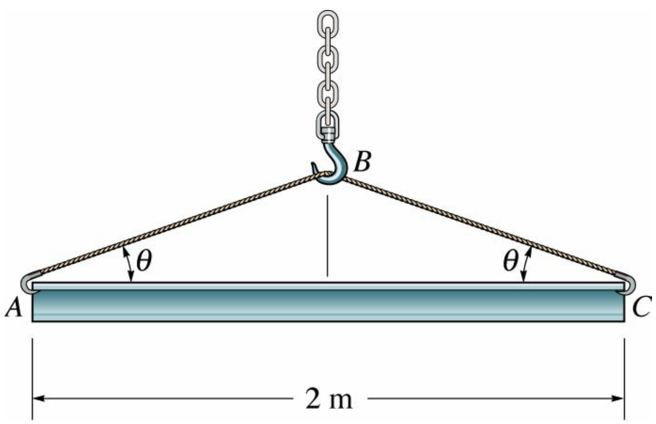
Forces in Structures and Machines

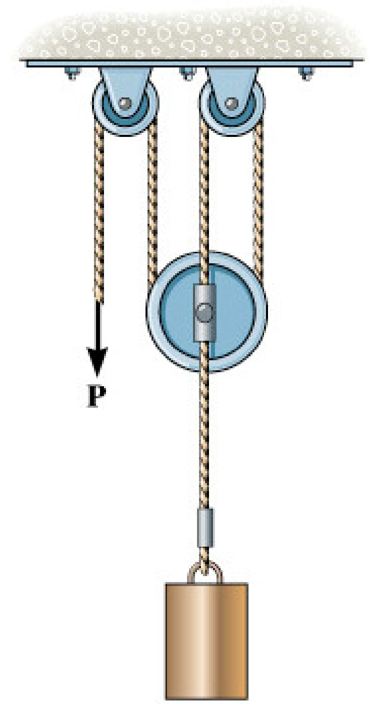
Resultant Force

Components of a Force

Moment of a Force

Equilibrium of Forces and Moments



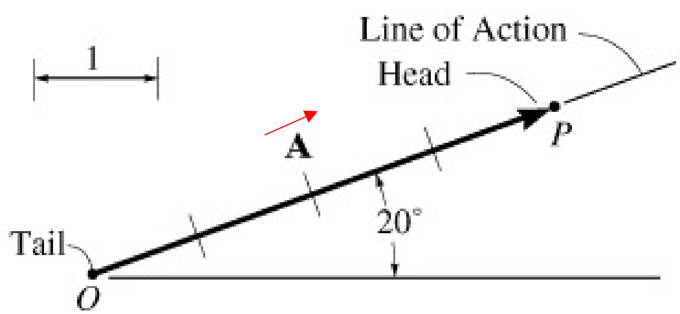


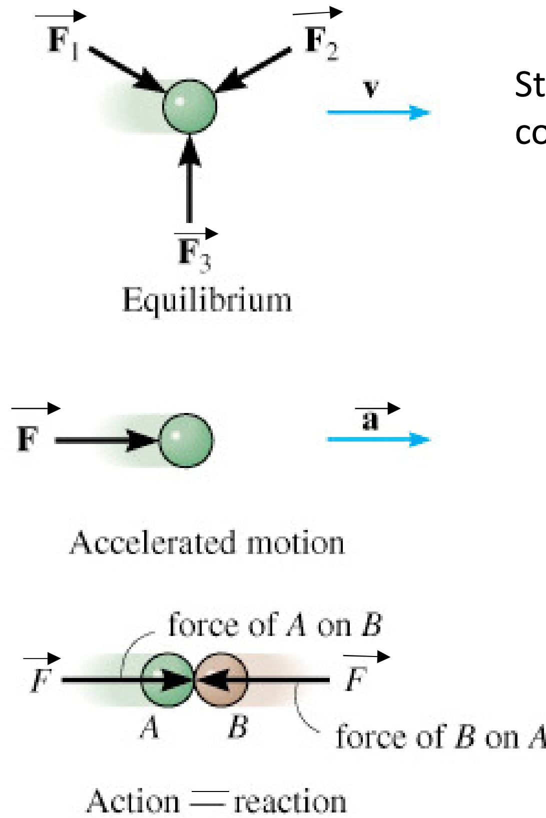
A force represents the action of one body on another and is characterized by its point of application, its magnitude and its direction. The direction of a force is defined by the line of action and the sense of the force.

Force is a vector.

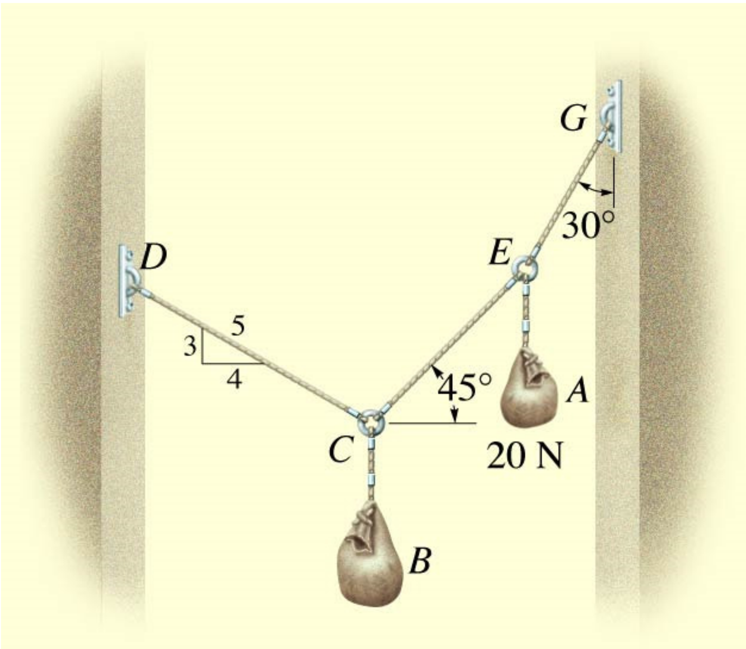
Vectors are defined as mathematical expressions possessing magnitude and direction, which add according to the «parallelogram law».

Vectors are represented by arrows in the figures, illustrations. In a text a vectoral quantity is written by drawing a short arrow above the letter used to represent it.

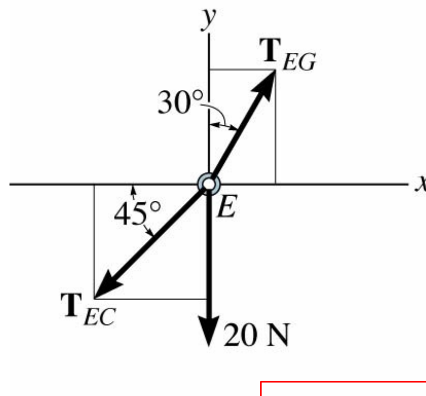




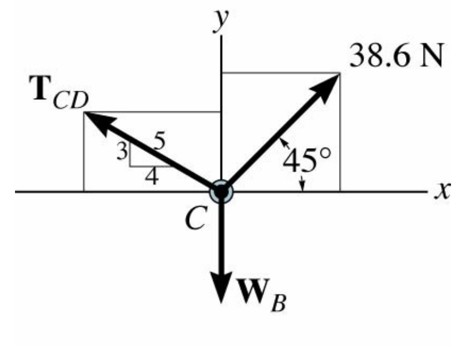
Stationary or moving with a constant speed along a linear path


G
30°
D
E
5
3
4
45°
A
C
20 N
B

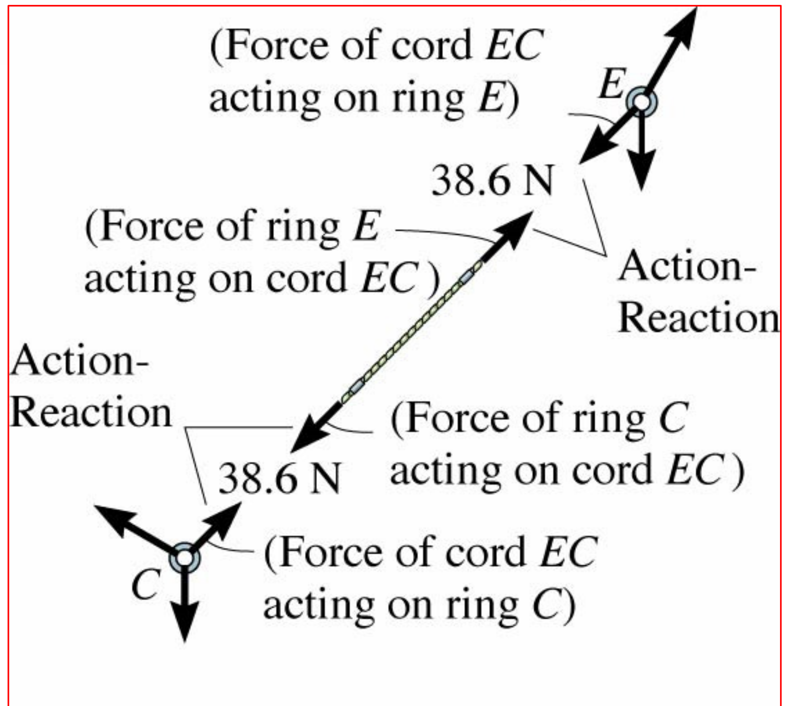
(a)


y
$\mathbf{T}_{EG}$
30°
x
45°
E
$\mathbf{T}_{EC}$
20 N

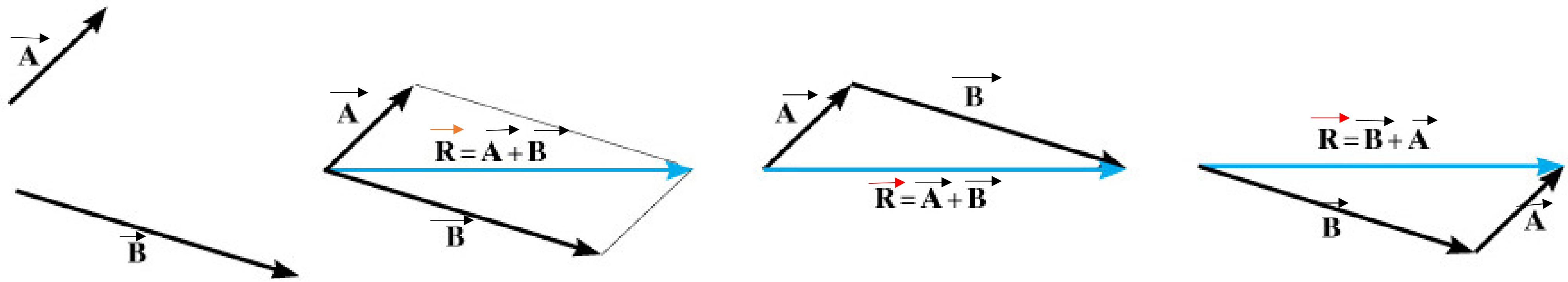
(b)


y
38.6 N
$\mathbf{T}_{CD}$
5
3
4
45°
x
C
$\mathbf{W}_B$

(c)


(Force of cord *EC* acting on ring *E*)
E
38.6 N
(Force of ring *E* acting on cord *EC*)
Action-Reaction
Action-Reaction
(Force of ring *C* acting on cord *EC*)
38.6 N
(Force of cord *EC* acting on ring *C*)
C

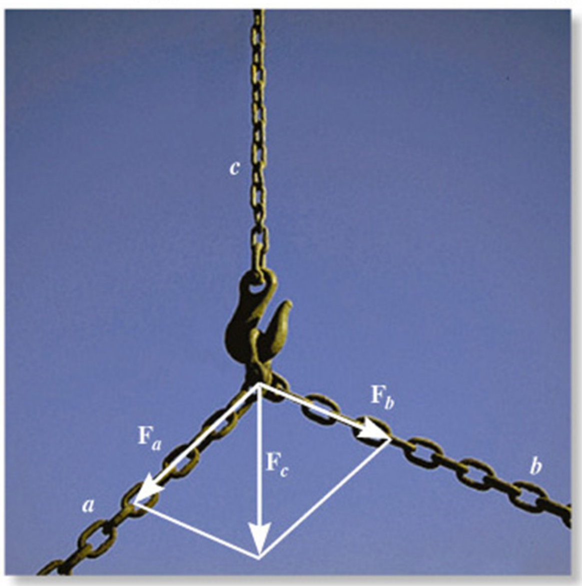
(d)

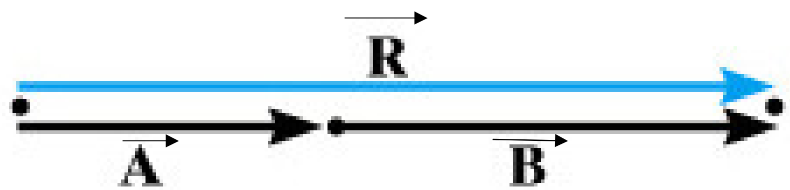


Parallelogram Law | Triangle construction | Triangle construction

(a) (b) (c) (d)

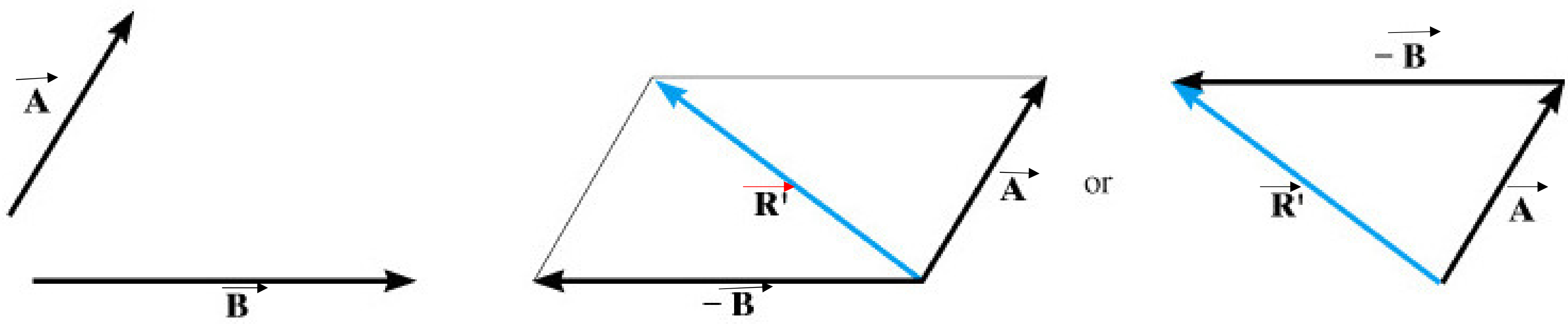
Vector Addition





$$\vec{R} = \vec{A} + \vec{B}$$

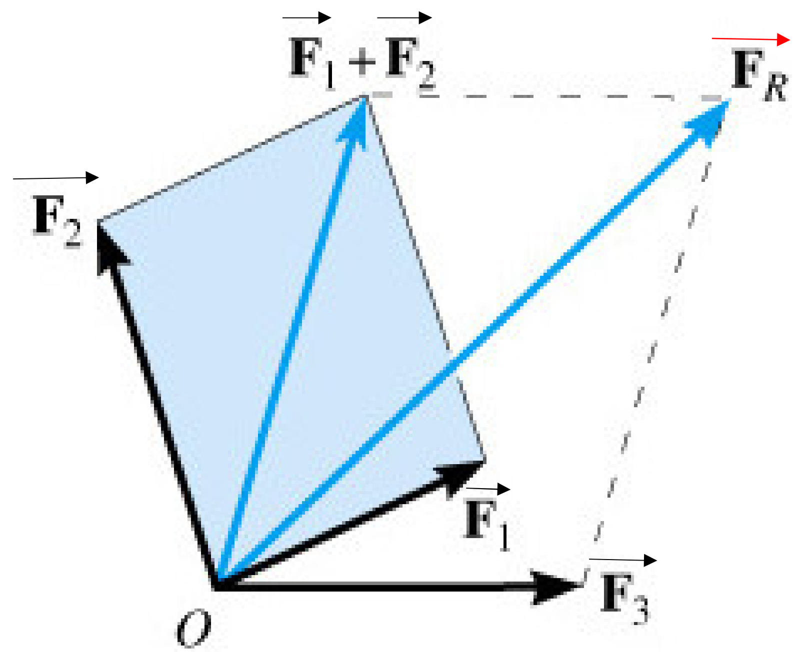
Addition of collinear vectors



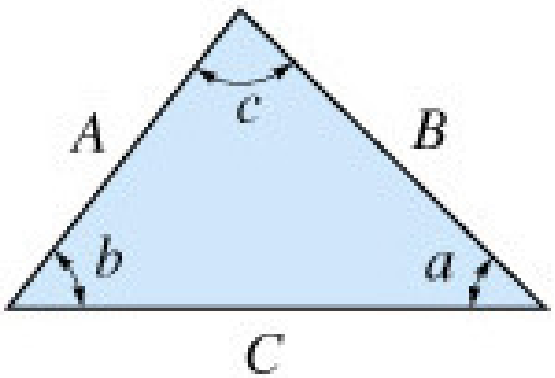
Parallelogram law

Triangle construction

Vector Subtraction



$\vec{F}_1 + \vec{F}_2$
$\vec{F}_R$
$\vec{F}_2$
$\vec{F}_1$
$\vec{F}_3$
O

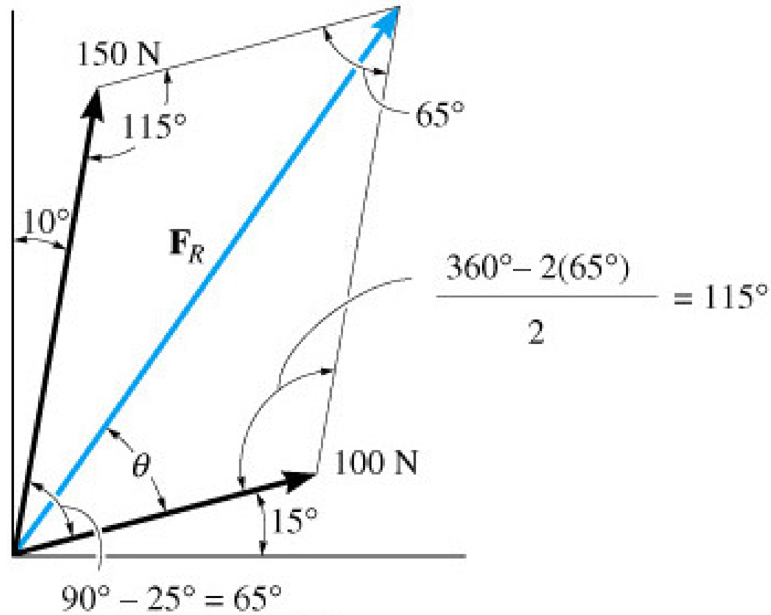


Sine law:

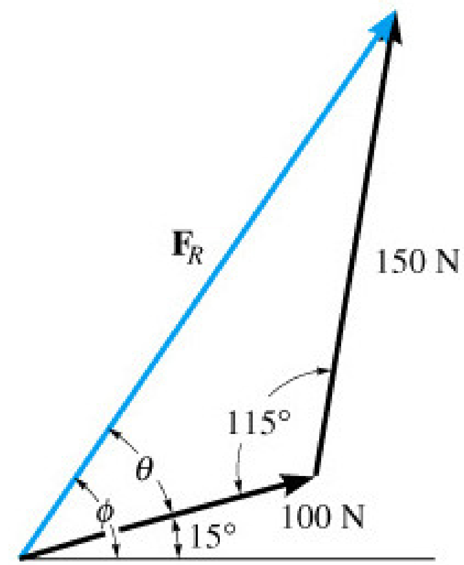
$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

Cosine law:

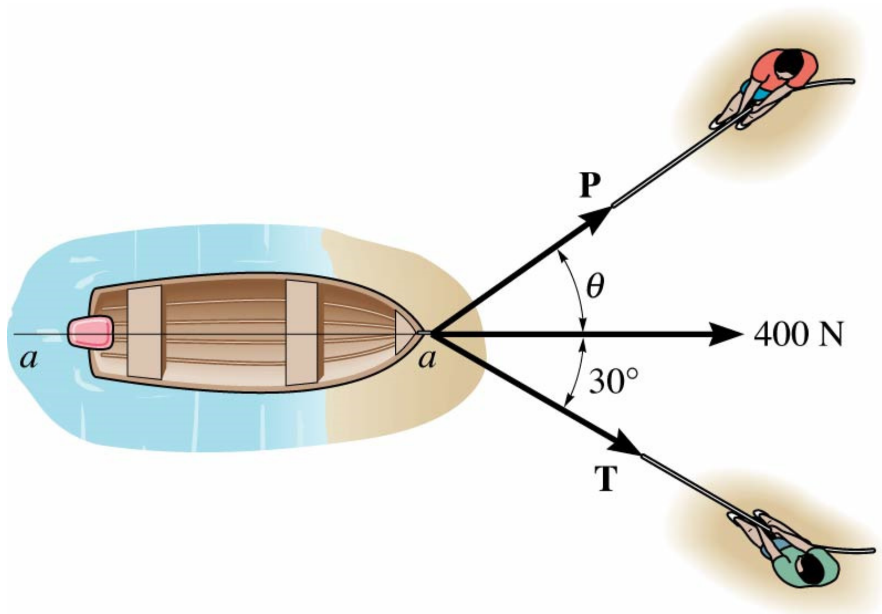
$$C = \sqrt{A^2 + B^2 - 2AB\cos c}$$

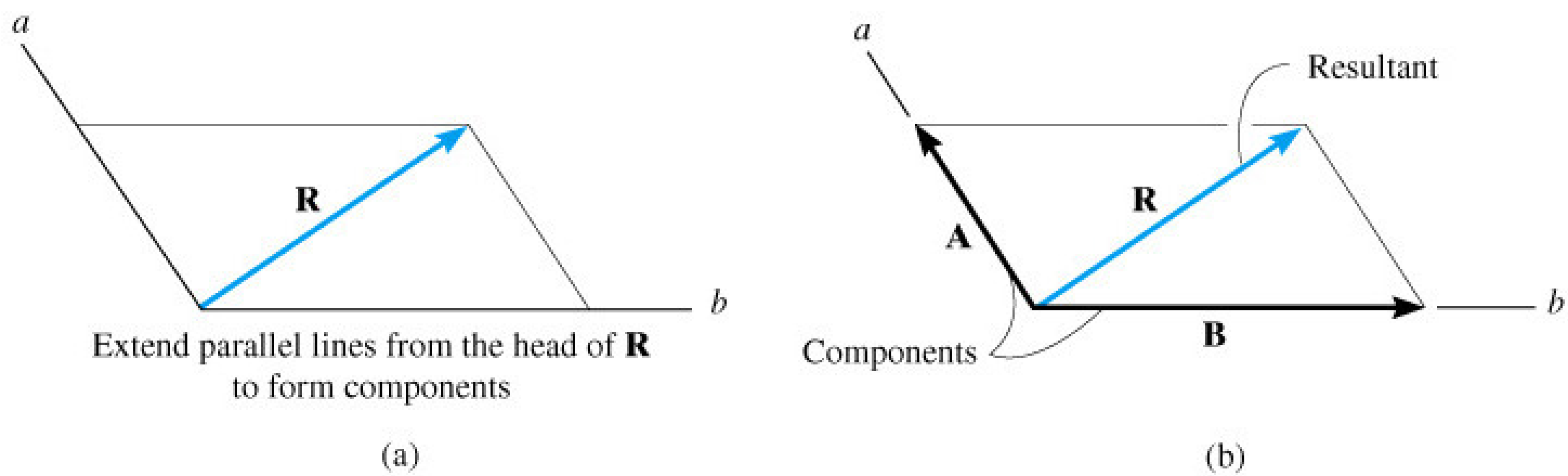


(b)



(c)


P
θ
400 N
30°
T
a
a

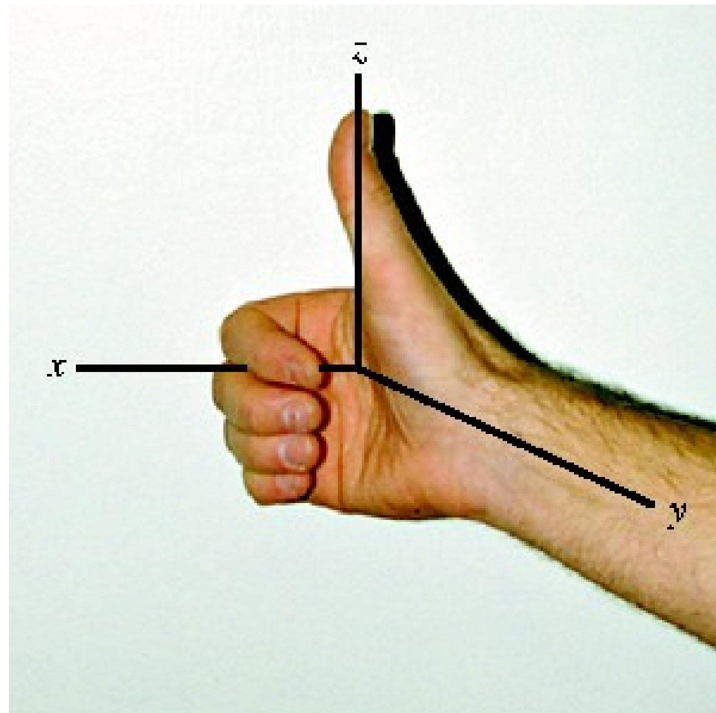


Resolution of a vector

- Right-Handed Coordinate System

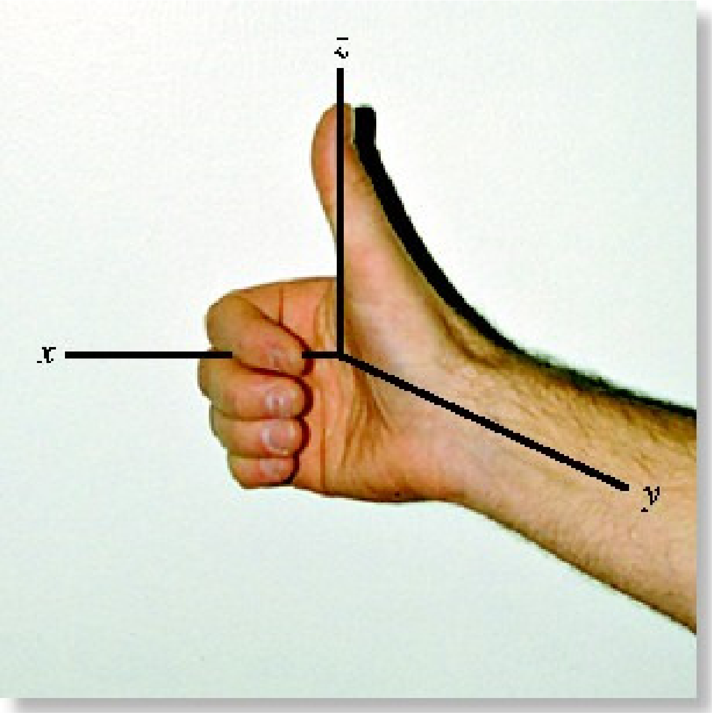
A rectangular or Cartesian coordinate system is said to be right-handed provided:

- Thumb of right hand points in the direction of the positive z axis when the right-hand fingers are curled about this axis and directed from the positive x towards the positive y axis



- Right-Handed Coordinate System

  - z-axis for the 2D problem would be perpendicular, directed out of the page.



Cartesian Vectors

- Unit Vector

  - Direction of $\vec{\mathbf{A}}$ can be specified using a unit vector

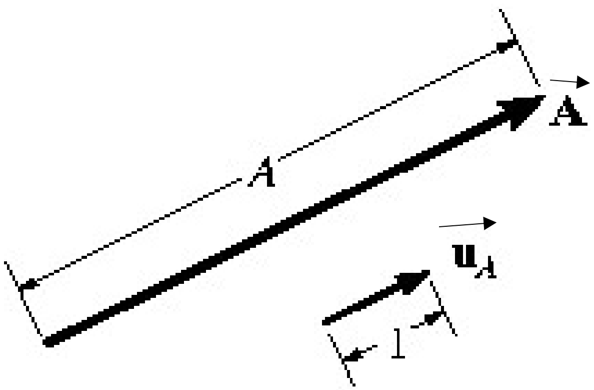
  - Unit vector has a magnitude of 1

  - If $\vec{\mathbf{A}}$ is a vector having a magnitude of $A \neq 0$, unit vector having the same direction as $\vec{\mathbf{A}}$ is expressed by

$$\vec{\mathbf{u}}_A = \vec{\mathbf{A}} / A$$

So that

$$\vec{\mathbf{A}} = A\,\vec{\mathbf{u}}_A$$



2.5 Cartesian Vectors

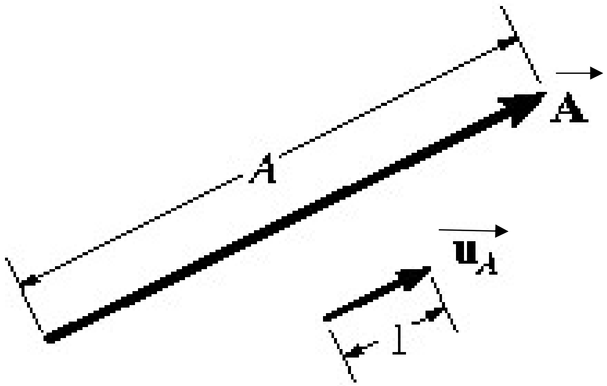
• Unit Vector

- Since $\vec{\mathbf{A}}$ is of a certain type, like force vector, a proper set of units are used for the description

- Magnitude *A* has the same sets of units, hence unit vector is dimensionless

- *A* (a positive scalar) defines magnitude of $\vec{\mathbf{A}}$

- $\vec{\mathbf{u}}_A$ defines the direction and sense of $\vec{\mathbf{A}}$

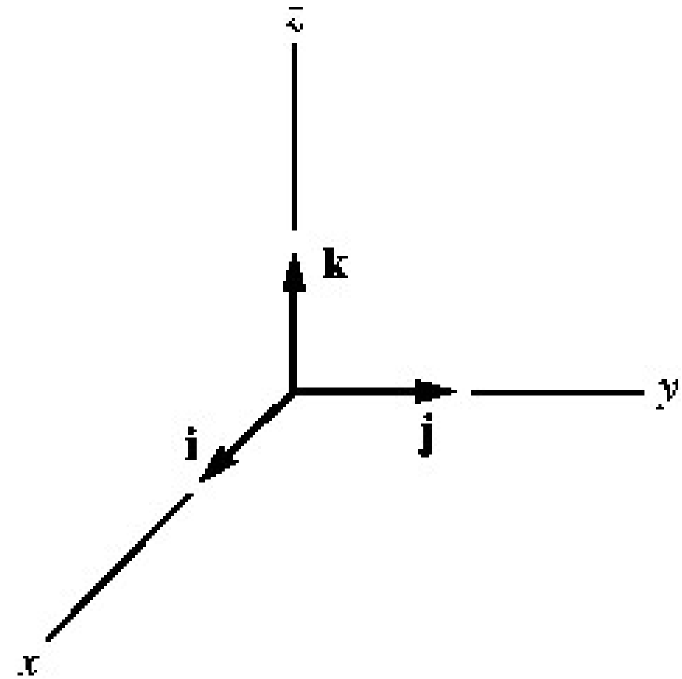


Cartesian Vectors

- Cartesian Unit Vectors

  - Cartesian unit vectors, $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$ are used to designate the directions of *z, y* and *z* axes

  - Sense (or arrowhead) of these vectors are described by a plus or minus sign (depending on pointing towards the positive or negative axes)



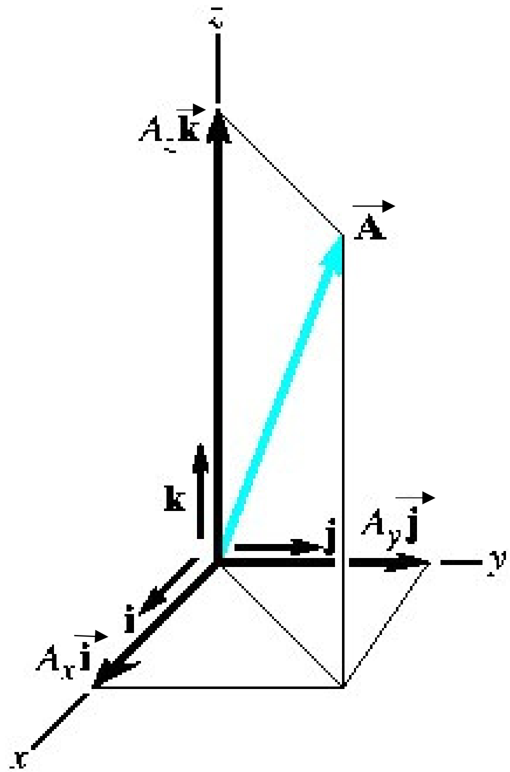
Cartesian Vectors

- Cartesian Vector Representations

  - Three components of $\vec{\mathbf{A}}$ act in the positive $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$ and $\vec{\mathbf{k}}$ directions

$$\vec{\mathbf{A}} = A_x\vec{\mathbf{i}} + A_y\vec{\mathbf{j}} + A_z\vec{\mathbf{k}}$$

*Note the magnitude and direction of each components are separated, easing vector algebraic operations.



Cartesian Vectors

- Magnitude of a Cartesian Vector

  - From the colored triangle,

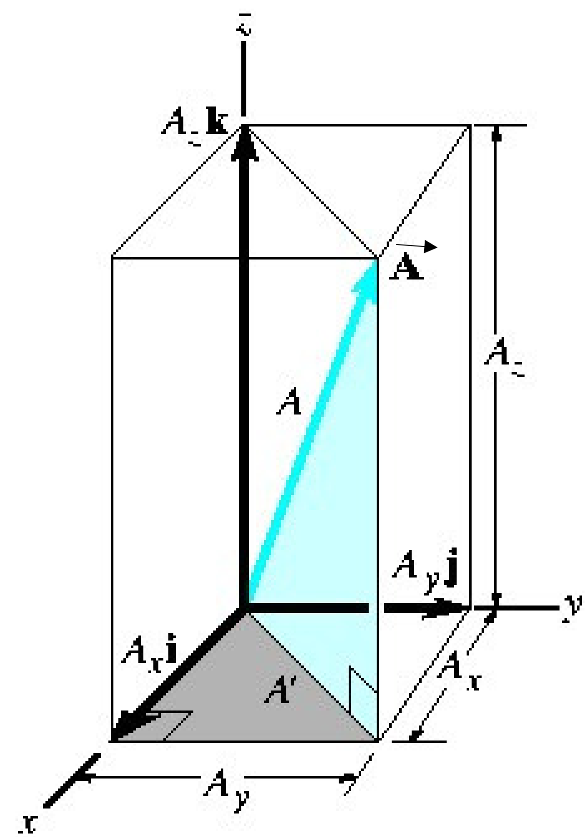
$$A = \sqrt{A'^2 + A_z^2}$$

  - From the shaded triangle,

$$A' = \sqrt{A_x^2 + A_y^2}$$

  - Combining the equations gives magnitude of **A**

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

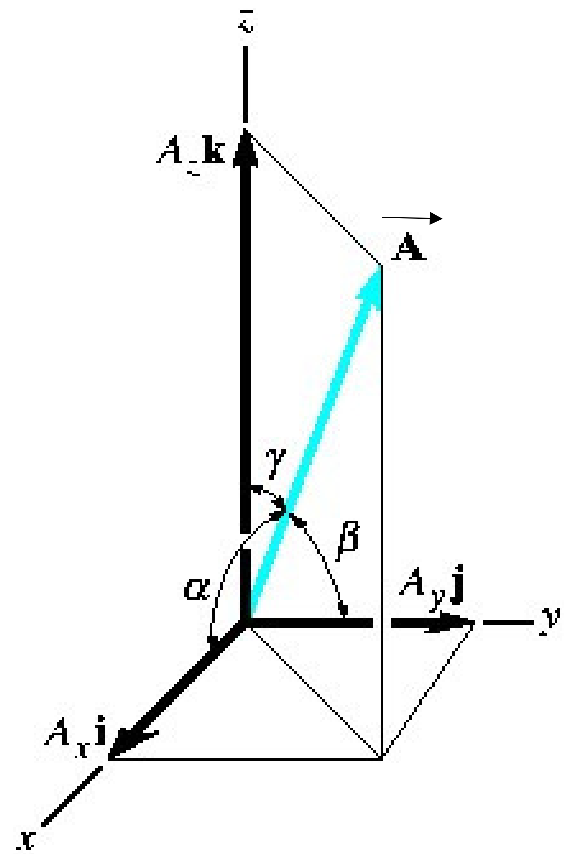


Cartesian Vectors

- Direction of a Cartesian Vector

  - Orientation of $\vec{A}$ is defined as the coordinate direction angles α, β and γ measured between the tail of $\vec{A}$ and the positive x, y and z axes

  - $0^\circ \le \alpha, \beta$ and $\gamma \le 180^\circ$



Direction angles: $\alpha, \quad \beta, \quad \gamma$

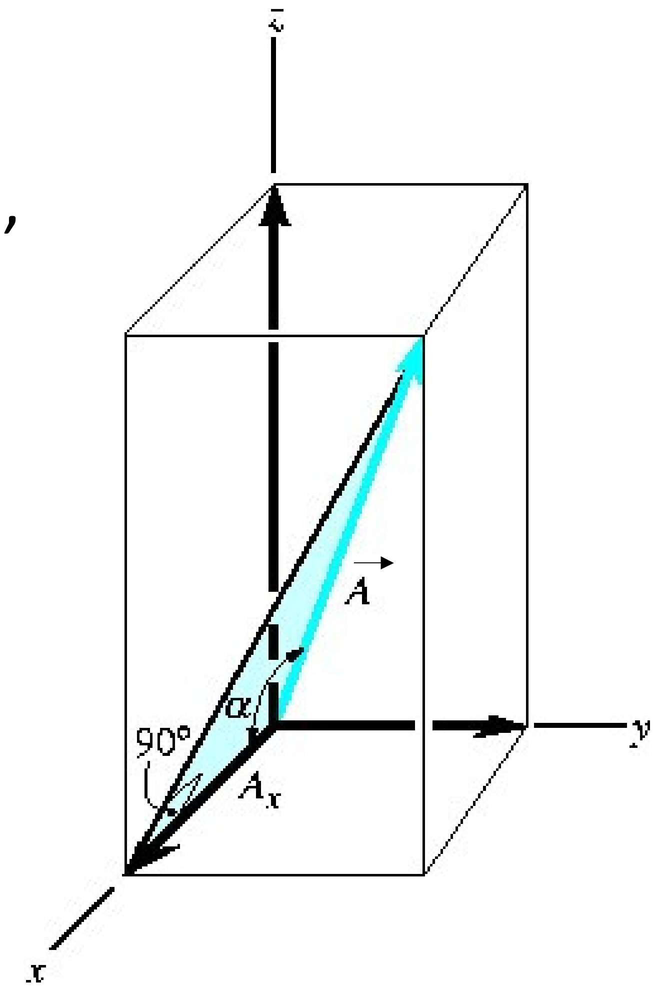
Direction Cosines: $\cos\alpha, \quad \cos\beta, \quad \cos\gamma$

Cartesian Vectors

- Direction of a Cartesian Vector

  - For angles α, β and γ (blue colored triangles), we calculate the *direction cosines* of $\vec{\mathbf{A}}$

$$\cos\alpha = \frac{A_x}{A}$$

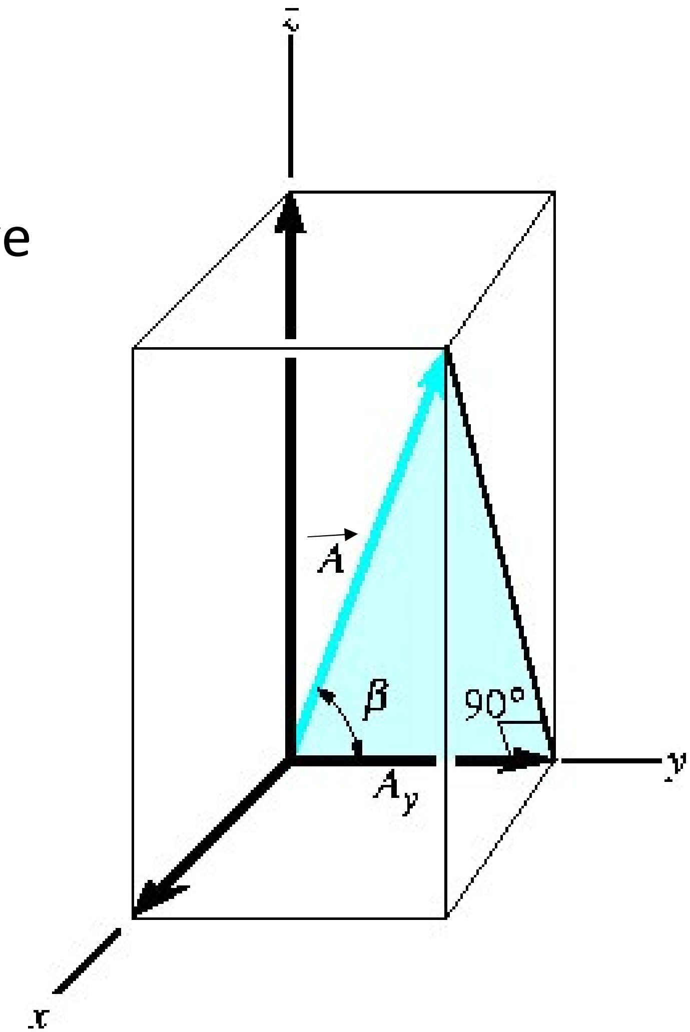


Cartesian Vectors

- Direction of a Cartesian Vector

  - For angles α, β and γ (blue colored triangles), we calculate the *direction cosines* of $\vec{\mathbf{A}}$

$$\cos\beta = \frac{A_y}{A}$$

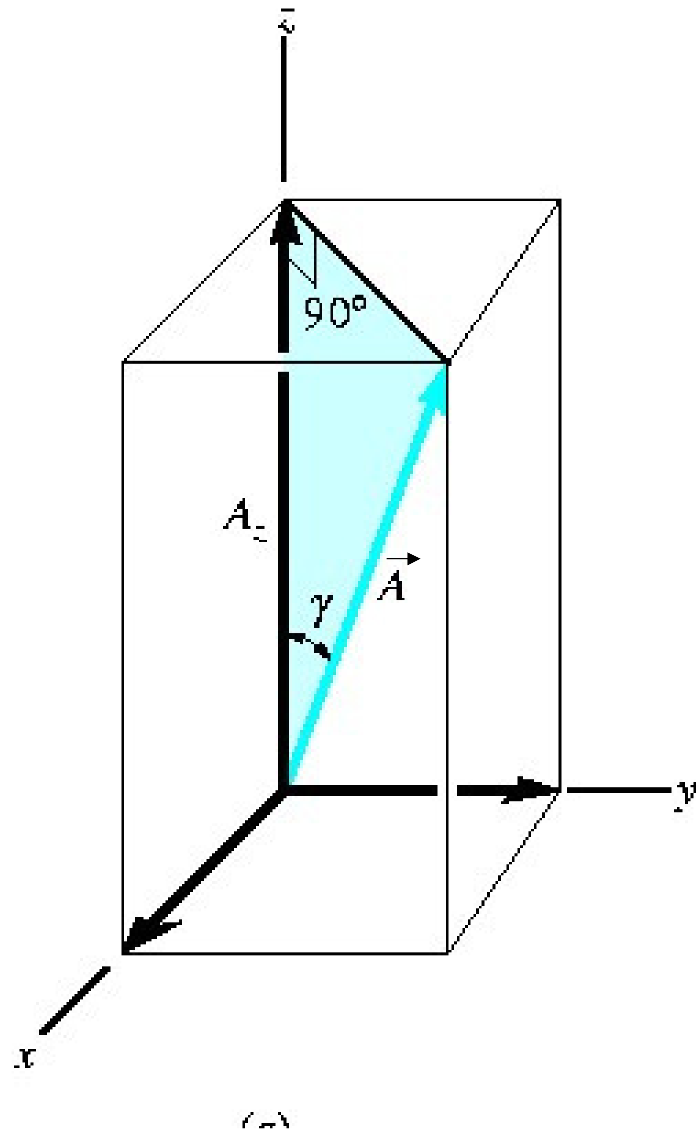


Cartesian Vectors

- Direction of a Cartesian Vector

  - For angles α, β and γ (blue colored triangles), we calculate the *direction cosines* of $\vec{\mathbf{A}}$

$$\cos\gamma = \frac{A_z}{A}$$



Cartesian Vectors

- Direction of a Cartesian Vector

  - Angles α, β and γ can be determined by the inverse cosines

  - Given

$$\vec{\mathbf{A}} = A_x\vec{\mathbf{i}} + A_y\vec{\mathbf{j}} + A_z\vec{\mathbf{k}}$$

  - then,

$$\begin{aligned}\mathbf{u}_A &= \vec{\mathbf{A}}/A \\ &= (A_x/A)\vec{\mathbf{i}} + (A_y/A)\vec{\mathbf{j}} + (A_z/A)\vec{\mathbf{k}}\end{aligned}$$

where $$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Cartesian Vectors

- Direction of a Cartesian Vector

  - $\vec{\mathbf{u}}_A$ can also be expressed as

$$\vec{\mathbf{u}}_A = \cos\alpha\,\vec{\mathbf{i}} + \cos\beta\,\vec{\mathbf{j}} + \cos\gamma\,\vec{\mathbf{k}}$$

  - Since $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ and magnitude of $u_A = 1$,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

  - **A** as expressed in Cartesian vector form

$$\begin{aligned}\vec{\mathbf{A}} &= A\vec{\mathbf{u}}_A \\ &= A\cos\alpha\,\vec{\mathbf{i}} + A\cos\beta\,\vec{\mathbf{j}} + A\cos\gamma\,\vec{\mathbf{k}} \\ &= A_x\vec{\mathbf{i}} + A_y\vec{\mathbf{j}} + A_Z\vec{\mathbf{k}}\end{aligned}$$

Addition of a System of Coplanar Forces

- Coplanar Force Resultants

  To determine resultant of several coplanar forces:

  - Resolve force into x and y components

  - Addition of the respective components using scalar algebra

  - Resultant force is found using the parallelogram law

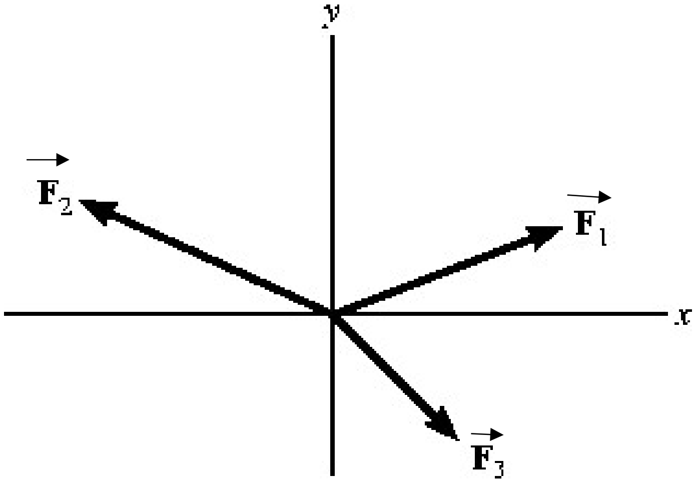
Addition of a System of Coplanar Forces

- ## Coplanar Force Resultants

  Example: Consider three coplanar forces

Cartesian vector notation

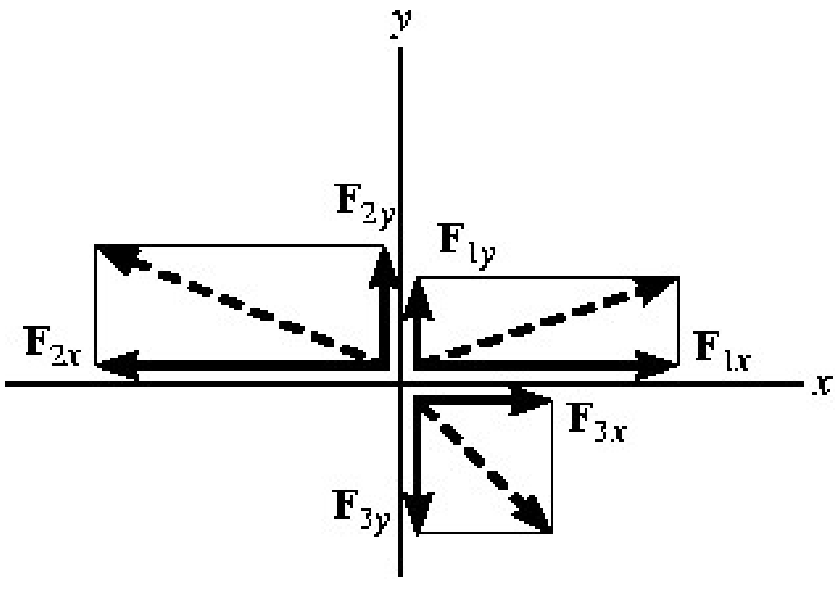
$$\vec{\mathbf{F}}_1 = F_{1x}\vec{\mathbf{i}} + F_{1y}\vec{\mathbf{j}}$$
$$\vec{\mathbf{F}}_2 = -F_{2x}\vec{\mathbf{i}} + F_{2y}\vec{\mathbf{j}}$$
$$\vec{\mathbf{F}}_3 = F_{3x}\vec{\mathbf{i}} - F_{3y}\vec{\mathbf{j}}$$



(a)

Addition of a System of Coplanar Forces

- Coplanar Force Resultants



(b)

Vector resultant is therefore

$$\vec{\mathbf{F}}_R = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3$$

$$= F_{1x}\vec{\mathbf{i}} + F_{1y}\vec{\mathbf{j}} - F_{2x}\vec{\mathbf{i}} + F_{2y}\vec{\mathbf{j}} + F_{3x}\vec{\mathbf{i}} - F_{3y}\vec{\mathbf{j}}$$

$$= (F_{1x} - F_{2x} + F_{3x})\vec{\mathbf{i}} + (F_{1y} + F_{2y} - F_{3y})\vec{\mathbf{j}}$$

$$= (F_{Rx})\vec{\mathbf{i}} + (F_{Ry})\vec{\mathbf{j}}$$

Addition of a System of Coplanar Forces

- Coplanar Force Resultants

  If scalar notation are used

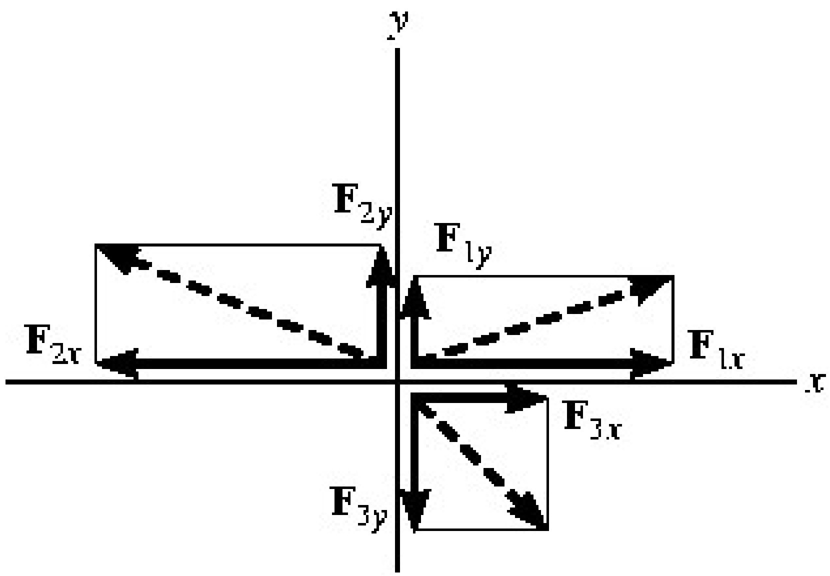
  $$F_{Rx} = (F_{1x} - F_{2x} + F_{3x})$$

  $$F_{Ry} = (F_{1y} + F_{2y} - F_{3y})$$

  In all cases,

  $$F_{Rx} = \sum F_x$$

  $$F_{Ry} = \sum F_y$$



(b)

* Take note of sign conventions

Addition of a System of Coplanar Forces

- Coplanar Force Resultants

  - Positive scalars = sense of direction along the positive coordinate axes

  - Negative scalars = sense of direction along the negative coordinate axes

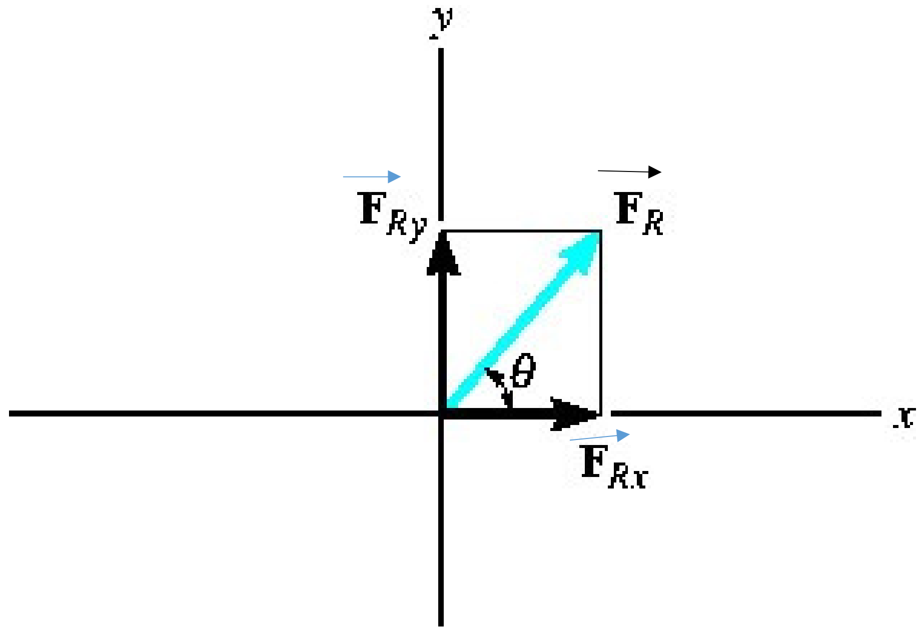
  - Magnitude of $\vec{\mathbf{F}}_R$ can be found by Pythagorean Theorem

$$F_R = \sqrt{F^2{}_{Rx} + F^2{}_{Ry}}$$

Addition of a System of Coplanar Forces

- Coplanar Force Resultants
  - Direction angle θ (orientation of the force) can be found by trigonometry

$$\theta = \tan^{-1}\left|\frac{F_{Ry}}{F_{Rx}}\right|$$



CONCURRENT FORCES

- For equilibrium

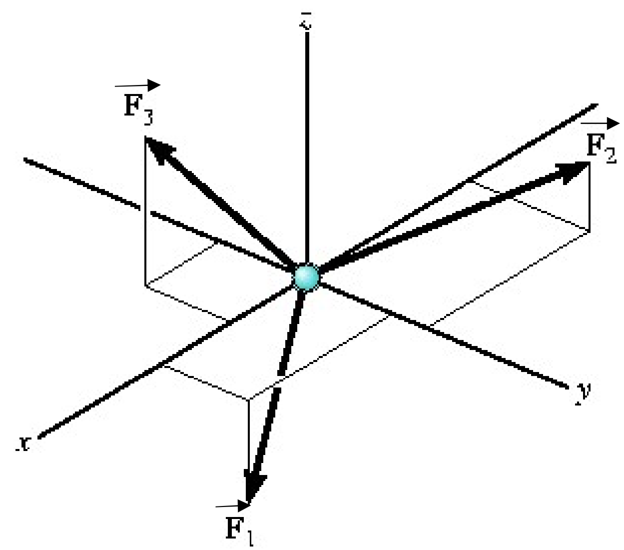
$$\sum \vec{\mathbf{F}} = 0$$

- Resolving into **i**, **j**, **k** components

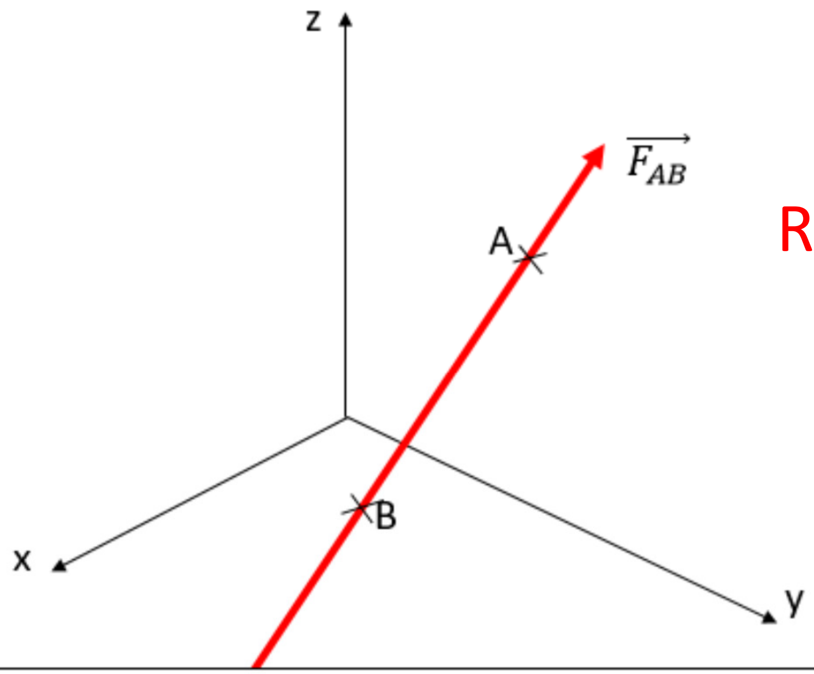
$$\sum F_x \vec{\mathbf{i}} + \sum F_y \vec{\mathbf{j}} + \sum F_z \vec{\mathbf{k}} = 0$$

- Three scalar equations representing algebraic sums of the x, y, z forces

$$\sum F_x \vec{\mathbf{i}} = 0$$
$$\sum F_y \vec{\mathbf{j}} = 0$$
$$\sum F_z \vec{\mathbf{k}} = 0$$



- Make use of the three scalar equations to solve for unknowns such as angles or magnitudes of forces



Resolve force into its components

$$\overrightarrow{F_{AB}} = F_{AB}\vec{u}$$

$$\vec{u} = \frac{(x_A - x_B)\vec{\imath} + (y_A - y_B)\vec{\jmath} + (z_A - z_B)\vec{k}}{\sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}} = \cos\alpha\,\vec{\imath} + \cos\beta\,\vec{\jmath} + \cos\gamma\,\vec{k}$$

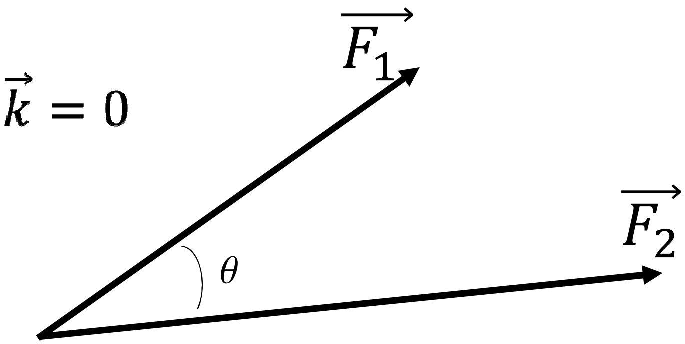
$$\overrightarrow{F_{AB}} = F_{ABx}\vec{\imath} + F_{ABy}\vec{\jmath} + F_{ABz}\vec{k}$$

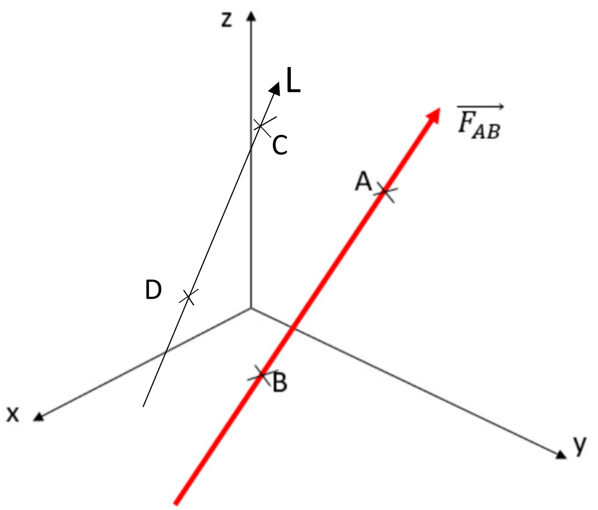
Scalar product of two vectors

$$\overrightarrow{F_1}\,.\,\overrightarrow{F_2} = F_1 * F_2 \cos\theta$$

$$\vec{\imath}\,.\,\vec{\imath} = 1 * 1 * \cos 0 = 1,\;\; \vec{\jmath}\,.\,\vec{\jmath} = 1,\;\; \vec{k}\,.\,\vec{k} = 1$$

$$\vec{\imath}\,.\,\vec{\jmath} = 1 * 1 * \cos 90^0 = 0, \qquad \vec{\imath}\,.\,\vec{k} = 0, \qquad \vec{\jmath}.\vec{k} = 0$$





$\overrightarrow{F_{AB}} = F_{AB}\vec{u}$

$$\vec{u} = \frac{(x_A - x_B)\vec{\imath} + (y_A - y_B)\vec{\jmath} + (z_A - z_B)\vec{k}}{\sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}} = \cos\alpha\,\vec{\imath} + \cos\beta\,\vec{\jmath} + \cos\gamma\,\vec{k}$$

$\overrightarrow{F_{AB}} = F_{ABx}\vec{\imath} + F_{ABy}\vec{\jmath} + F_{ABz}\vec{k}$

Projection of a vector (e.g. Force) on a given axis, L

$$F_{AB\,L} = \overrightarrow{F_{AB}}\,.\,\overrightarrow{u_L}$$

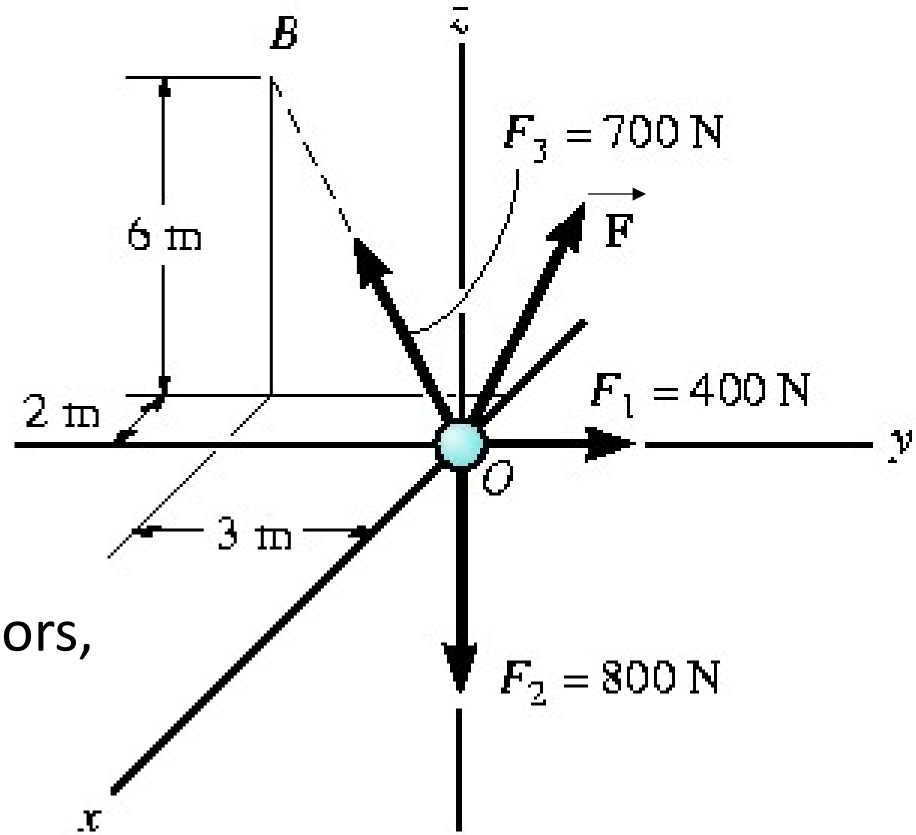
$$\vec{u}_L = \frac{(x_c - x_D)\vec{\imath} + (y_C - y_D)\vec{\jmath} + (z_C - z_D)\vec{k}}{\sqrt{(x_c - x_D)^2 + (y_C - y_D)^2 + (z_C - z_D)^2}}$$

- Procedure for Analysis

Equations of Equilibrium

- Apply $\sum F_x = 0$, $\sum F_y = 0$ and $\sum F_z = 0$ when forces can be easily resolved into x, y, z components
- When geometry appears difficult, express each force as a Cartesian vector. Substitute vectors into $\sum \vec{\mathbf{F}} = 0$ and set $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, $\vec{\mathbf{k}}$ components = 0
- Negative results indicate that the sense of the force is opposite to that shown in the FBD.

Determine the magnitude and coordinate direction angles of force $\vec{F}$ that are required for equilibrium of the particle *O*.



(a)

Equations of Equilibrium

Expressing each forces in Cartesian vectors,

$$\vec{F}_1 = \{400\vec{j}\} \text{ [N]}$$
$$\vec{F}_2 = \{-800\vec{k}\} \text{ [N]}$$
$$\vec{F}_3 = F_3(\vec{r}_B / r_B)$$
$$= \{-200\vec{i} - 300\vec{j} + 600\vec{k}\} \text{ [N]}$$
$$F = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

Solution

For equilibrium,

$\sum \vec{\mathbf{F}} = 0; \qquad \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \vec{\mathbf{F}}_3 + \vec{\mathbf{F}} = 0$

$$400\vec{\mathbf{j}} - 800\vec{\mathbf{k}} - 200\vec{\mathbf{i}} - 300\vec{\mathbf{j}} + 600\vec{\mathbf{k}}$$

$$+ F_x\vec{\mathbf{i}} + F_y\vec{\mathbf{j}} + F_z\vec{\mathbf{k}} = 0$$

$\sum F_x = 0; \qquad -200 + F_x = 0 \rightarrow F_x = 200\text{N}$

$\sum F_y = 0; \qquad 400 - 300 + F_y = 0 \rightarrow F_y = -100\text{N}$

$\sum F_z = 0; \qquad -800 + 600 + F_z = 0 \rightarrow F_z = 200\text{N}$

$$\vec{F} = 200\vec{i} - 100\vec{j} - 200\vec{k} \ [N]$$

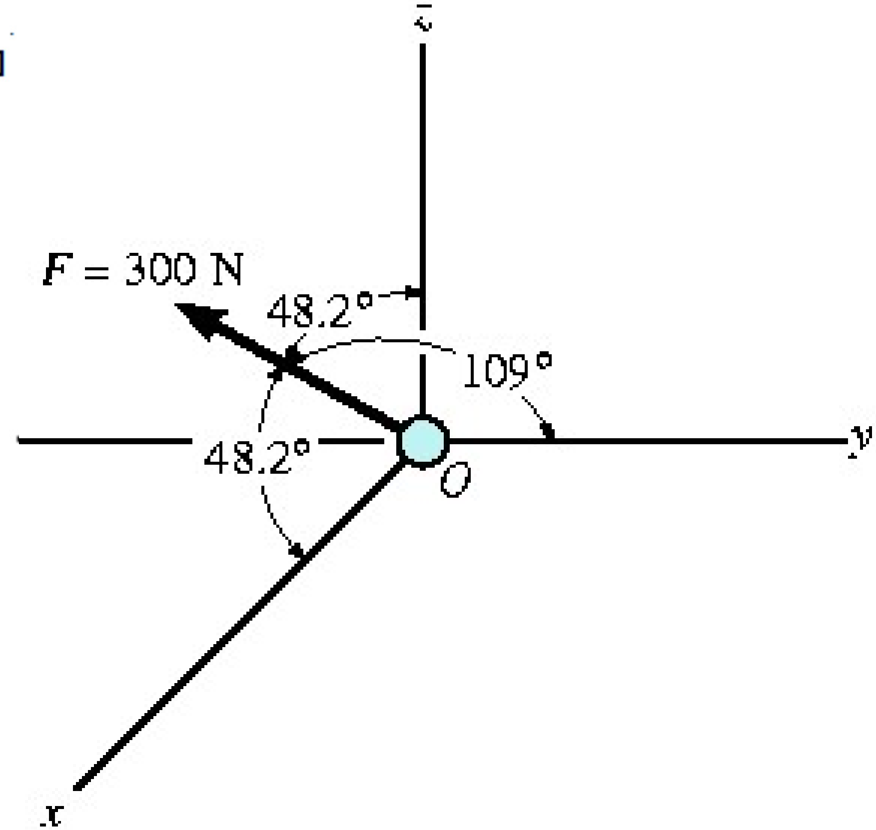
$$F = \sqrt{(200)^2 + (-100)^2 + (200)^2} = 300 \ [N]$$

$$\vec{u}_F = \frac{\vec{F}}{F} = \frac{200}{300}\vec{i} - \frac{100}{300}\vec{j} - \frac{200}{300}\vec{k}$$

$$\alpha = \cos^{-1}\left(\frac{200}{300}\right) = 48.2^\circ$$

$$\beta = \cos^{-1}\left(\frac{-100}{300}\right) = 109^\circ$$

$$\gamma = \cos^{-1}\left(\frac{200}{300}\right) = 48.2^\circ$$

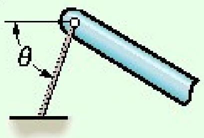
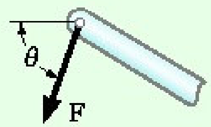
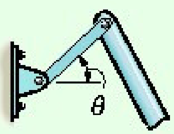
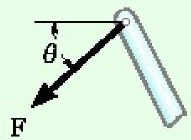
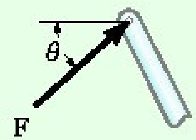
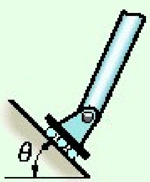
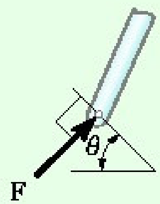
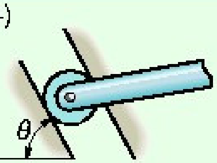
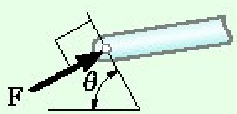
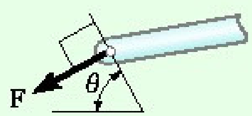


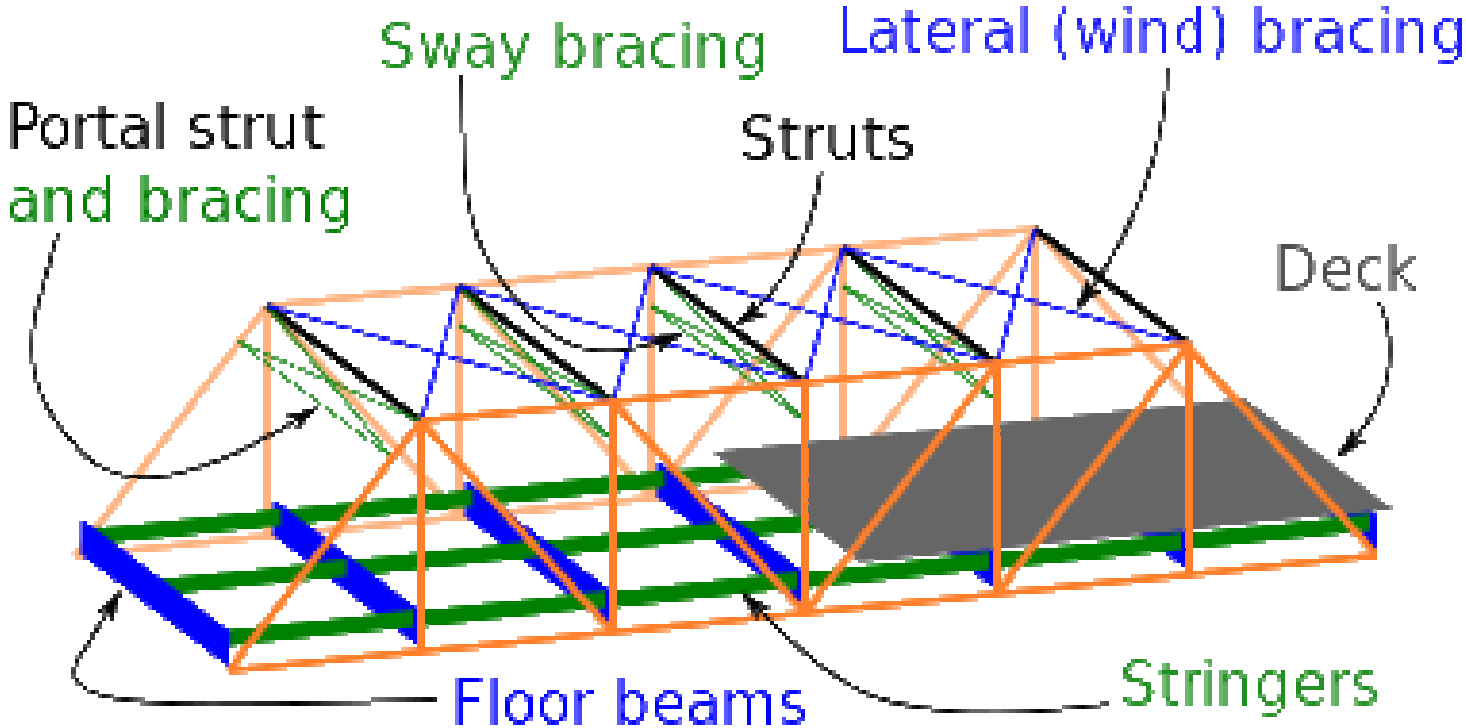
(c)

Procedure for Drawing a FBD(Free-Body Diagram)

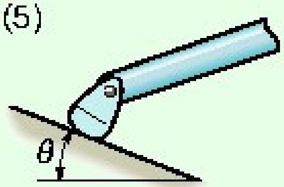
1. Draw outlined shape
   - Isolate object from its surroundings
2. Show all the forces
   - Indicate all the forces
3. Identify each forces
   - Known forces should be labeled with proper magnitude and direction
   - Letters are used to represent magnitude and directions of unknown forces

Supports for Rigid Bodies Subjected to Two-Dimensional Force Systems

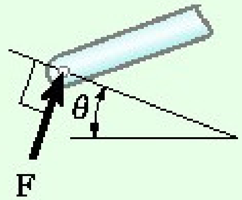
Types of Connection	Reaction	Number of Unknowns
(1)  cable		One unknown. The reaction is a tension force which acts away from the member in the direction of the cable.
(2)  weightless link	 or 	One unknown. The reaction is a force which acts along the axis of the link.
(3)  roller		One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.
(4)  roller or pin in confined smooth slot	 or 	One unknown. The reaction is a force which acts perpendicular to the slot.


Sway bracing
Lateral (wind) bracing
Portal strut
and bracing
Struts
Deck
Floor beams
Stringers

(5)

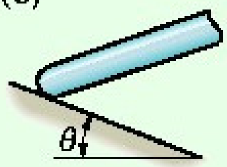


rocker

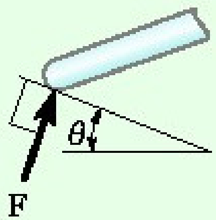


One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

(6)

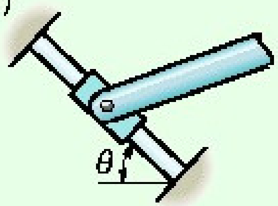


smooth contacting surface

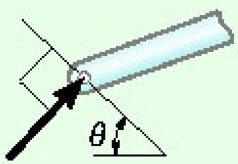


One unknown. The reaction is a force which acts perpendicular to the surface at the point of contact.

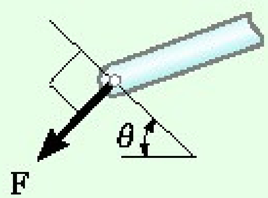
(7)



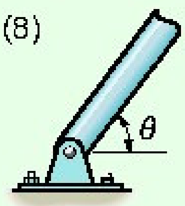
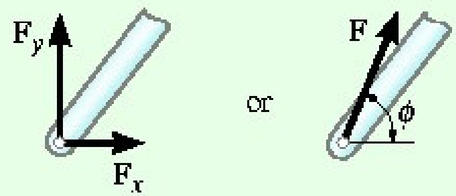
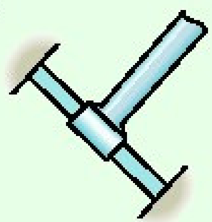
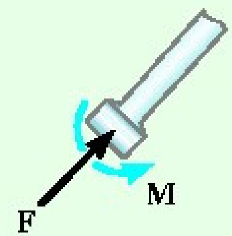
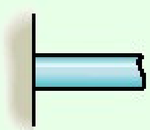
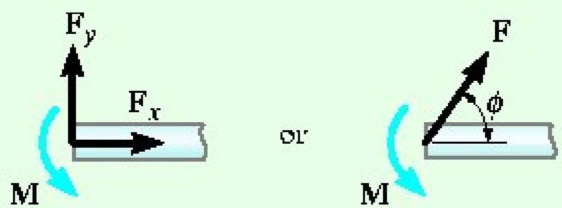
member pin connected to collar on smooth rod



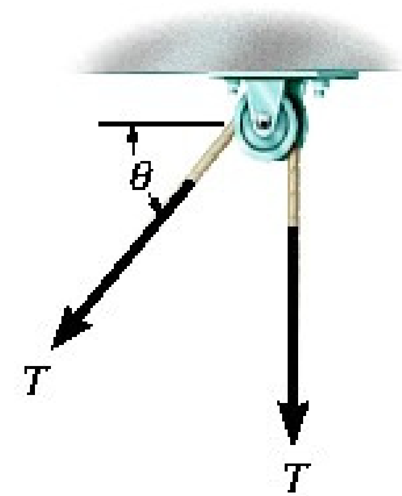
or



One unknown. The reaction is a force which acts perpendicular to the rod.

Types of Connection	Reaction	Number of Unknowns
(8)  smooth pin or hinge		Two unknowns. The reactions are two components of force, or the magnitude and direction ϕ of the resultant force. Note that ϕ and θ are not necessarily equal [usually not, unless the rod shown is a link as in (2)].
(9)  member fixed connected to collar on smooth rod		Two unknowns. The reactions are the couple moment and the force which acts perpendicular to the rod.
(10)  fixed support		Three unknowns. The reactions are the couple moment and the two force components, or the couple moment and the magnitude and direction ϕ of the resultant force.

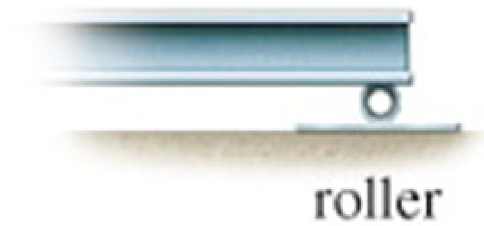
- Cables and Pulley
  - Cables (or cords) are assumed to have negligible weight and they cannot stretch
  - A cable only support tension or pulling force
  - Tension always acts in the direction of the cable
  - Tension force in a continuous cable must have a constant magnitude for equilibrium



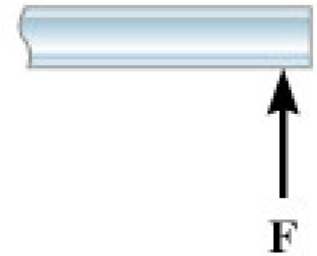
Cable is in tension

- The bucket is held in equilibrium by the cable
- Force in the cable = weight of the bucket
- Isolate the bucket for FBD
- Two forces acting on the bucket, weight **W** and force $\boldsymbol{F_c}$ of the cable
- Resultant of forces = 0

$$\mathbf{W} = \boldsymbol{F_c}$$

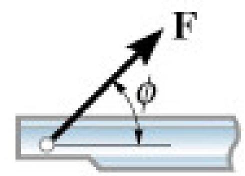


(a)



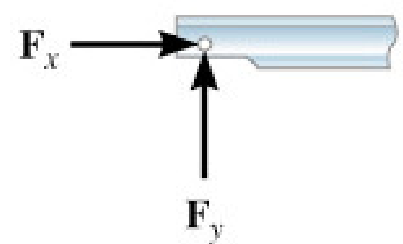
(b)

(a)



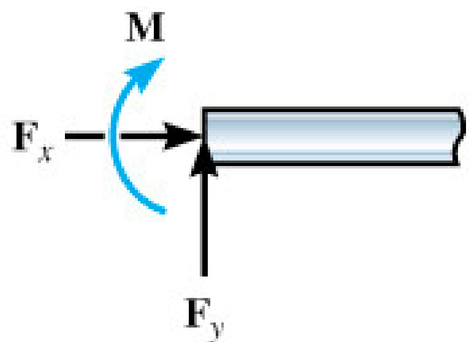
(b)

or

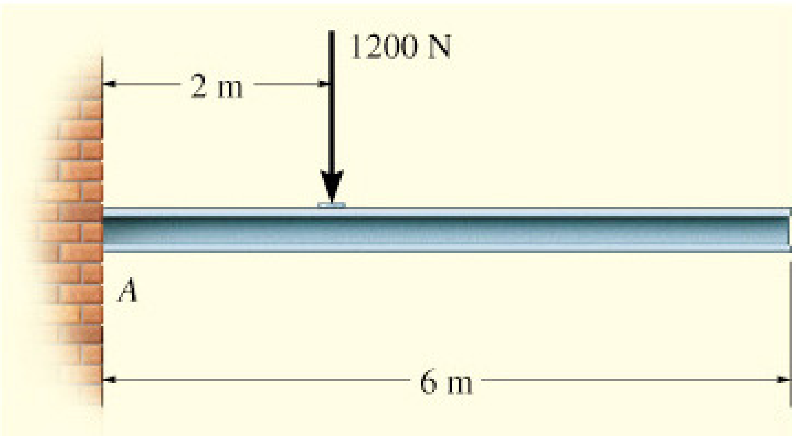


(c)

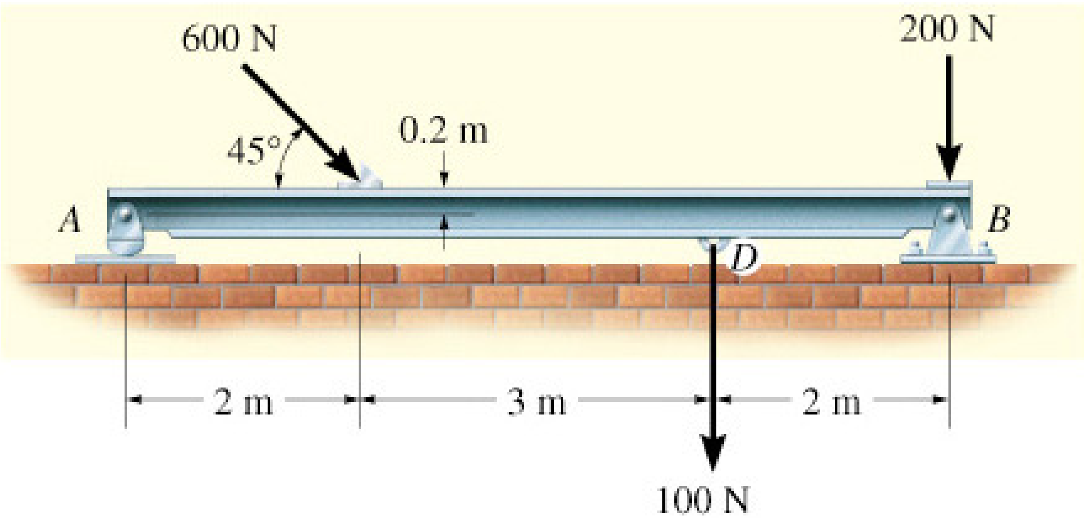
(a)



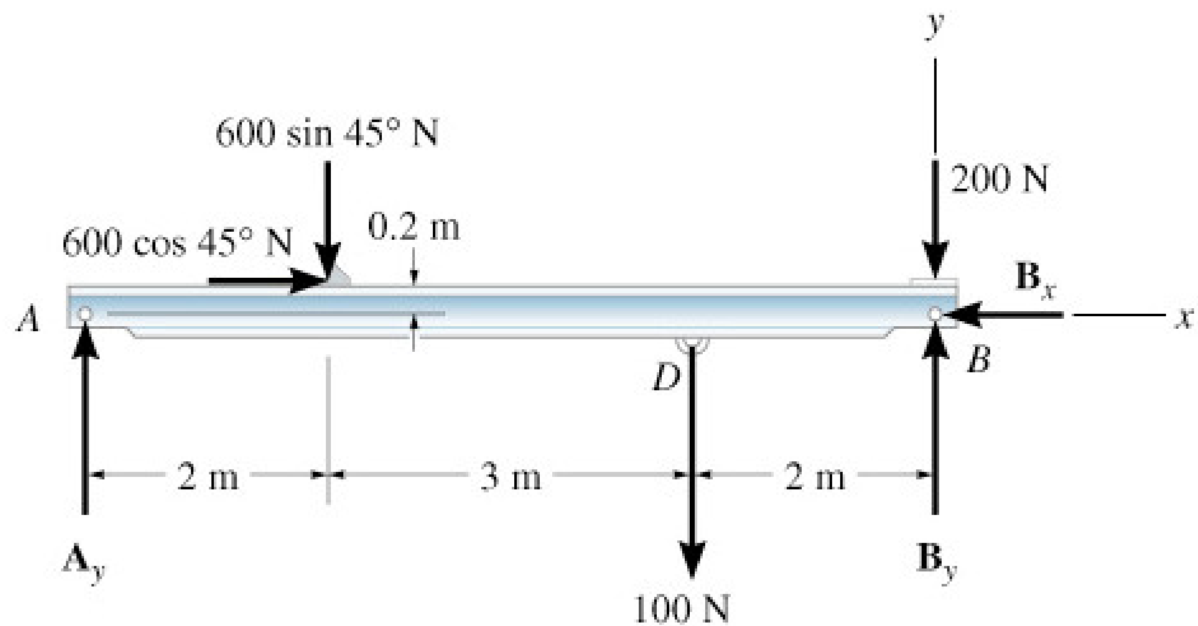
(b)



(a)


600 N
45°
0.2 m
200 N
A
B
D
2 m
3 m
2 m
100 N

(a)


y
600 sin 45° N
200 N
600 cos 45° N
0.2 m
$\mathbf{B}_x$
x
A
B
D
2 m
3 m
2 m
$\mathbf{A}_y$
$\mathbf{B}_y$
100 N

(b)

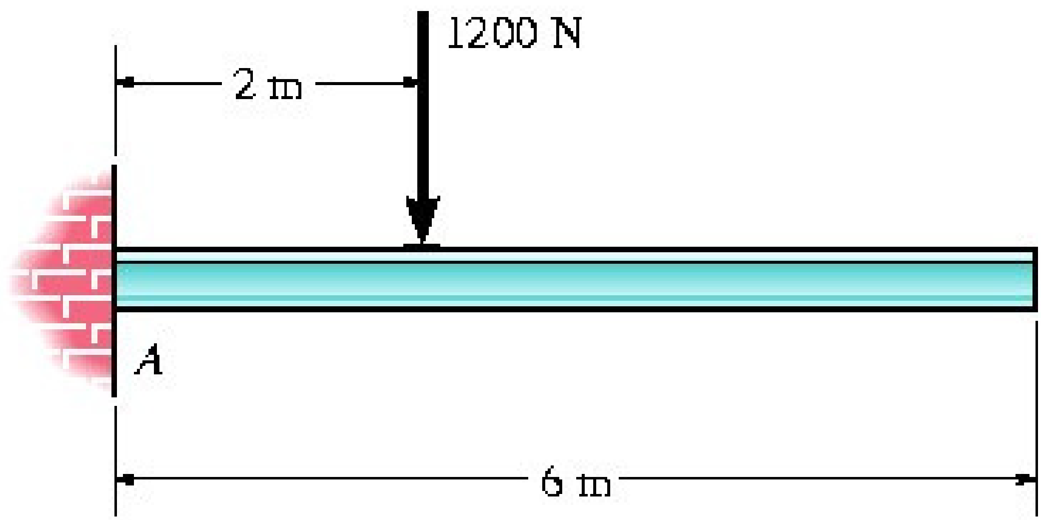
Weight and Center of Gravity

- When a body is subjected to gravity, each particle has a specified weight
- For entire body, consider gravitational forces as a system of parallel forces acting on all particles within the boundary
- The system can be represented by a single resultant force, known as weight $\vec{\mathbf{W}}$ of the body
- Location of the force application is known as the center of gravity

*Center of gravity occurs at the geometric center or centroid for uniform body of homogenous material

Example

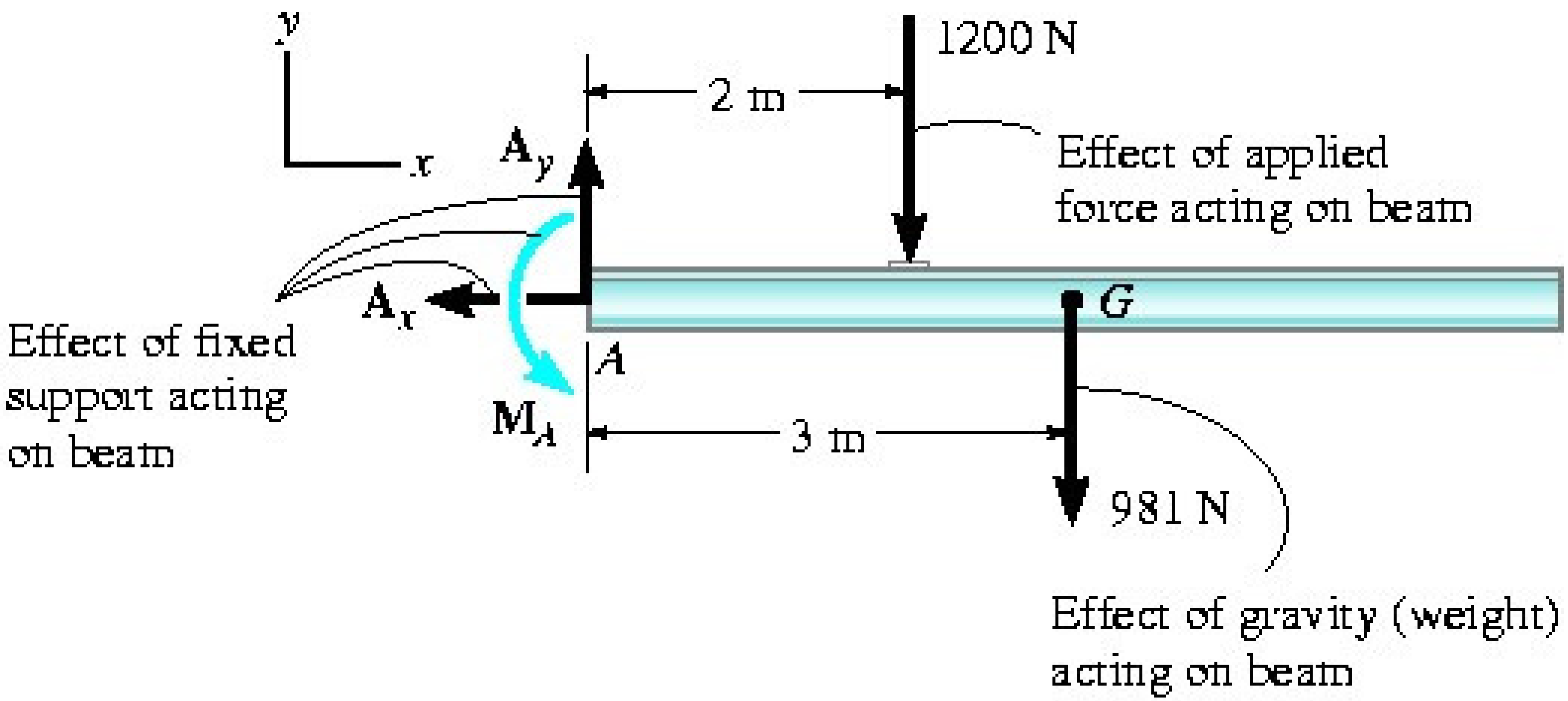
Draw the free-body diagram of the uniform beam. The beam has a mass of 100kg.



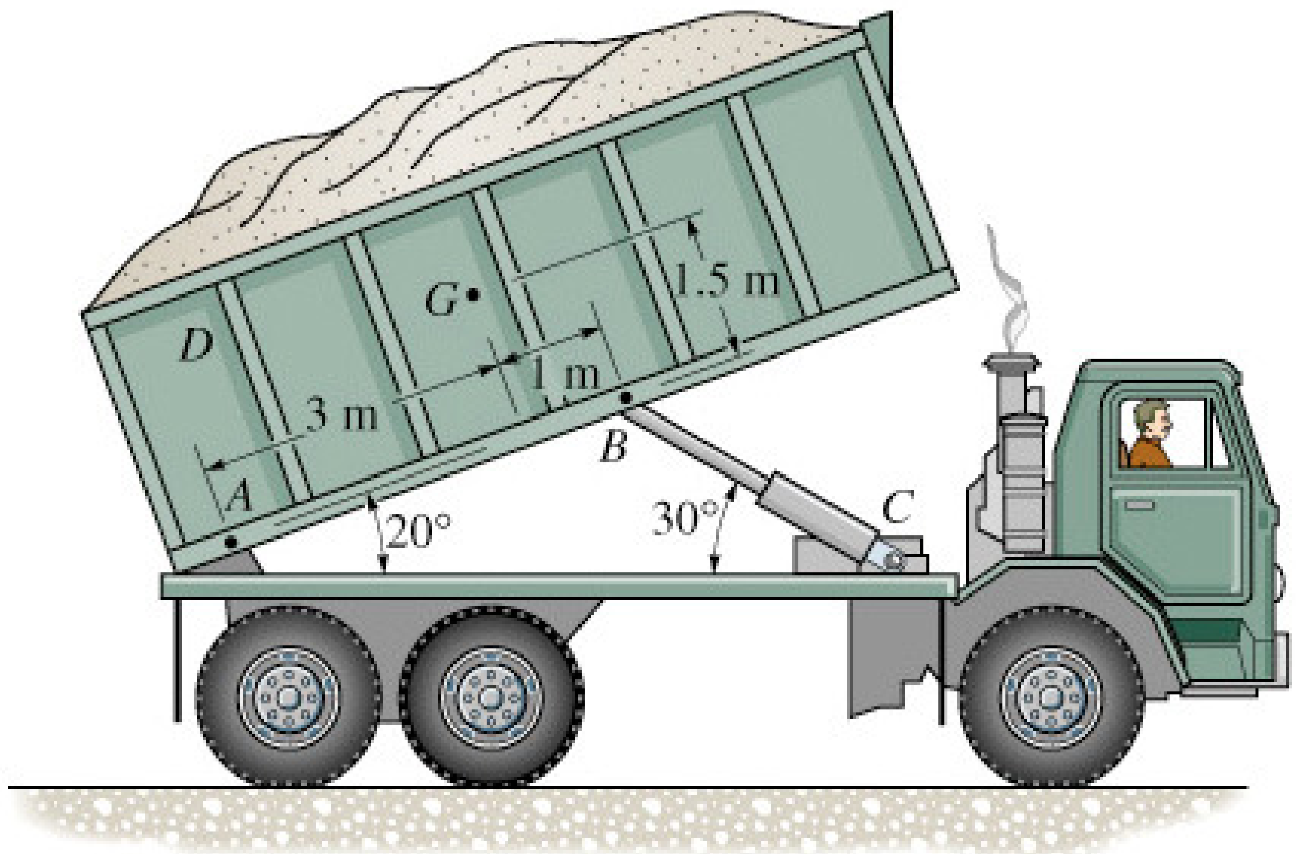
(a)

Solution

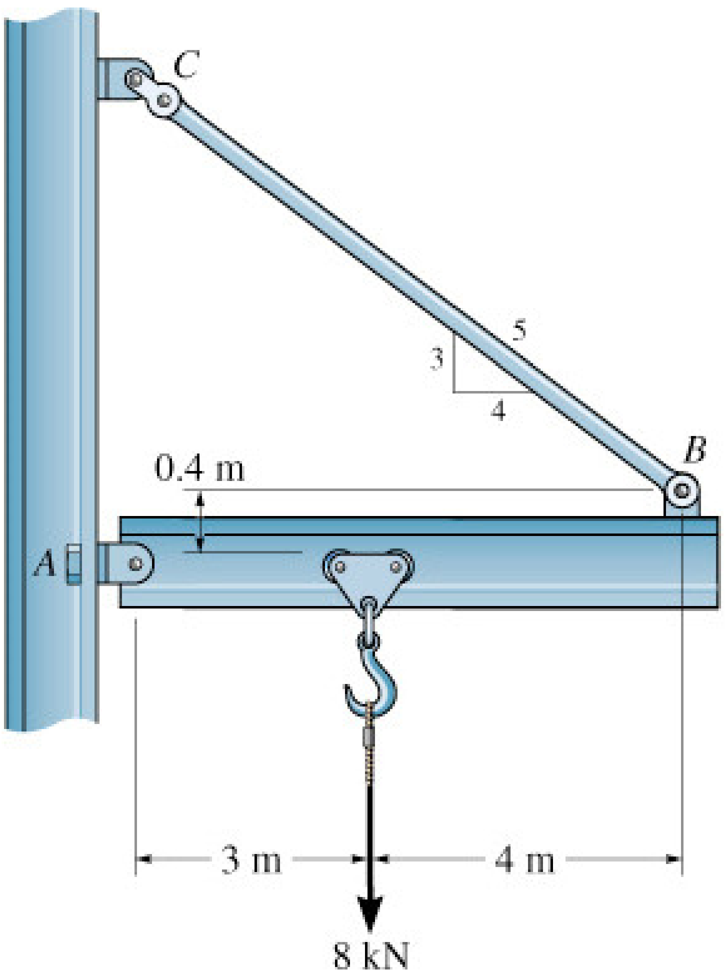
Free-Body Diagram

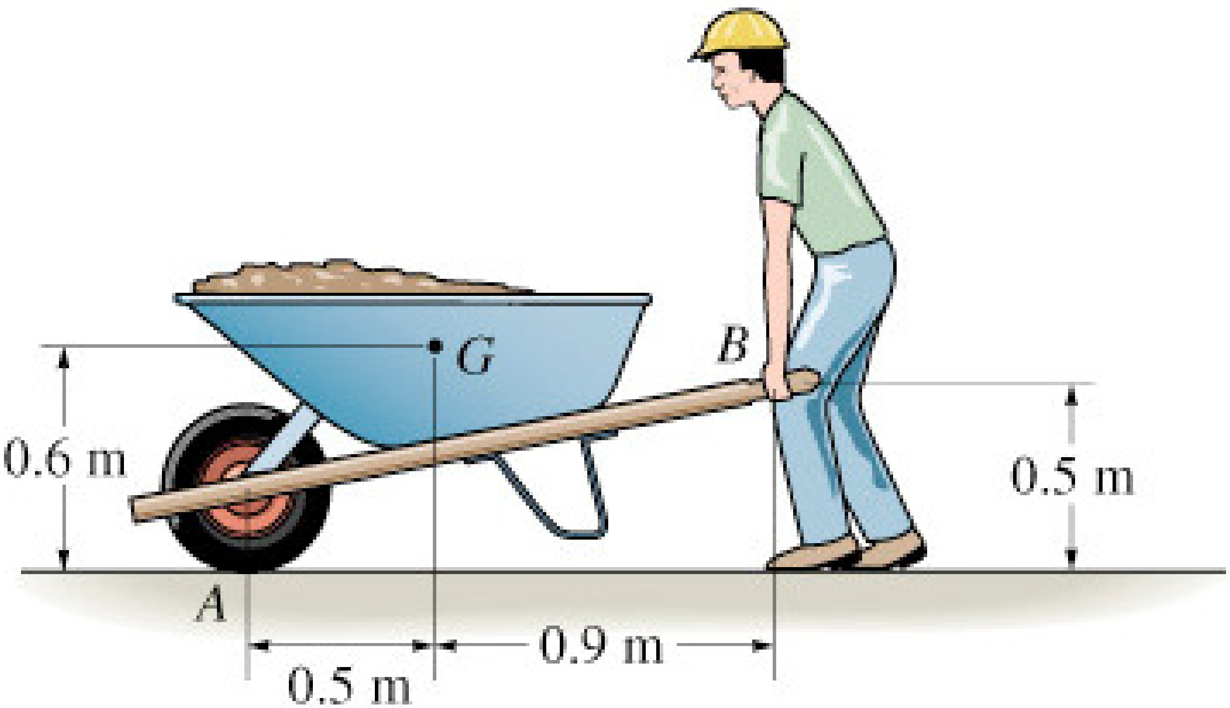


(b)



G
D
1.5 m
1 m
3 m
B
A
C
20°
30°

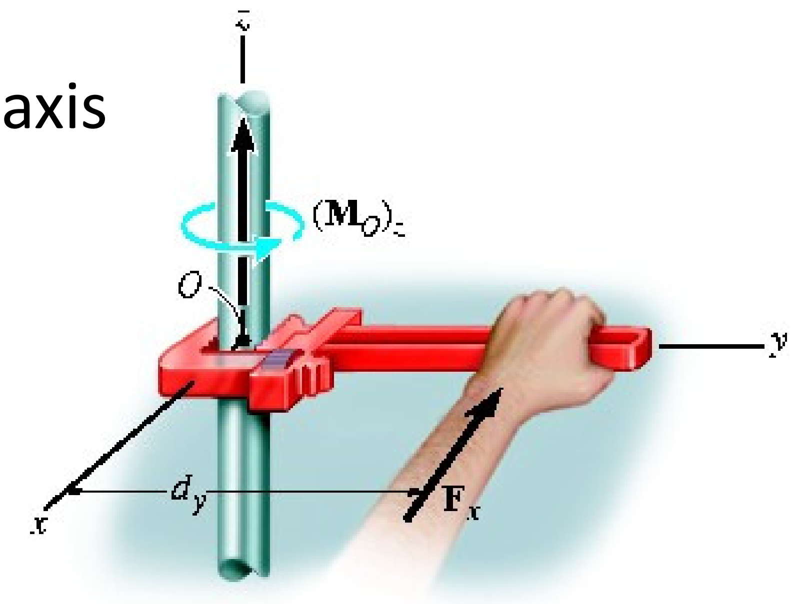

C
5
3
4
B
0.4 m
A
3 m
4 m
8 kN


G
B
0.6 m
0.5 m
A
0.5 m
0.9 m

Moment of a Force

Moment of a force about a point or axis – a measure of the tendency of the force to cause a body to rotate about the point or axis

Consider horizontal force $\mathbf{F}_x$, which acts perpendicular to the handle of the wrench and is located d_y from the point O



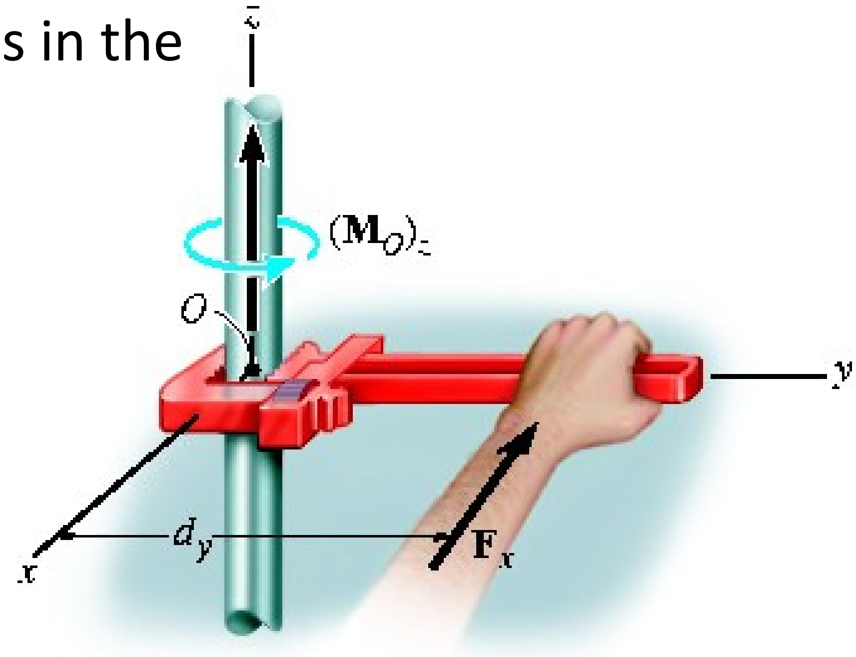
(a)

$\mathbf{F}_x$ tends to turn the pipe about the z axis

The larger the force or the distance d_y, the greater the turning effect

In General

- Consider the force $\vec{\mathbf{F}}$ and the point O which lies in the shaded plane
- The moment $\vec{\mathbf{M}}_O$ about point O, or about an axis passing through O and perpendicular to the plane, is a *vector* quantity
- Moment $\vec{\mathbf{M}}_O$ has its specified magnitude and direction



(a)

Moment is used as a measure of the effect of a force to rotate the body on which the force acts.

A force does not always cause rotation, but tends to rotate

Magnitude

For magnitude of $\mathbf{M}_O$,

$$\mathbf{M}_O = d * F$$

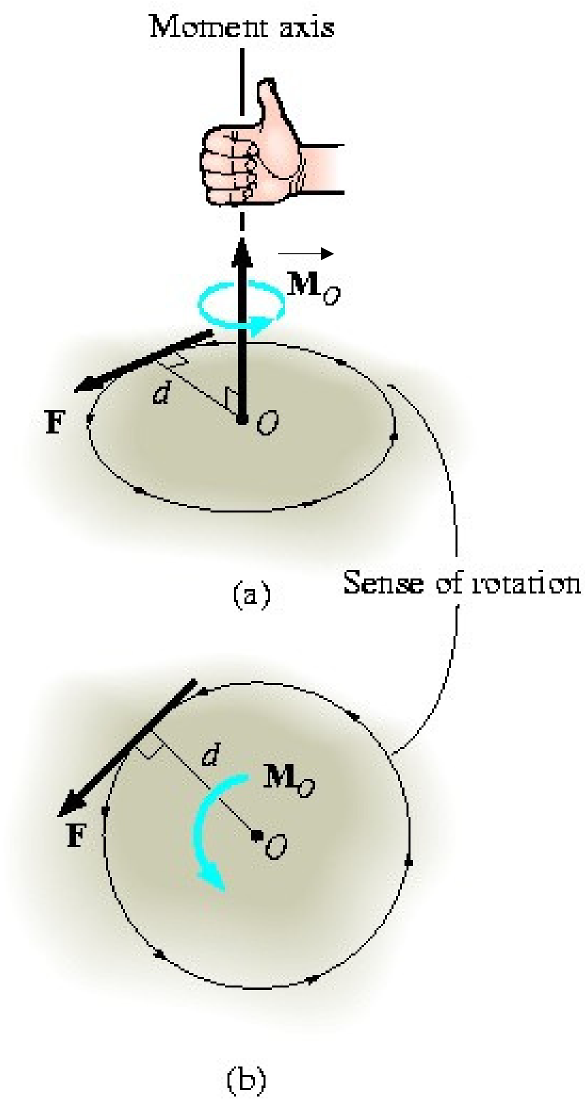
where d: moment arm or perpendicular distance from the axis at point O to its line of action of the force

Units for moment is [Nm]

Direction

- Direction of $\mathbf{M}_O$ is specified by using "right hand rule"

  - fingers of the right hand are curled to follow the sense of rotation when force rotates about point O, then open thumb. Thumb will be in the sense of moment vector

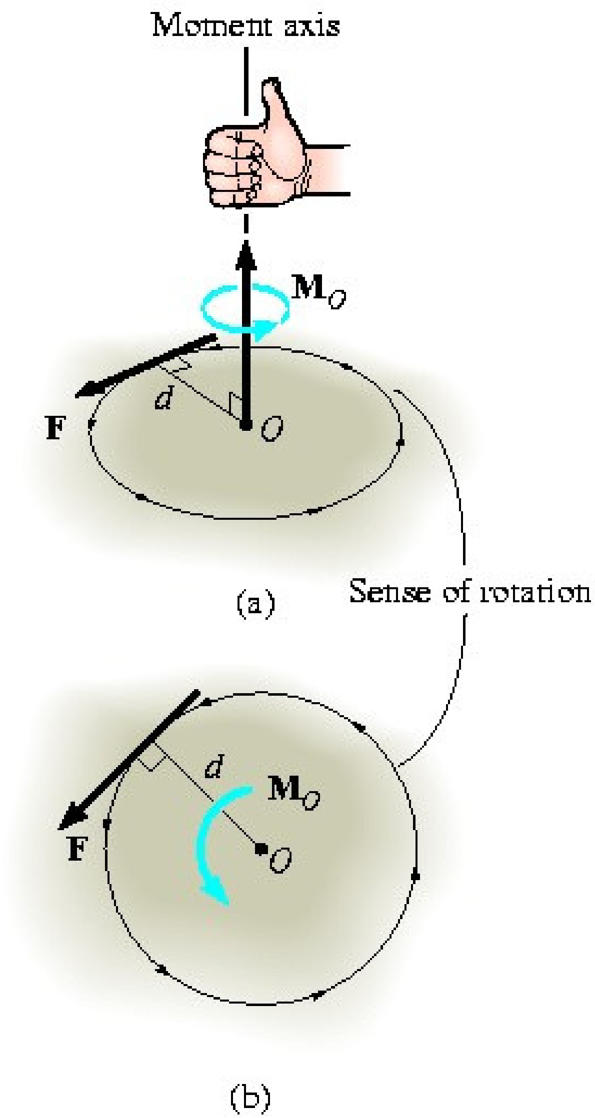


$\vec{\mathbf{M}}_O$ is shown by a vector arrow with a curl to distinguish it from force vector

Example (Fig b)

$\vec{\mathbf{M}}_O$ is represented by the counterclockwise curl, which indicates the action of $\vec{\mathbf{F}}$

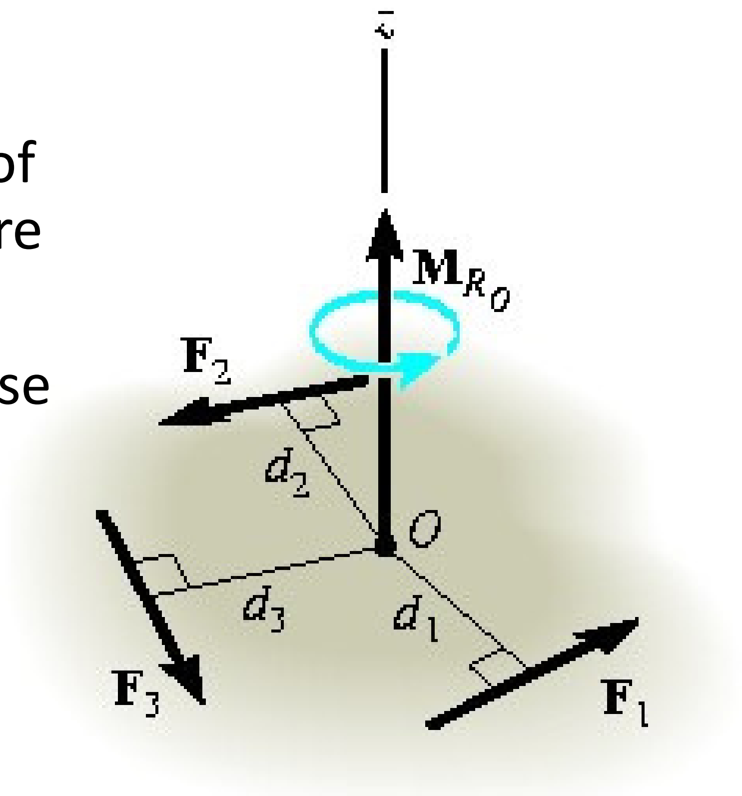
- Moment vector is perpendicular to the plane containing $\vec{\mathbf{F}}$ and d
- Moment axis intersects the plane at point O



Resultant Moment of a System of Coplanar Forces

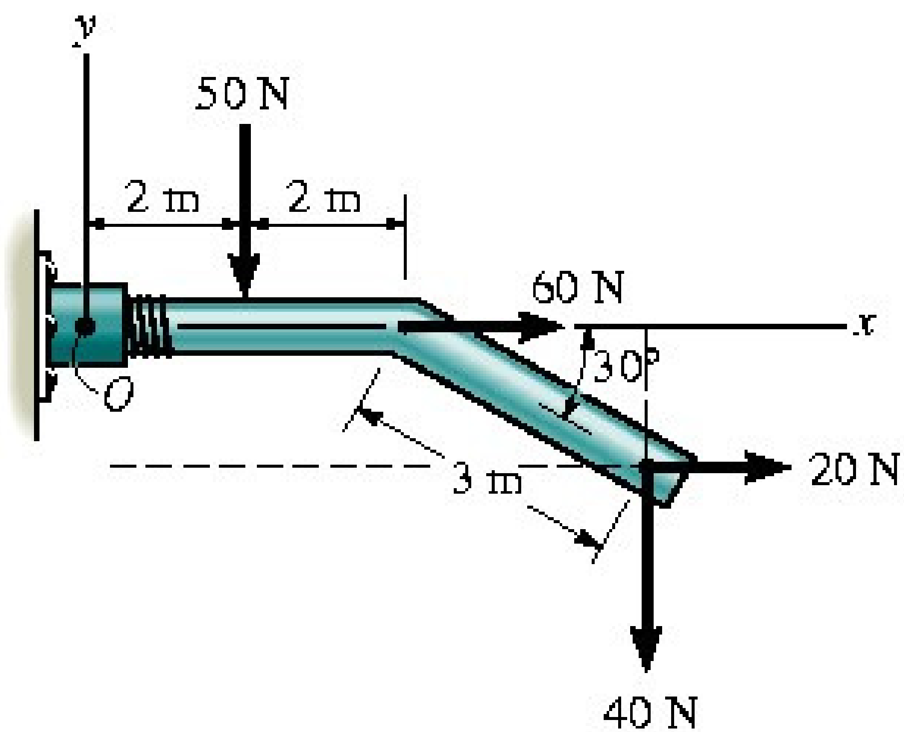
- Resultant moment, $\mathbf{M}_{Ro}$ = addition of the moments of all the forces algebraically since all moment forces are collinear

$\mathbf{M}_{Ro} = \sum(d * F)$ taking counterclockwise to be positive



Moment of a Force – 2Dimensional case

Example Determine the resultant moment of the four forces acting on the rod about point O

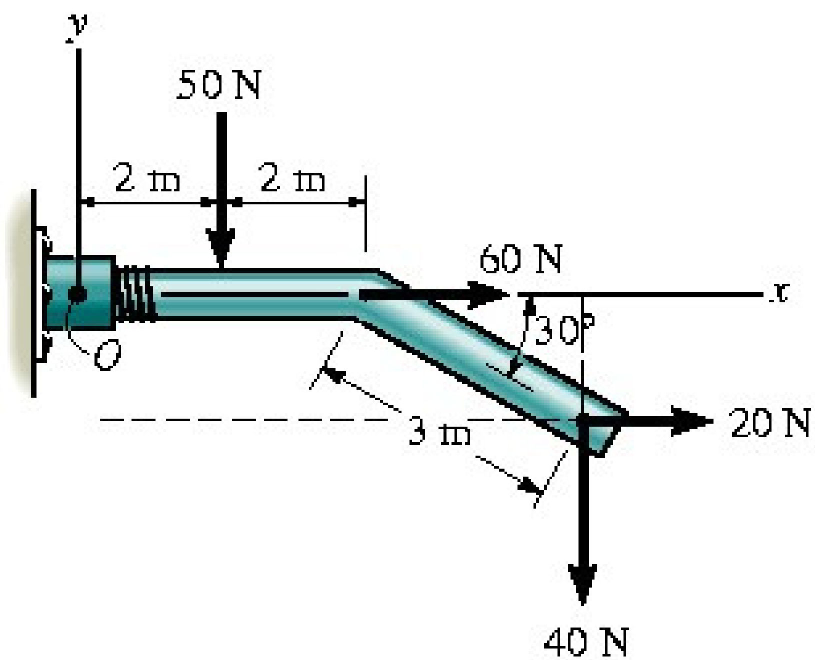


Solution

Positive moments acts in the +k direction, CCW

- $\sum M_o = \sum d * F = -(2 * 50) + (0 * 60) + ((3 * \sin 30) * 20) - ((4 + 3 * \cos 30) * 40) = -334[Nm]$

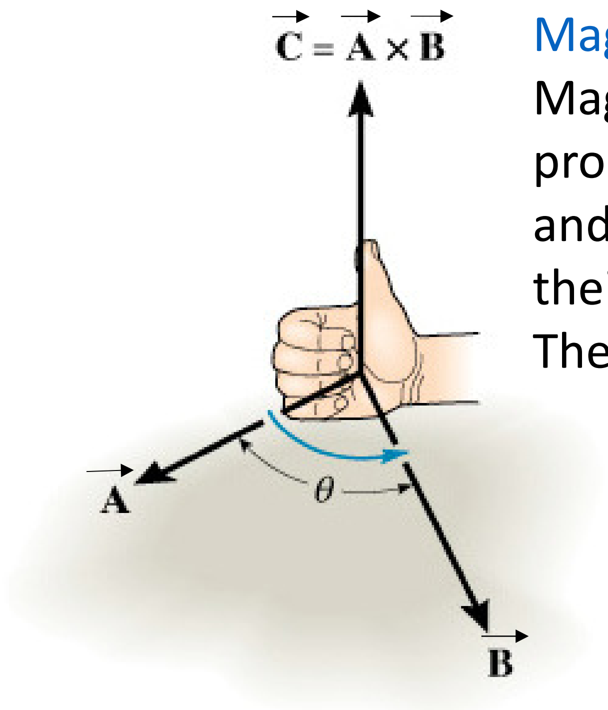
Note: Negative moment means rotation tends in clockwise direction. The resultant rotation effect of forces is to rotate the rod in the clockwise direction



VECTOR PRODUCT
or CROSS PRODUCT

- Cross product of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ yields $\vec{\mathbf{C}}$, which is written as

$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ Read as "**C** equals **A** cross **B**"

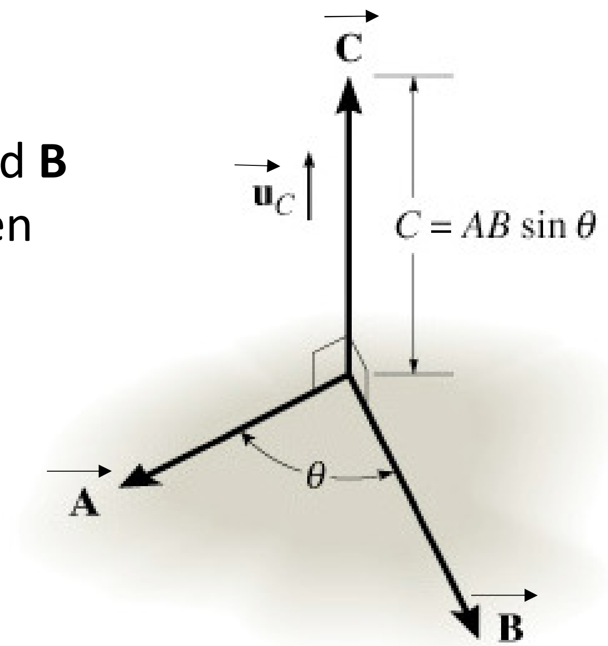


Magnitude

Magnitude of **C** is defined as the product of the magnitudes of **A** and **B** and the sine of the angle θ between their tails

Therefore,

$$C = A*B*\sin\theta$$



Cross Product

Direction

- Vector $\vec{\mathbf{C}}$ has a direction that is perpendicular to the plane containing $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ such that $\vec{\mathbf{C}}$ is specified by the right hand rule

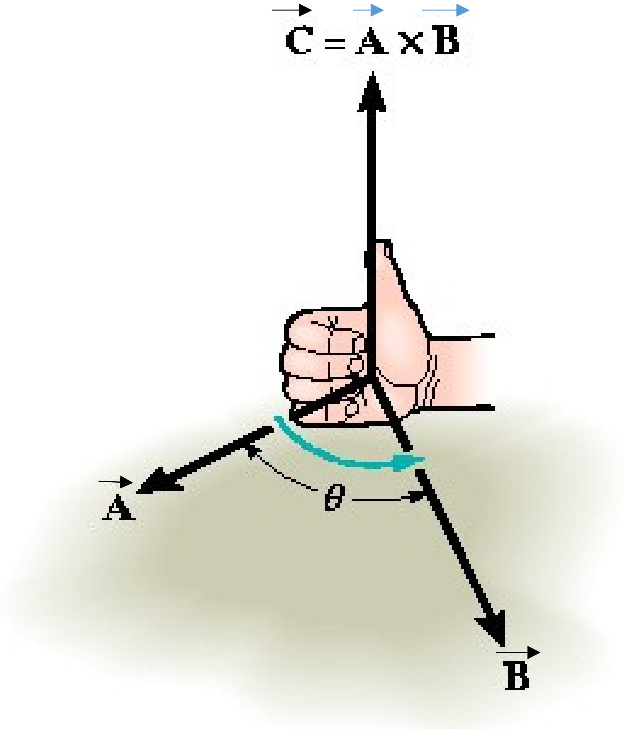
  - Curling the fingers of the right

  hand from vector $\vec{\mathbf{A}}$ (cross) to

  vector $\vec{\mathbf{B}}$

  - Thumb points in the direction of

  vector $\vec{\mathbf{C}}$



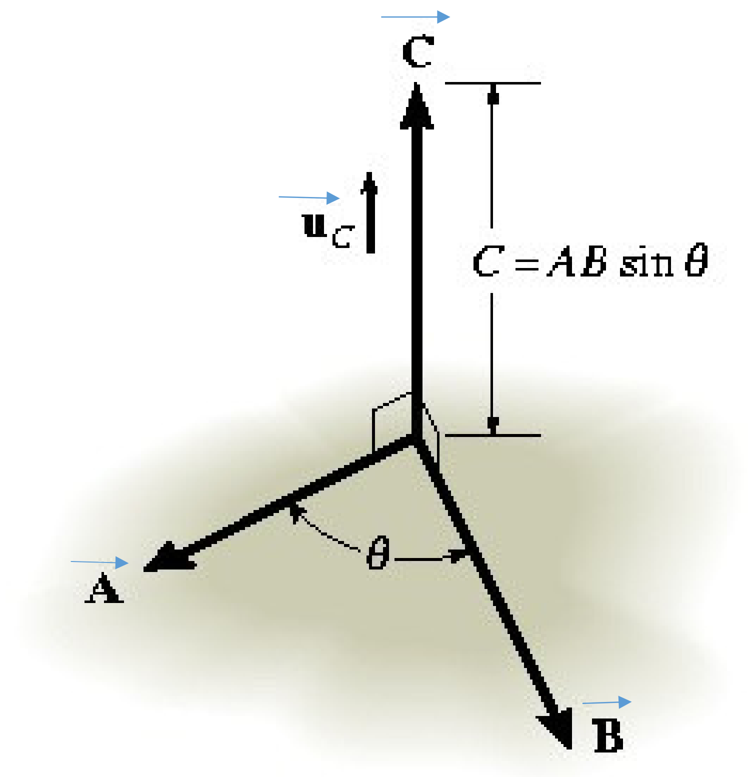
Cross Product

- Expressing vector $\vec{\mathbf{C}}$ when magnitude and direction are known

  $$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}} = (AB\sin\theta)\vec{\mathbf{u}}_C$$

  where scalar $AB\sin\theta$ defines the magnitude of vector $\vec{\mathbf{C}}$

  unit vector $\vec{\mathbf{u}}_C$ defines the direction of vector $\vec{\mathbf{C}}$



Cross Product

Laws of Operations

1. Commutative law is not valid

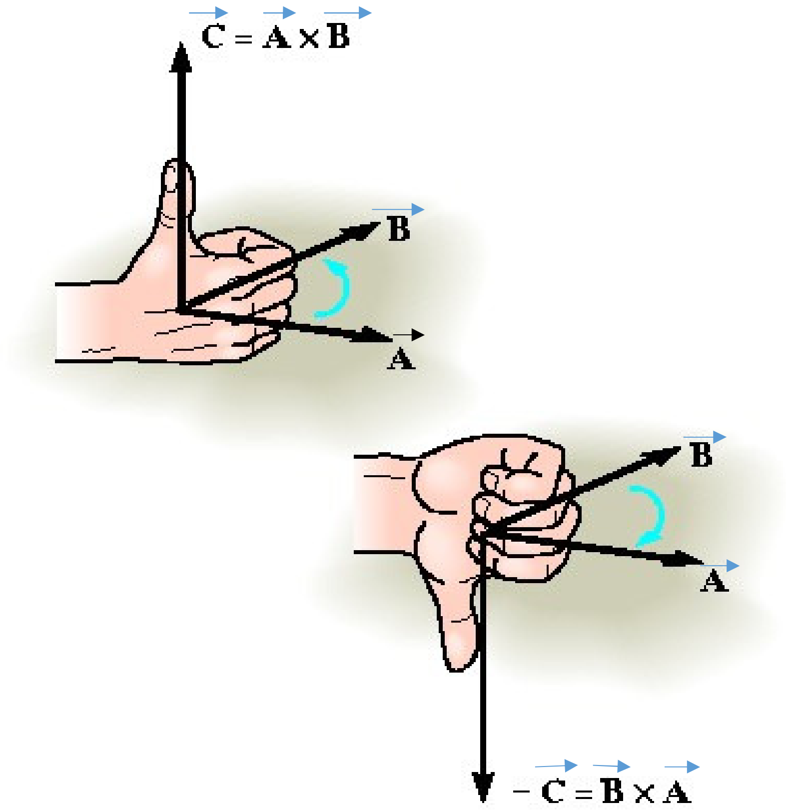
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} \neq \vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

Rather,

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\,\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

- Shown by the right hand rule
- Cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ yields a vector opposite in direction to $\vec{\mathbf{C}}$

$$\vec{\mathbf{B}} \times \vec{\mathbf{A}} = -\vec{\mathbf{C}}$$



Cross Product

Laws of Operations

2. Multiplication by a Scalar

$$a(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = (a\vec{\mathbf{A}}) \times \vec{\mathbf{B}} = \vec{\mathbf{A}} \times (a\vec{\mathbf{B}}) = (\vec{\mathbf{A}} \times \vec{\mathbf{B}})a$$

3. Distributive Law

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{D}}) = (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) + (\vec{\mathbf{A}} \times \vec{\mathbf{D}})$$

- Proper order of the cross product must be maintained since they are not commutative

Cross Product

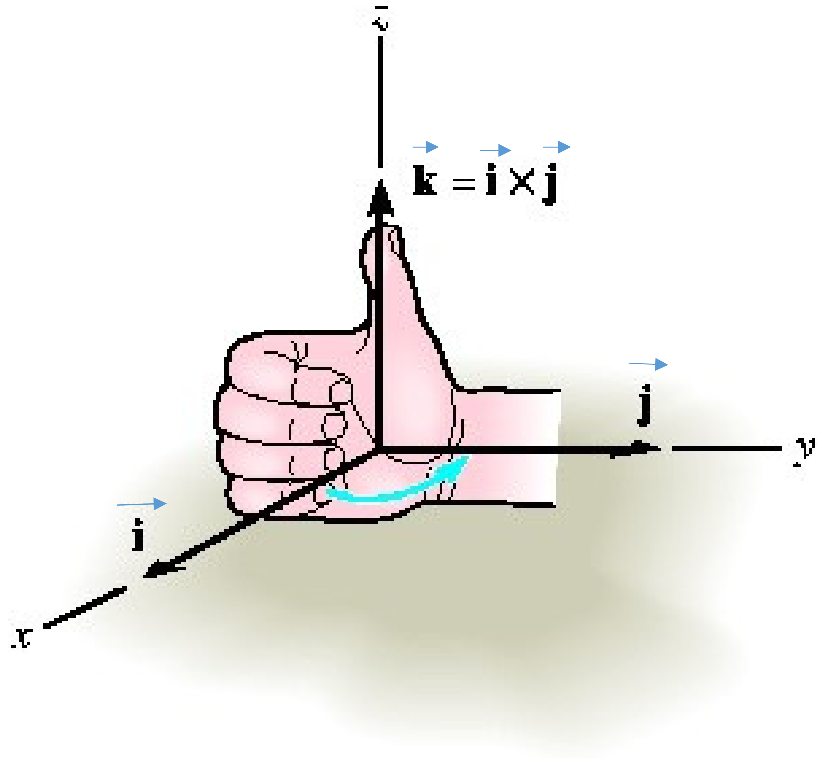
Cartesian Vector Formulation

- Use $C = AB \sin\theta$ on pair of Cartesian unit vectors

Example

For $\vec{\mathbf{i}} \times \vec{\mathbf{j}}$, $(i)(j)(\sin 90^\circ)$

$= (1)(1)(1) = 1$



Cross Product

Laws of Operations

- In a similar manner,

$$\vec{\mathbf{i}} \times \vec{\mathbf{j}} = \vec{\mathbf{k}} \qquad \vec{\mathbf{i}} \times \vec{\mathbf{k}} = -\vec{\mathbf{j}} \qquad \vec{\mathbf{i}} \times \vec{\mathbf{i}} = \mathbf{0}$$

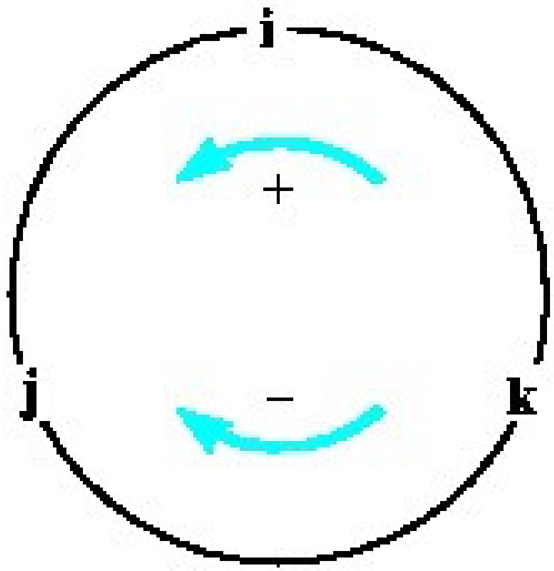
$$\vec{\mathbf{j}} \times \vec{\mathbf{k}} = \vec{\mathbf{i}} \qquad \vec{\mathbf{j}} \times \vec{\mathbf{i}} = -\vec{\mathbf{k}} \qquad \vec{\mathbf{j}} \times \vec{\mathbf{j}} = \mathbf{0}$$

$$\vec{\mathbf{k}} \times \vec{\mathbf{i}} = \vec{\mathbf{j}} \qquad \vec{\mathbf{k}} \times \vec{\mathbf{j}} = -\vec{\mathbf{i}} \qquad \vec{\mathbf{k}} \times \vec{\mathbf{k}} = \mathbf{0}$$

- Use the circle for the results.

  Crossing CCW yield positive

  and CW yields negative results



Cross Product

Laws of Operations

- Consider cross product of vector **A** and **B**

$$\vec{\mathbf{A}} \text{ X } \vec{\mathbf{B}} = (A_x\vec{\mathbf{i}} + A_y\vec{\mathbf{j}} + A_z\vec{\mathbf{k}}) \text{ X } (B_x\vec{\mathbf{i}} + B_y\vec{\mathbf{j}} + B_z\vec{\mathbf{k}})$$

$$= A_xB_x\,(\mathbf{i} \text{ X } \mathbf{i}) + A_xB_y\,(\mathbf{i} \text{ X } \mathbf{j}) + A_xB_z\,(\mathbf{i} \text{ X } \mathbf{k})$$

$$+ A_yB_x\,(\mathbf{j} \text{ X } \mathbf{i}) + A_yB_y\,(\mathbf{j} \text{ X } \mathbf{j}) + A_yB_z\,(\mathbf{j} \text{ X } \mathbf{k}) + A_zB_x\,(\mathbf{k} \text{ X } \mathbf{i}) + A_zB_y\,(\mathbf{k} \text{ X } \mathbf{j}) + A_zB_z\,(\mathbf{k} \text{ X } \mathbf{k})$$

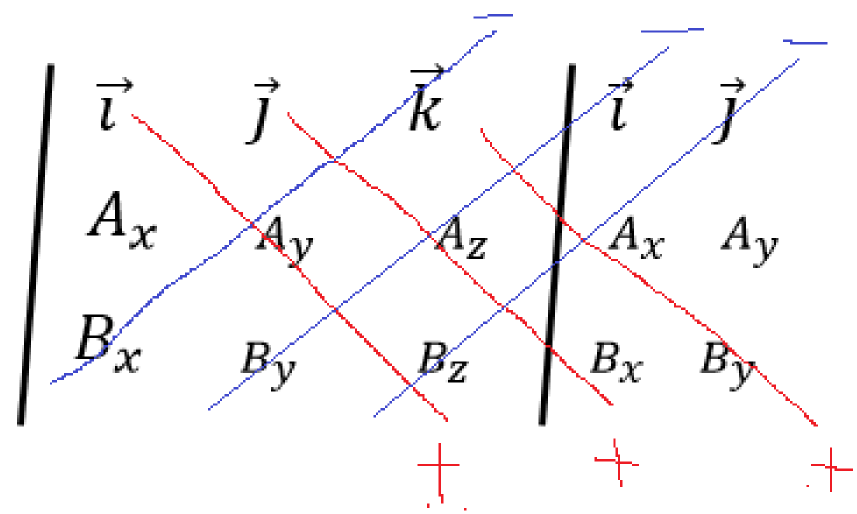
$$= (A_yB_z - A_zB_y)\vec{\mathbf{i}} - (A_xB_z - A_zB_x)\vec{\mathbf{j}} + (A_xB_y - A_yB_x)\vec{\mathbf{k}}$$

Cross Product

Laws of Operations

- In determinant form,

$$\vec{A} X \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} =$$



$$(A_y * B_z)\vec{i} + (A_z * B_x)\vec{j} + (A_x * B_y)\vec{k} - (B_x * A_y)\vec{k} - (B_y * A_z)\vec{i} - (B_z * A_x)\vec{j} =$$

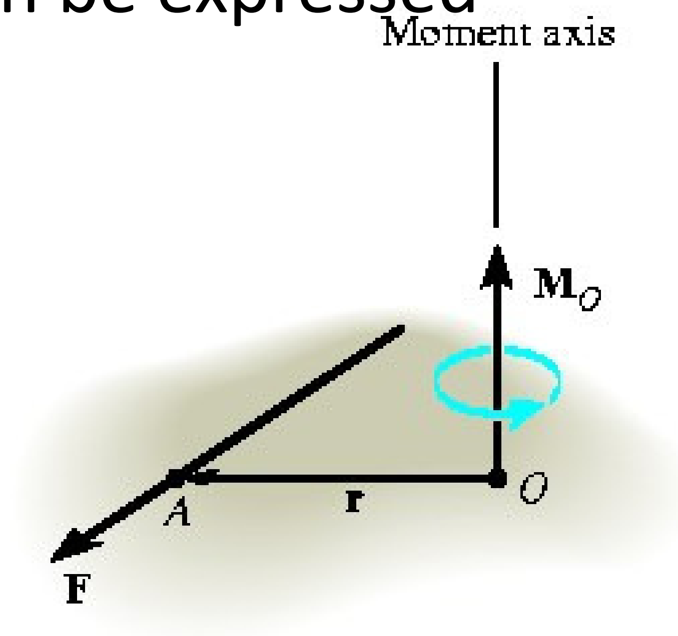
$$=(A_y * B_z - B_y * A_z)\vec{i} + (A_z * B_x - B_z * A_x)\vec{j} + (A_x * B_y - B_x * A_y)\vec{k}$$

Moment of Force - Vector Formulation

- Moment of force $\vec{\mathbf{F}}$ about point O can be expressed using cross product

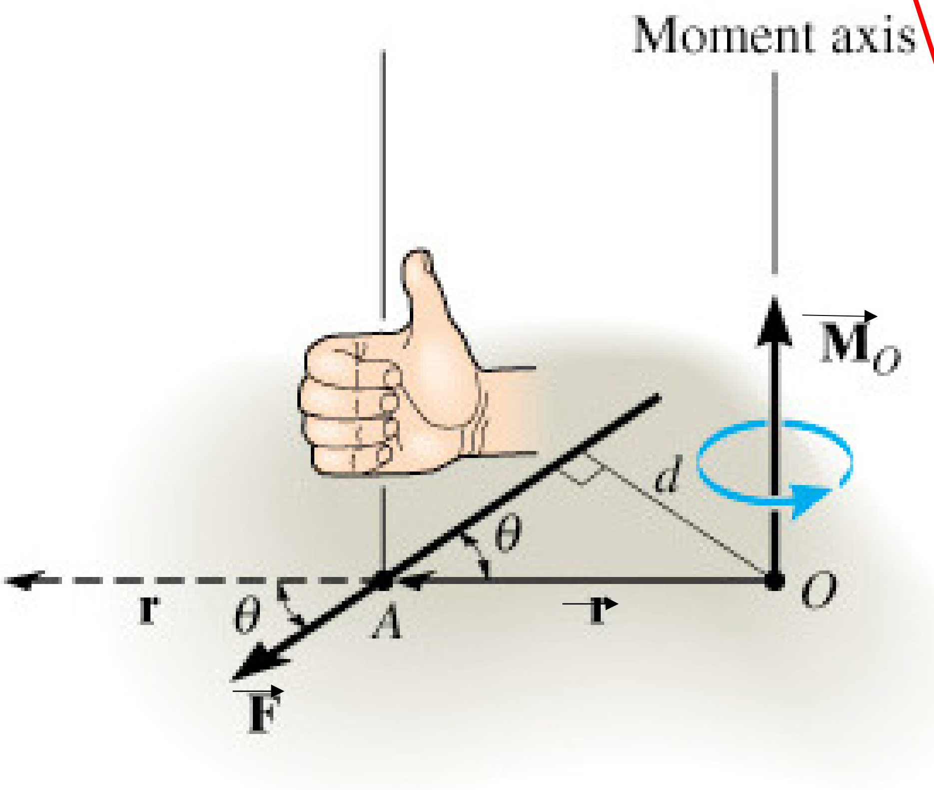
$$\vec{\mathbf{M}}_O = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

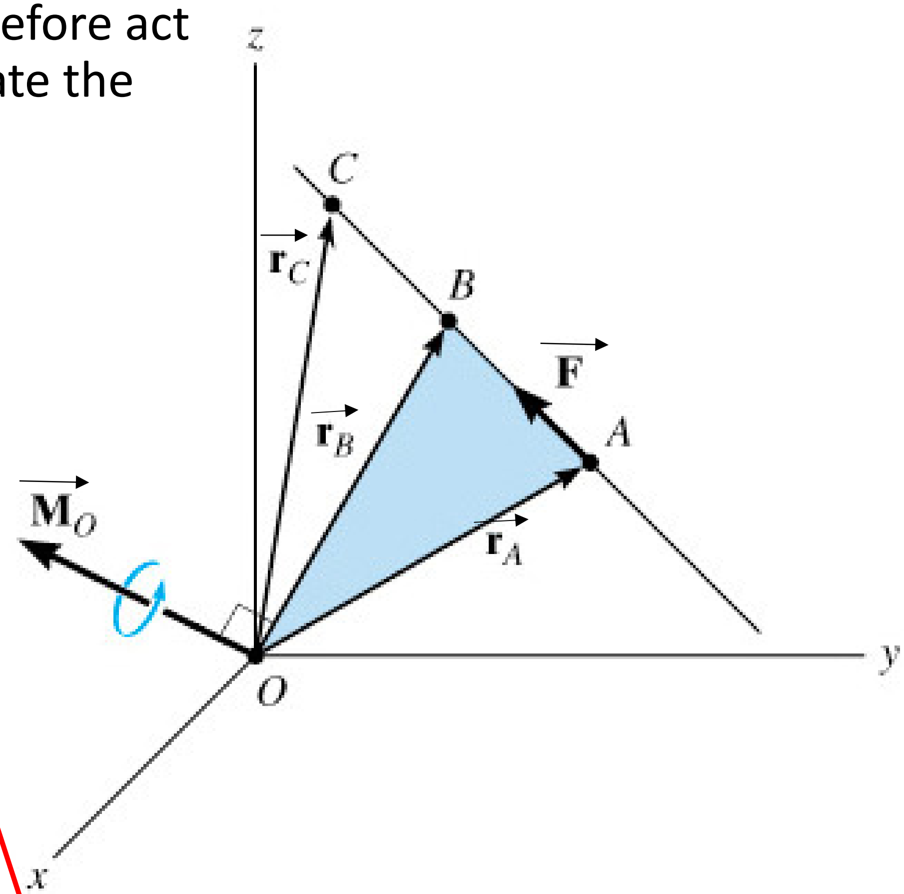
where **r** represents position vector from O to any point lying on the line of action of $\vec{\mathbf{F}}$



(a)

$\overrightarrow{\mathbf{F}}$ has the properties of a sliding vector and therefore act at any point along its line of action and still create the same moment about O. i.e.





$$\overrightarrow{M_o^F} = \overrightarrow{r_A} x \vec{F} = \overrightarrow{r_B} x \vec{F} = \overrightarrow{r_C} x \vec{F}$$

Magnitude

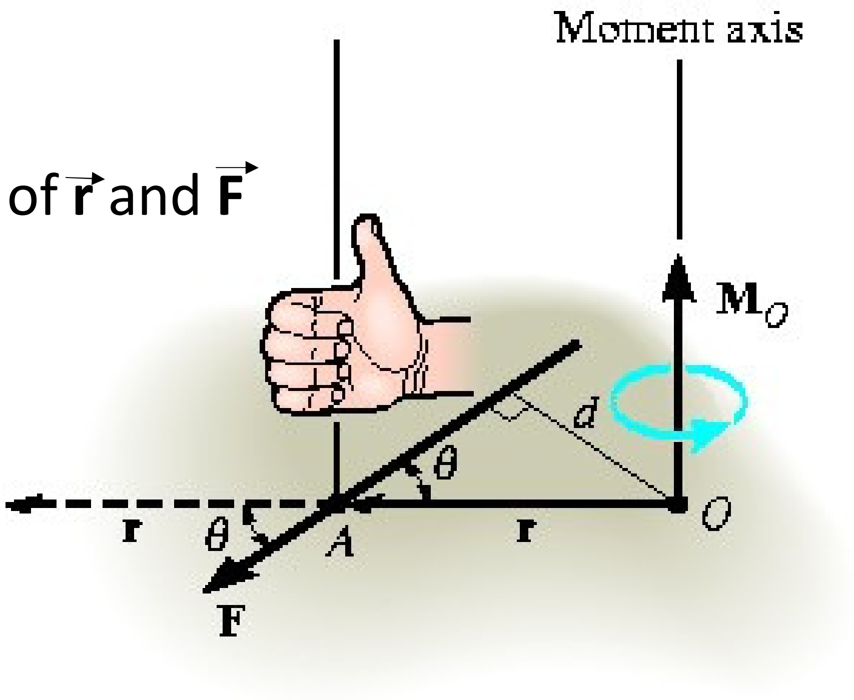
- For magnitude of cross product,

$$M_O = rF \sin\theta$$

where θ is the angle measured between tails of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$

- Treat $\vec{\mathbf{r}}$ as a sliding vector. Since $d = r \sin\theta$,

$$M_O = rF \sin\theta = F\,(r\sin\theta) = Fd$$



Moment of Force - Vector Formulation

Direction

- Direction and sense of $\overrightarrow{\mathbf{M}_O}$ are determined by right-hand rule

  - Extend $\overrightarrow{\mathbf{r}}$ to the dashed position

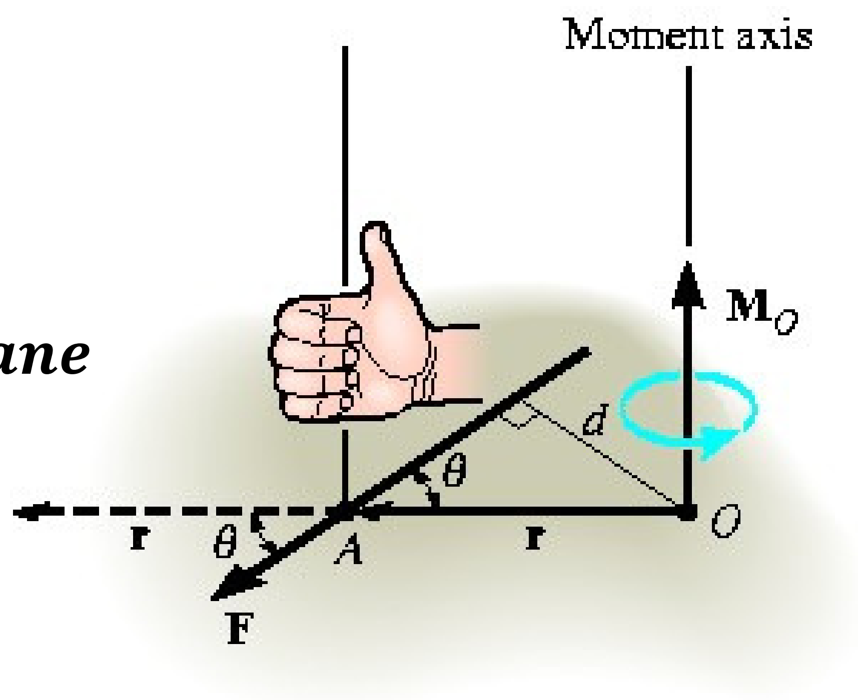
  - Curl fingers from $\overrightarrow{\mathbf{r}}$ towards $\overrightarrow{\mathbf{F}}$

  - Direction of $\overrightarrow{\mathbf{M}_O}$ is the same

  as the direction of the thumb

$\overrightarrow{\boldsymbol{M}_o}$ ***is perpendicular to the plane***

which is defined by $\vec{r}$ ***and*** $\vec{F}$

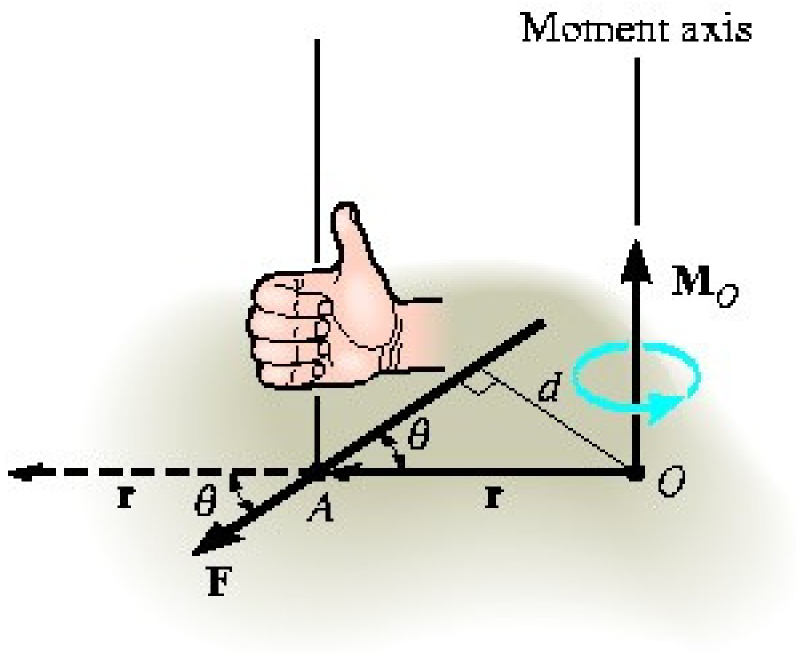


Moment of Force - Vector Formulation

Direction

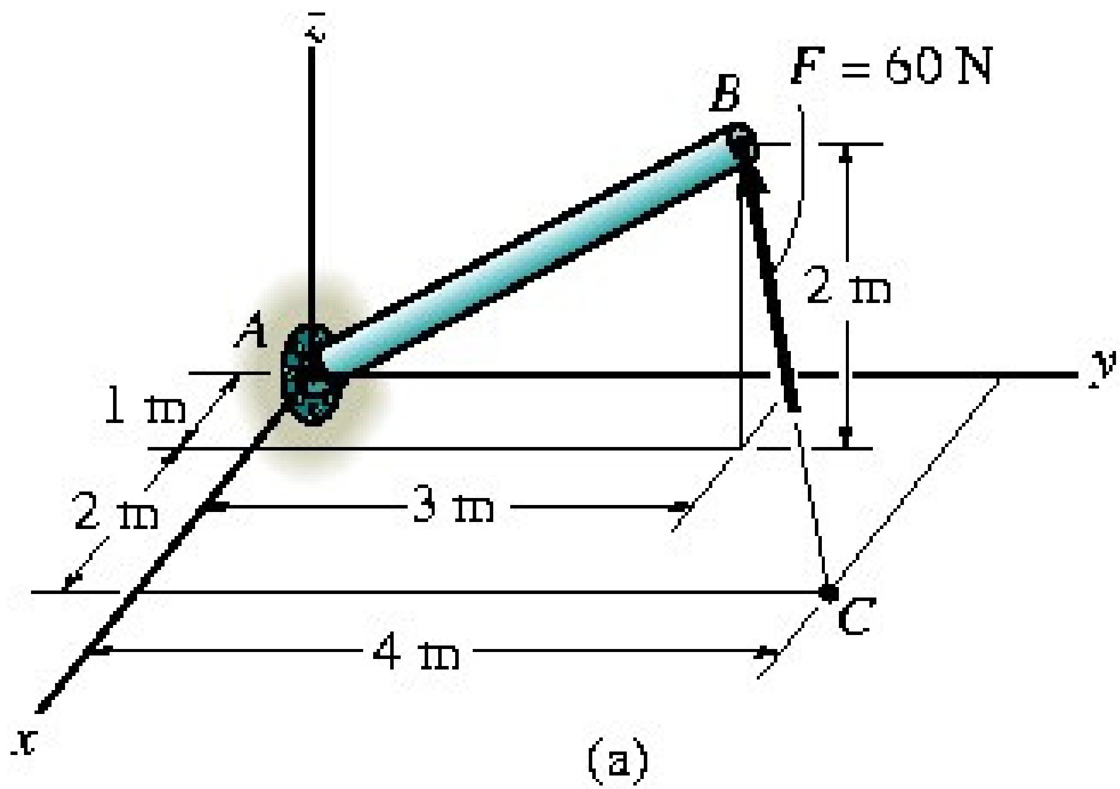
*Note:

- "curl" of the fingers indicates the sense of rotation
- Maintain proper order of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$ since cross product is not commutative



Moment of Force - Vector Formulation

Example The pole is subjected to a 60N force that is directed from C to B. Determine the magnitude of the moment created by this force about the support at A



Moment of Force - Vector Formulation

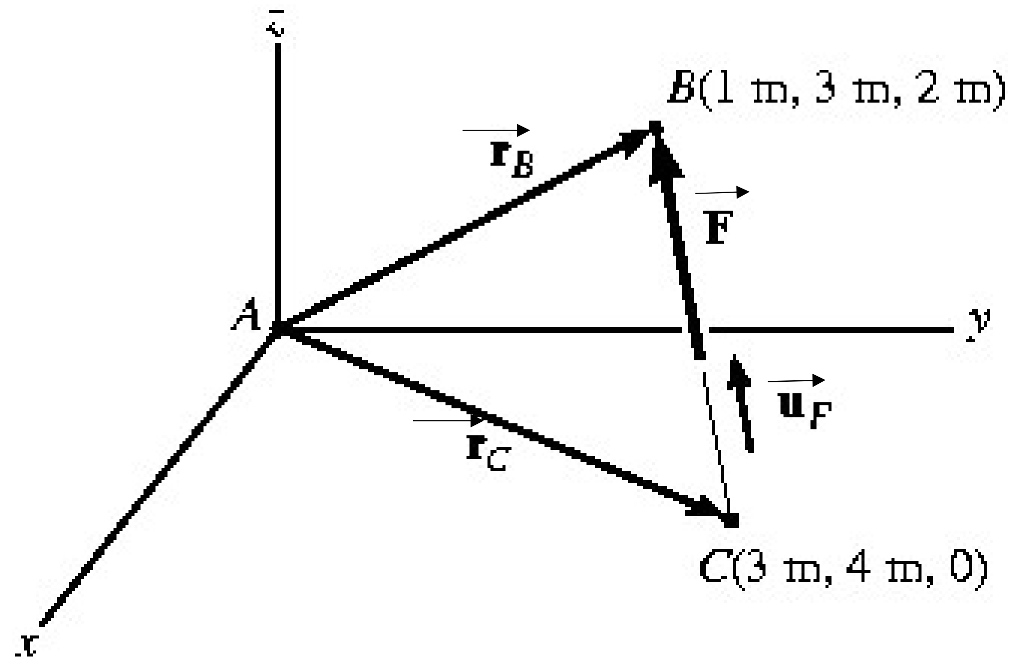
Solution

- Either one of the two position vectors can be used for the solution, since $\vec{\mathbf{M}}_A = \vec{\mathbf{r}}_B \times \vec{\mathbf{F}}$ or $\vec{\mathbf{M}}_A = \vec{\mathbf{r}}_C \times \vec{\mathbf{F}}$
- Position vectors are represented as

  $\vec{\mathbf{r}}_B = \{1\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}\}$ m and

  $\vec{\mathbf{r}}_C = \{3\mathbf{i} + 4\mathbf{j}\}$ m
- Force $\vec{\mathbf{F}}$ has magnitude 60N

  and is directed from C to B



(b)

Moment of Force - Vector Formulation

Solution

$$\vec{F} = (60N)\vec{u}_F$$

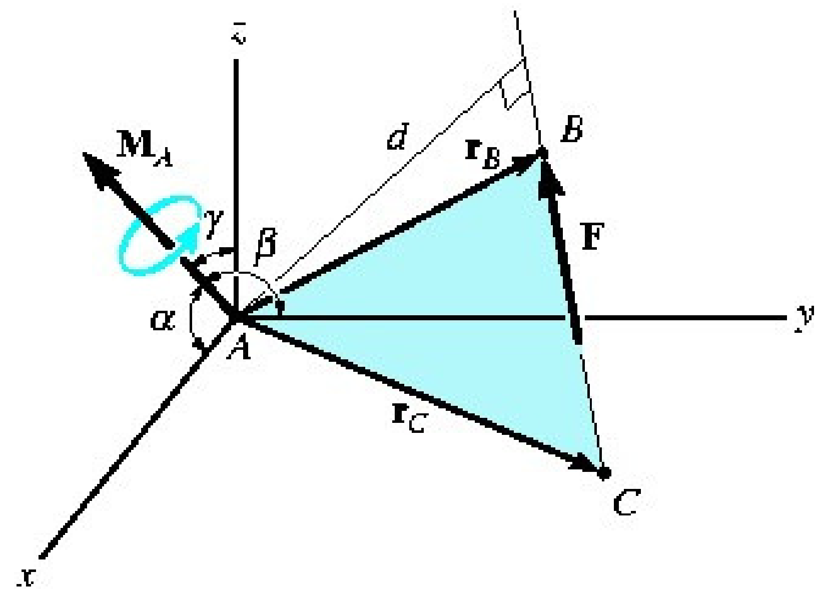
$$= (60N)\left[\frac{(1-3)\vec{i} + (3-4)\vec{j} + (2-0)\vec{k}}{\sqrt{(-2)^2 + (-1)^2 + (2)^2}}\right]$$

$$= \{-40\vec{i} - 20\vec{j} + 40\vec{k}\}N$$

Substitute into determinant formulation

$$\vec{M}_A = \vec{r}_B X \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 2 \\ -40 & -20 & 40 \end{vmatrix}$$

$$= [3(40) - 2(-20)]\vec{i} - [1(40) - 2(-40)]\vec{j} + [1(-20) - 3(40)]\vec{k}$$



(c)

$$\overrightarrow{M_A} = 160\vec{i} - 120\vec{j} + 100\vec{k} \quad [Nm]$$

$$M_A = \sqrt{(160)^2 + (120)^2 + (100)^2} = 224[Nm]$$

Direction angles of $\vec{M}_A$

$$\cos\alpha = \frac{160}{224}\ , \qquad \cos\beta = \frac{-120}{224}\ , \qquad \cos\gamma = \frac{100}{224}$$

Moment of a Force about a Specified Axis

Vector Analysis

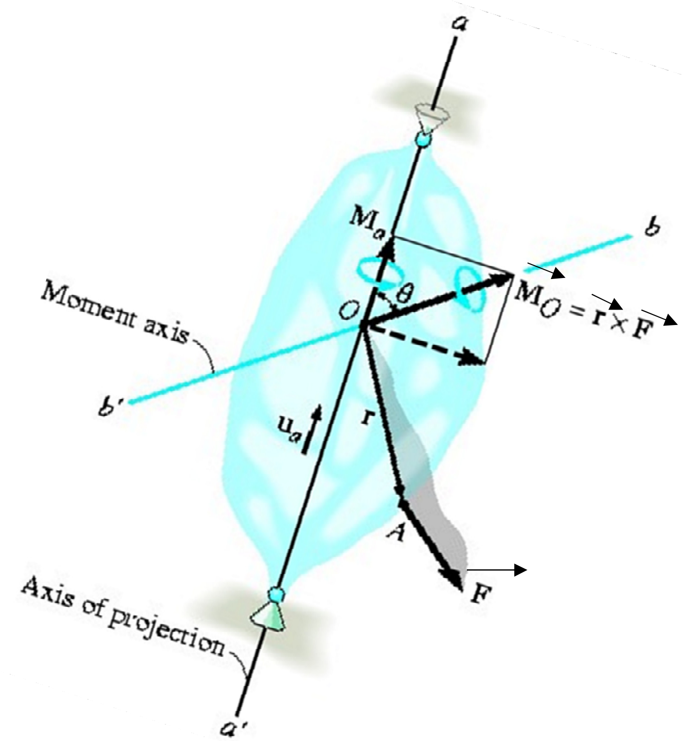
- Consider body subjected to force $\overrightarrow{\mathbf{F}}$ acting at point A
- To determine moment, $\mathbf{M}_{aa'}$, moment with respect to axis aa'

  - For moment of $\overrightarrow{\mathbf{F}}$ about any arbitrary point O that lies on the aa' axis

$$\overrightarrow{\mathbf{M}_O} = \overrightarrow{\boldsymbol{r}_A} \text{ X } \overrightarrow{\mathrm{F}}$$

$$position\ vector\ for\ point\ A, \vec{\mathbf{r}}_{\mathbf{A}}\ is\ with\ respect\ to\ point\ O$$

- $\overrightarrow{\mathbf{M}_O}$ acts along the moment axis bb', so projected $\mathbf{M}_O$ on the aa' axis is $M_{aa'}$



Moment of a Force about a Specified Axis

Vector Analysis

- For magnitude of $\vec{M}_{aa'}$

$$M_{aa'} = \overrightarrow{\mathbf{M_O}}.\overrightarrow{\mathbf{u_{aa'}}}$$

where $\overrightarrow{\mathbf{u_{aa'}}}$ is a unit vector that defines the direction of aa' axis

$$M_A = \vec{\mathbf{u}}_a \cdot (\vec{\mathbf{r}} \times \vec{\mathbf{F}})$$

- In determinant form,

$$M_{aa'} = (u_{ax}\vec{i} + u_{ay}\vec{j} + u_{az}\vec{k}) \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

Moment of a Force about a Specified Axis

Vector Analysis

- Or expressed as,

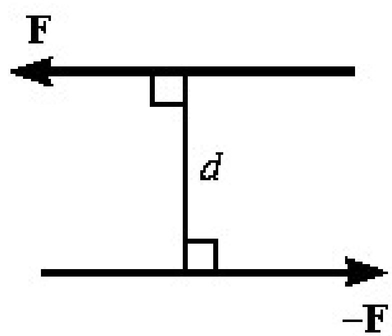
$$M_{aa'} = \vec{u}_{ax} \cdot (\vec{r} X \vec{F}) = \begin{vmatrix} u_{ax} & u_{ay} & u_{az} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

where u_{ax}, u_{ay}, u_{az} represent the x, y, z components of the unit vector defining the direction of aa' axis and r_x, r_y, r_z represent that of the position vector drawn from any point O on the aa' axis and F_x, F_y, F_z represent that of the force vector

Moment of a Couple

- Couple
  - two parallel forces
  - same magnitude but opposite direction
  - separated by perpendicular distance d
- Resultant force = 0
- Tendency to rotate in specified direction
- Couple moment = sum of moments of both couple forces about any arbitrary point

TORQUE



Moment of a Couple

Example

Position vectors $\vec{\mathbf{r}}_A$ and $\vec{\mathbf{r}}_B$ are directed from O to A and B, lying on the line of action of $\vec{\mathbf{F}}$ and $-\vec{\mathbf{F}}$

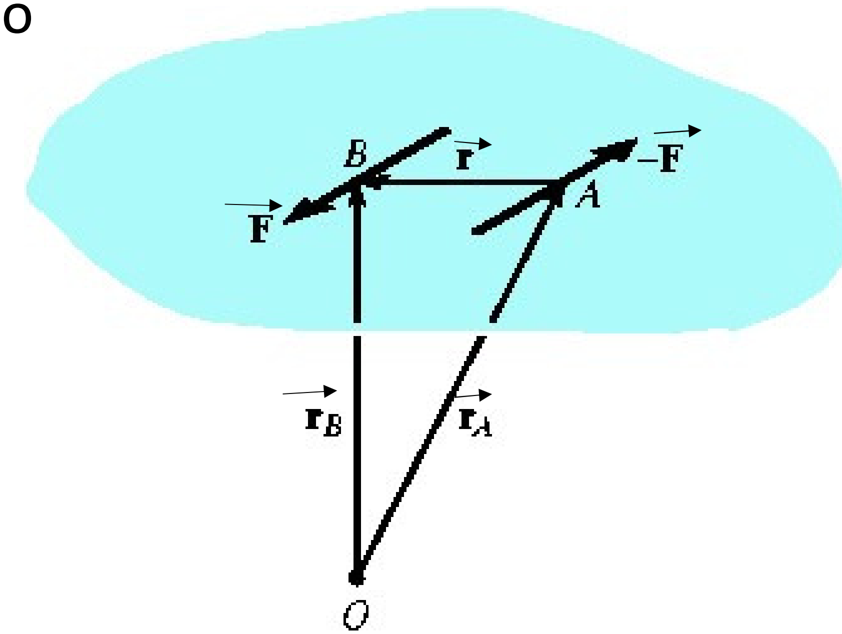
- Couple moment about O

  $\vec{\mathbf{M}} = \vec{\mathbf{r}}_A \text{ X } (-\vec{\mathbf{F}}) + \vec{\mathbf{r}}_A \text{ X } (\vec{\mathbf{F}})$

- Couple moment about A

  $\vec{\mathbf{M}} = \vec{\mathbf{r}} \text{ X } \vec{\mathbf{F}}$

  since moment of –F about A = 0



Moment of a Couple

- A couple moment is a free vector

  - It can act at any point since $\vec{\mathbf{M}}$ depends only on the position vector $\vec{\mathbf{r}}$ directed between forces and not position vectors $\vec{\mathbf{r}}_A$ and $\vec{\mathbf{r}}_B$, directed from O to the forces

  - Unlike moment of force, it do not require a definite point or axis

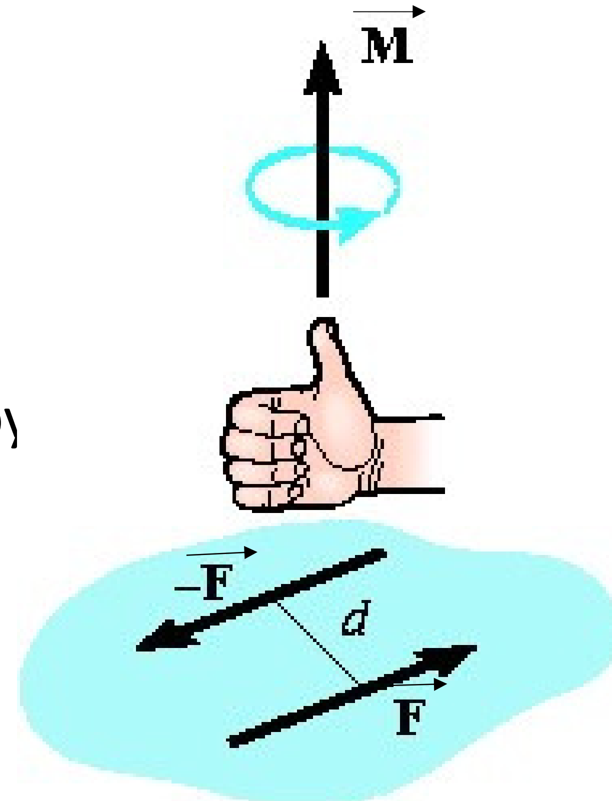
Moment of a Couple

Scalar Formulation

- Magnitude of couple moment

$$M = Fd$$

- Direction and sense are determined by right hand rule
- In all cases, $\vec{\mathbf{M}}$ acts perpendicular to plane containing the forces



Equivalent Couples

- Two couples are equivalent if they produce the same moment
- Since moment produced by the couple is always perpendicular to the plane containing the forces, forces of equal couples either lie on the same plane or plane parallel to one another