraic equations:

$$\begin{bmatrix} u_2(x_0) & y_{p_2}(x_0) \\ u_2'(x_0) & y_{p_2}'(x_0) \end{bmatrix} \begin{bmatrix} c_4 \\ c_2 \end{bmatrix} = \frac{1}{2}\sqrt{\mathcal{N}_0} \begin{bmatrix} u_1(x_0) \\ u_1'(x_0) \end{bmatrix} \tag{4.36}$$

for the determination of $c_2$ and $c_4$. Hence, we find that

$$c_2 = \frac{\sqrt{\mathcal{N}_0}}{W\{u_2(x_0), y_{p_2}(x_0)\}}, \tag{4.37}$$

$$c_4 = \tfrac{1}{2} c_2 W\{u_2(x_0), y_{p_2}(x_0)\}, \tag{4.38}$$

where $W\{F(t), G(t)\}$ denotes the Wronskian conventionally defined by

$$W = \det\begin{bmatrix} F(t) & G(t) \\ F'(t) & G'(t) \end{bmatrix}. \tag{4.39}$$

Consequently, substituting the complete form of the $X_0(x)$ and $Y_0(x)$ into (4.3) and (4.4), respectively, the approximate solution of the flow problem which satisfies the original boundary conditions (2.11) and (2.12) is obtained. Therefore, the desired flow quantities may be derived by means of the function $\Phi(x,\eta)$ and the stream function $\Psi(x,\eta)$. As an example, from (3.45) the drag can be written as

$$D = \tfrac{1}{6}\alpha_0 \Delta_0 \int_{x_0}^{\infty} X_0^2(x)\,\mathrm{d}x = \tfrac{1}{6}\alpha_0 \Delta_0 c_2^2 I_0(x_0), \tag{4.40}$$

where $I_0(x_0)$ is

$$\begin{aligned} I_0(x_0) = \int_{x_0}^{\infty} u_2^2(x)\,\mathrm{d}x &= -\tfrac{1}{15}x_0(3x_0^2 - 2) + \tfrac{8}{15}\mathscr{E}(x_0) \\ &+ \tfrac{2}{15}[(3x_0^2 + 1)u_1(x_0) - 4\ln\tfrac{1}{4}\,u_1(x_0)]\ \operatorname{arc\,cot} x_0 \\ &- \tfrac{1}{15}x_0(3x_0^4 + 10x_0^2 + 15)\ \operatorname{arc\,cot}^2 x_0. \end{aligned} \tag{4.41}$$

It is clear that the drag is an implicit function of the design parameters $\alpha_i$, except $\alpha_0$.

### 4.1. An illustrative application: Prolate spheroid

It may be interesting to apply the first approximate solution to a particular body for which the drag is known, in order to test the precision of the present theory. A prolate spheroid is considered to this end (Fig. 2). The dimensionless drag force on a prolate spheroid is given exactly by

$$D_{\text{exact}} = \frac{\tfrac{4}{3}a\beta^2}{\frac{1+\beta^2}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 1}, \tag{4.42}$$

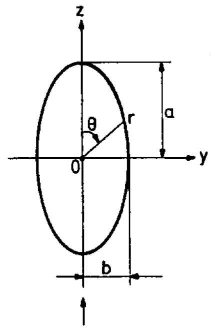


Fig. 2. Stokes flow past a prolate spheroid.

where

$$\beta = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{4.43}$$

in which $a$ is longest of the two semiaxes [13, 7]. According to the aforementioned normalization in Section 2, the volume of the body may be equated to $\frac{4}{3}\pi$ in order to deal with a unit characteristic length, i.e., on putting $l = 1$ we find that

$$a = \left(\frac{b}{a}\right)^{-2/3}. \tag{4.44}$$

This condition also implies that $\alpha_0$ in Eq. (3.19) can be written in terms of the rest shape parameters. If we look at the volume expression in spherical coordinates

$$V = 2\pi \int_0^{\pi} \int_0^{F(\theta)} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \tag{4.45}$$

which is equivalent to

$$V = \tfrac{2}{3}\pi \int_{-1}^{1} F^3(\eta)\,\mathrm{d}\eta = \tfrac{2}{3}\pi\alpha_0^3 \int_{-1}^{1} [1 + f(\eta)]^3 \,\mathrm{d}\eta, \tag{4.46}$$

and set $l = 1$, or $V = \frac{4}{3}\pi$, $\alpha_0$ results in

$$\alpha_0 = (1 + V_1 + V_2 + V_3)^{-1/3}, \tag{4.47}$$

where

$$V_1 = \frac{3}{2} \sum_{i=1}^{\infty} \frac{1 + (-1)^i}{i+1} \alpha_i, \tag{4.48}$$

$$V_2 = \frac{3}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1 + (-1)^{i+j}}{i+j+1} \alpha_i \alpha_j, \tag{4.49}$$

$$V_3 = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1 + (-1)^{i+j+k}}{i+j+k+1} \alpha_i \alpha_j \alpha_k. \tag{4.50}$$

Table 1
The comparison of the drag obtained by employing the first approximation of the present method for a prolate spheroid as a function of the ratio $b/a$

| $\frac{b}{a}$ | $M$ | $D$ | $D_{\text{exact}}$ | Error |
|---|---|---|---|---|
| 1.0 | 0 | 1.0 | 1.0 | 0. |
| 0.95 | 4 | 0.993 899 | 0.993 428 | 0.000 474 |
| 0.9 | 6 | 0.988 966 | 0.987 065 | 0.001 926 |
| 0.75 | 12 | 0.981 789 | 0.969 932 | 0.012 226 |
| 0.5 | 32 | 0.999 548 | 0.955 569 | 0.046 024 |

We now need to derive the expansion of the surface equation of the prolate spheroid

$$r^2\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}\right)=1 \tag{4.51}$$

in terms of $\eta$. This therefore leads to

$$r=\frac{b}{\sqrt{1-\beta^2\eta^2}}=b\left[1+2\sum_{i=1}^{\infty}\frac{\eta^{2i}\beta^{2i}}{2^{2i}}\frac{\Gamma(2i)}{\Gamma(i)\Gamma(i+1)}\right] \tag{4.52}$$

which, by comparison with (3.19), shows that

$$b=\alpha_0, \tag{4.53}$$

$$\alpha_{2i-1}=0, \qquad \alpha_{2i}=2\left(\frac{\beta}{2}\right)^{2i}\frac{\Gamma(2i)}{\Gamma(i)\Gamma(i+1)}, \quad i=1,2,\ldots \tag{4.54}$$

It is evident that the vanishing of the shape parameters with odd indices is due to the central symmetry of the body.

Hence, from (4.40), the first approximate drag of the prolate spheroid can be obtained for a given $b/a$. Results are presented in Table 1 as a function of $b/a$. The $M$ values show how many terms have been taken from the series expansion of the shape function $f(\eta)$. They are determined in such a way that the drag remains constant up to six significant digits. It may be observed that the number $M$ increases because of the relatively slow convergence of $f(\eta)$ in (4.52) as the ratio $b/a$ decreases. The table also includes the exact drag values and relative errors calculated by $(D-D_{\text{exact}})/D_{\text{exact}}$ for comparison.

## 5. Conclusion and a discussion on the convergence of the method

In this work, a new approach has been presented to solve the Stokes equation for a wide class of axisymmetrical bodies of arbitrary shapes. The first approximate drag calculations for flow past prolate spheroids having various $b/a$ ratios yield quite encouraging results which show that the method can satisfactorily be used in physical applications in a simple and concise manner. Actually, it is shown from Table 1 that the qualitative behavior of the solution $[N=1,M]$ is in a good agreement with that of the exact solution in the range of $0.75\leqslant b/a\leqslant 1$. Such an approximation, however, starts to lose its efficiency for $b/a\leqslant 0.5$. So, for a prolate spheroid having a small $b/a$ ratio it is necessary to increase the dimension of the Hilbert subspace $N$ in order to achieve more accuracy.

Giving further particular comments on the method itself may be unnecessary since the mathematical presentation is clear and fairly detailed. However, it is worthwhile to state that the method is closely related to the classical eigenfunction expansion in a spherical coordinate system. Such a classical expansion of the stream function in terms of Legendre or Gegenbauer polynomials, however, does not allow one to find the solution for an arbitrary body. As a result, the validity of our procedure for a wide class of axisymmetrical bodies of arbitrary shapes seems to make a significant contribution to the classical theory. Actually, the embedding of the shape function into the independent variable through a transformation standardizes the scheme. Therefore, the orthogonal expansion over resulting coordinates gives the possibility of taking care of the contributions of the separate terms in the series representation of the shape function in a systematic way.

The following stage of this work is the investigation of general solutions of the resulted system of ordinary differential equations, (4.1) and (4.2), in a finite-dimensional subspace. The construction of the truncated solutions of the vector-valued functions $\boldsymbol{Y}(\xi)$ and $\boldsymbol{X}(\xi)$ of order $N$ will be presented in the second part of this series of papers. Here we shall not give a rigorous proof on the convergence of those solutions as $N \to \infty$, but shall at least make it plausible.

The infinite series expansion of the stream function may be written in the form

$$\Psi(\xi,\eta) = (1-\eta^2)\sum_{k=0}^{\infty} Y_k(\xi)\mathscr{G}_k(\eta) = (1-\eta^2)\sum_{k=0}^{N-1} Y_k(\xi)\mathscr{G}_k(\eta) + (1-\eta^2)^{1/2}R_N(\xi,\eta), \tag{5.1}$$

where $R_N(\xi,\eta)$ is the remainder defined by

$$R(\xi,\eta) = \sum_{k=N}^{\infty} Y_k(\xi)[(1-\eta^2)^{1/2}\mathscr{G}_k(\eta)]. \tag{5.2}$$

If, for a fixed $\xi$, the norm of the remainder,

$$\delta_N(\xi) = \int_{-1}^{1} R_N^2(\xi,\eta)\,\mathrm{d}\eta \geqslant 0 \tag{5.3}$$

is considered, then we have to show that $\delta_N(\xi)$ is bounded for all $\xi \geqslant 1$ saving only the point at infinity. Using the orthonormality of the Gegenbauer polynomials and the following identity:

$$Y_k(\xi) = \int_{-1}^{1} \Psi(\xi,\eta)\mathscr{G}_k(\eta)\,\mathrm{d}\eta, \quad k = 0,1,\ldots, \tag{5.4}$$

we obtain the relation

$$\delta_N(\xi) = \int_{-1}^{1} \Psi^2(\xi,\eta)\frac{\mathrm{d}\eta}{1-\eta^2} - \sum_{k=0}^{N-1} Y_k^2(\xi). \tag{5.5}$$

Since $\delta_N \geqslant 0$, we find the Bessel-like inequality

$$\sum_{k=0}^{N-1} Y_k^2(\xi) \leqslant \int_{-1}^{1} \Psi^2(\xi,\eta)\frac{\mathrm{d}\eta}{1-\eta^2}. \tag{5.6}$$

We may deduce from a physical consideration that the integral on the right-hand side exists. Indeed, it is well known that the stream function vanishes along the symmetry axis of the body when $\theta = 0$ and $\pi$, or at $\eta = \mp 1$ in the transformed domain. For this reason we introduced the weighting factor

$1 - \eta^2$ in the expansion of the stream function. Therefore, the stream function is square integrable with respect to the weight $(1 - \eta^2)^{-1}$ in the domain of $\eta \in [-1, 1]$, resulting a function of $\xi$, $S(\xi)$ say, which is related to the $\xi$-dependence of $\Psi^2(\xi, \eta)$. Hence, $S(\xi)$ is a continuous function of $\xi$ since the differential operator in (3.5) has no singularity when $\tau \geqslant 0$, or equivalently when $\xi \geqslant 1$. As a consequence, each $Y_k(\xi)$ in (5.4) is uniformly convergent from the usual Schwarz's inequality. Furthermore, we have for $1 \leqslant \xi < \infty$

$$\sum_{k=0}^{N-1} Y_k^2(\xi) \leqslant S(\xi) \leqslant m, \tag{5.7}$$

where $m$ is a positive real number. The last inequality implies also that the norm of the remainder $\delta_N$ vanishes as $N \to \infty$. Therefore, we suffer no trouble to consider the truncated problem for computational purposes.

At the left-end point $\xi = 1$, the expansion of $\Psi(\xi, \eta)$ is convergent, and the no-slip condition is fulfilled by the conditions $Y_k(1) = 0$ and $Y_k'(1) = 0$. On the other hand, the situation with the limiting value of $\xi$ as $\xi \to \infty$ is rather different. We need not, and this is not really so, to show that $\Psi(\xi, \eta)$ remains finite as $\xi \to \infty$. Unboundedness of the stream function as $\xi \to \infty$ may also be seen from the Stokes' solution (3.66) of the sphere problem. What we have to show as $\xi \to \infty$ may be derived according to the boundary conditions in (3.52). Employing the expressions (3.42) and (3.43) we see that the stream function obeys the conditions

$$\lim_{\xi \to \infty} \frac{1}{\xi} \frac{\partial \Psi}{\partial \xi} = -\alpha_0^2 (1 - \eta^2) \tag{5.8}$$

and

$$\lim_{\xi \to \infty} \frac{1}{\xi^2} \frac{\partial \Psi}{\partial \eta} = \alpha_0^2 \eta \tag{5.9}$$

for all $\eta$. So the infinite expansion of the stream function in terms of Gegenbauer polynomials is no longer valid as $\xi \to \infty$. However, imposing the condition (3.56) on $Y_k(\xi)$, the expression in (5.7) can be treated by means of our formalism. More specifically, for large enough values of $\xi$ we may write

$$\frac{1}{\xi} \frac{\partial \Psi}{\partial \xi} \equiv (1 - \eta^2) \frac{1}{\xi} Y_0'(\xi) \mathscr{G}_0(\eta) \tag{5.10}$$

which consists of only the first term of the expansion of $\Psi(\xi, \eta)$. Similarly, the condition (3.60) is sufficient to satisfy (5.8). Therefore, our stream function satisfies the conditions at infinity. Moreover, the conditions (3.56) and (3.60) make sure the square integrability of $X_k(\xi)$ in $\xi \in [1, \infty)$ and the existence of the drag given by (3.45), because of the differential relation (3.40) between $X_k(\xi)$ and $Y_k(\xi)$.

Finally, it is noteworthy to indicate that a two-dimensional array of approximants $[N, M]$ will be obtained similar to the Pade table. On the one hand, the dimension of the vector space, i.e., $N$ may be extended up to any desired order of truncation to characterize the exact solution more accurately. On the other hand, the shape function contains infinite number of design parameters. Hence, a specified body can be represented accurately by an appropriate selection of the number of the parameters, say $M$. Another useful aspect of proposing the shape function with such a parametric structure is that it gives us a flexibility to adjust some or all of the parameters in order to optimize a fluid mechanical problem such as the minimization of the drag [27, 28].

## Acknowledgements

The authors are especially grateful to Professor E.S. Şuhubi for his advice and criticism throughout this work.

## References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970) 253, 771 and 997.
[2] V. Barcillon, in: C.E. Pearson, Ed., *Handbook of Applied Mathematics* (Van Nostrand Reinhold, New York, 1974) 344.
[3] G.K. Batchelor, Slender-body theory for particles of arbitrary cross-section in Stokes flow, *J. Fluid Mech.* **44** (1970) 419.
[4] B.D. Bowen and J.H. Masliyah, Drag force on isolated axisymmetric particles in Stokes flow, *Can. J. Chem. Eng.* **51** (1973) 8.
[5] H. Brenner, The Stokes resistance of a slightly deformed sphere, *Chem. Eng. Sci.* **19** (1964) 519.
[6] F. Chorlton, *Textbook of Fluid Dynamics* (Van Nostrand, London, 1967) 343.
[7] A.T. Chwang and T. Yao-Tsu Wu, Hydromechanics of low-Reynolds-number flow. II. Singularity method for Stokes flows, *J. Fluid Mech.* **67** (1975) 787.
[8] W.D. Collins, A note on the axisymmetric Stokes flow past a spherical cap, *Mathematika* **10** (1963) 72.
[9] R.G. Cox, The motion of long slender bodies in a viscous fluid. Part 1. General theory, *J. Fluid Mech.* **44** (1970) 791.
[10] R.G. Cox, The motion of long slender bodies in a viscous fluid. Part 2. Shear flow, *J. Fluid Mech.* **45** (1971) 625.
[11] M.J. Gluckman, S. Weinbaum and R. Pfeffer, Axisymmetric slow viscous flow past an arbitrary convex body of revolution, *J. Fluid Mech.* **55** (1972) 677.
[12] M.D. Greenberg, *Application of Green's Functions in Science and Engineering* (Prentice-Hall, Englewood Cliffs, NJ, 1971).
[13] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics* (Noordhoff, Leyden, 1973) 141, 99, 123 and 155.
[14] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow* (Gordon and Breach, London, 1963).
[15] H. Lamb, *Hydrodynamics* (Cambridge Univ. Press, Cambridge, 1967) 602.
[16] S.H. Lee and L.G. Leal, Low-Reynolds-number flow past cylindrical bodies of arbitrary cross-sectional shape, *J. Fluid Mech.* **164** (1986) 401.
[17] A.J. Oberbeck, Über stationäre Flüssigkeitsbewegungen mit Berücksichtigung der inner Reibung, *J. Reine Angew. Math.* **81** (1876) 62.
[18] V. O'brien, Form factors for deformed spheroids in Stokes flow, *A.I.Ch.E. J.* **14** (1968) 870.
[19] F.K. Odqvist, Über die Randwertaufgaben der Hydrodynamik Zäher Flüssigkeiten, *Math. Z.* **32** (1930) 329.
[20] C.W. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik* (Akademische Verlag, Leipzig, 1927).
[21] L.E. Payne and W.H. Pell, The Stokes flow problem for a class of axially symmetric bodies, *J. Fluid Mech.* **7** (1960) 529.
[22] H. Ramkissoon, Stokes flow past a slightly deformed fluid sphere, *ZAMP* **37** (1986) 859.
[23] R.A. Sampson, On Stokes's current function, *Phil. Trans. A* **182** (1891) 449.
[24] A.M.O. Smith and J.L. Hess, Calculation of potential flow about arbitrary bodies, *Prog. Aero. Sci.* **8** (1966) 1.
[25] I.N. Sneddon, *Special Functions of Mathematical Physics and Chemistry* (Oliver and Boyd, Edinburgh, 1966) 25.
[26] G.G. Stokes, On the effect of the internal friction of fluids on pendulums, *Trans. Camb. Phil. Soc.* **9** (1851) 8.
[27] H. Taşeli, Viscous-drag evaluation via a theoretical method and shape optimization in axisymmetric Stokes flow, Ph.D. Thesis, Istanbul Technical University, 1987.
[28] H. Taşeli, M. Demiralp and K. Kafali, Drag minimization in Stokes flow, *Int. J. Eng. Sci.* **27** (1989) 633.
[29] G.K. Youngren and A. Acrivos, Stokes flow past a particle of arbitrary shape: a numerical method of solution, *J. Fluid Mech.* **69** (1975) 377.