Math 513 Representation Theory of Finite Groups
Fall 2013 HW 1, due Monday Oct. 14

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**1)** Let $R$ be a ring with identity and $e$ in $R$ be an idempotent (i.e. $e^2 + e$ ). *If $e$ is in the center of $R$ , it is called a central idempotent. If $e$ can be writte as sum of two idempotents, it is called a primitive idempotent. If $e, f$ in $R$ are idempotents with $ef = 0$ , they are called orthogonal.*

Show that

**i)** sum of two idempotents is an idempotent.

**ii)** sum of two central idempotents is a central idempotent.

**iii)** sum of two primitive idempotents is a primitive idempotent.

**iv)** sum of two primitive central idempotents is an orthogonal idempotent.

**v)** for an idempotent $e$ in $R$ , $R = Re$ if and only if $ae = a$ for all $a$ .

**vi)** $Ae \neq I + J$ for every right ideal $J, J$ of $R$ if and only if $e$ is not primitive.

**2)** Let $A := \mathbf{C}[G]$ , $\eta_G = \sum_{g \in G} g$ .

**i)** If $\mathrm{char}F \not| |G|$ , then $\eta_G$ is an idempotent.

**ii)** If $\mathrm{char}F||G|$ , then $\eta_G$ is nilpotent and hence the trivial submodule $F\eta_G$ has no complentary submodule.

**3)** Let $A = F[x]$ , $V$ be a vector space over $F$ . Show that $\rho : A \longrightarrow E$ given by

$$\rho(f(x))(T) = T \cdot f(x) \text{ where } T \cdot f(x) := f(T) \text{ for } T \in \mathrm{End}_F(V)$$

is a representation (i.e. $V$ is an $F[x]$ -module).

**4)** Show that for an $F$ - algebra $A$

**i)** $V$ is a cyclic $A$ -module if and only if $V \cong A/I$ where $I$ is a right ideal of $A$ .

**ii)** $V$ is an irreducible $A$ -module if and only if all non-zero vectors of $A$ is cyclic.

**iii)** Give an example (and verify) of an indecomposable module which is not cyclic.

**5)** Show the following. Let $A := \mathbf{C}[G]$ and,

**i)** $\rho : A \longrightarrow \mathbf{C}$ be a representation. $G/\ker\rho$ is abelian.

**ii)** $V$ be given by a representation $\rho : A \longrightarrow GL_n(\mathbf{C})$ . Assume that there are $g$ , $h$ in $G$ such that $\rho(g)$ does not commute with $\rho(h)$ . Show that $M$ is irreducible.

**iii)** $V$ be given by a representation $\rho : A \longrightarrow GL_n(\mathbf{C})$ . Assume that there are $g$ , $h$ in $G$ such that $\rho(g)$ does not commute with $\rho(h)$ . Show that $M$ is irreducible if and only if for every matrix $X$ over $\mathbf{C}$ satisfying $A\eta_G(g) = \eta_G(g)A$ for all $g$ in $G$ , there exists $\lambda_A$ in $\mathbf{C}$ with $\lambda I_n = X$ .

**4)** Show that converse of Schur's Lemma hold for a $\mathbf{C}[G]$ -module $M$ and give an example for which it fails (and verify).

**5)** Show that for an $F$ - algebra $A$ , the product $[\,,\,]$ defined by $[a, b] := ab - ba$ the following hold;

**i)** $[\,,\,]$ is bilinear and skew-symmetric hence $[a, a] = 0$ for all $a$, $b$ in $A$ .

**ii)** $[\,,\,]$ satisfies the Jacobi identity $[[a, b], c] + [[b, c]a, ] + [[c, a], b] = 0$ .

**iii)** Find the reason that $\mathbf{R}^3$ with $\times$ -product in is not an algebra but rather it satisfies the property given above for $[\,,\,]$ .

**6)** Show that for an $F$ - algebra $A$ , $\mathrm{End}_A(A^\circ) \cong A^{op}$ or ( $\cong A$ depending on what ? explain ) as rings.

**7)** Find at least three non-trivial conditions for $G$ to be abelian using that $G$ has a representation $G \longrightarrow GL_n(\mathbf{C})$ with image in the center of $GL_n(\mathbf{C})$ .