Let Π be a connected groupoid.

Let $F : \Pi \to SET$ be a functor. Then for any $b \in \Pi$ $F(b)$ is a $\Pi_b = \Pi(b,b)$-set.

If $\alpha \in \Pi(b,c)$ is a morphism from b to c, then $F(\alpha) : F(b) \to F(c)$ is a bijection.

If $G : \Pi \to SET$ is another functor and φ is a natural transformation then

$$\begin{tikzcd} F(b) \arrow[r, "F(\alpha)"] \arrow[d, "\varphi(b)"'] & F(c) \arrow[d, "\varphi(c)"] \\ G(b) \arrow[r, "G(\alpha)"'] & G(c) \end{tikzcd}$$

is a commutative diagram.

In particular, if $\alpha \in \Pi_b = \Pi(b,b)$, then

$$\begin{tikzcd} F(b) \arrow[r, "F(\alpha)"] \arrow[d, "\varphi(b)"'] & F(b) \arrow[d, "\varphi(b)"] \\ G(b) \arrow[r, "G(\alpha)"'] & G(b) \end{tikzcd}$$

is also commutative.

Hence, the map $\varphi(b) : F(b) \to G(b)$ is Π_b-equivariant.

So we have a functor $\varepsilon_b : [\Pi, SET] \to \Pi_b\text{-}SET$, where

$\Pi_b\text{-}SET$ is the category of Π_b-SET's with morphisms equivariant maps.

Now define a functor κ_b in the opposite direction: Fix some $b \in B$ and let A be a Π_b-set.

$\Phi(A) : \Pi \to SET$ is defined as follows:

For any $x \in B$, let $\Phi(A)(x) = A \times_{\pi} \Pi(b,x)$, where $A \times_{\pi} \Pi(b,x)$ is $A \times \Pi(b,x)$ modulo the relation $(ag,\gamma) \sim (a,g\gamma)$, for any $(a,g,\gamma) \in A \times \pi \times \Pi(b,x)$.

For any $\alpha \in \Pi(x,y)$ morphism in Π let

$$\Phi(A)(\alpha) : A \times_{\pi} \Pi(b,x) \longrightarrow A \times_{\pi} \Pi(b,y)$$

$$[(ag,\gamma)] \longmapsto [(ag,\gamma\alpha)]$$

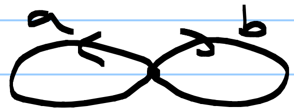
$$b \xrightarrow{\gamma} x \xrightarrow{\alpha} y$$

$$\begin{tikzcd}
(ag,\gamma) \arrow[d] & \sim & (a,g\gamma) \arrow[d] \\
(ag,\gamma\alpha) & \sim & (a,g\gamma\alpha)
\end{tikzcd}$$

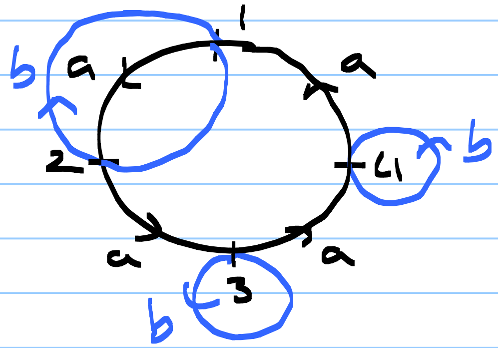
Examples:

1) k-fold covers $X=\bigvee_n S^1$. Since $\pi_1(X)=F_n$ any k-fold cover is determined by an action on F_n on $\{1,2,\dots,k\}$ i.e. a homomorphism $F_n \to S_k$.

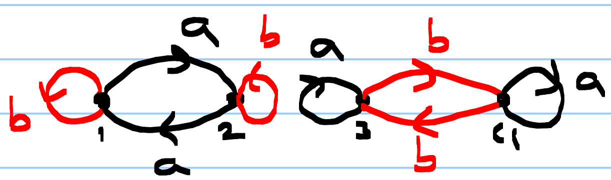
For example, let $n=2$ and $k=4$.

$F_2=\langle a,b\rangle$, X: 

i) $F_2 \to S_4$, $a \mapsto (1234)$, $b \mapsto (12)$

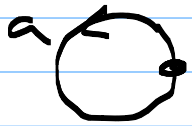
$\tilde{X}$: 

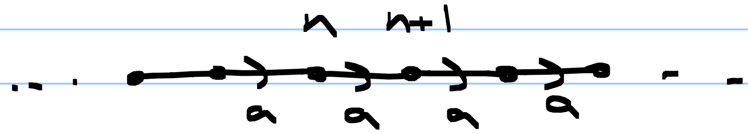
ii) $F_2 \to S_4$, $a \mapsto (12)$, $b \mapsto (34)$

$\tilde{X}$: 

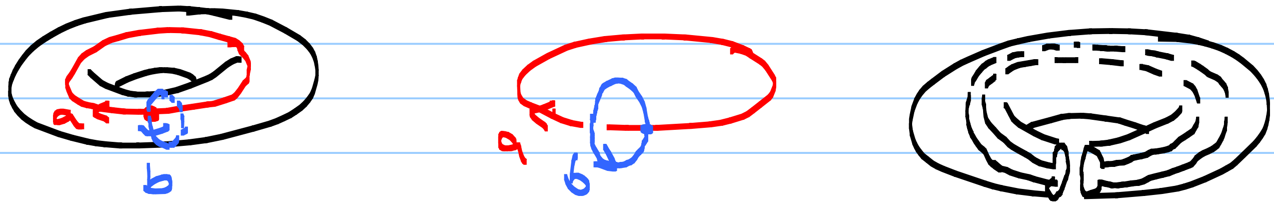
2) $X=S^1$, $\pi_1(X)\cong\mathbb{Z}$, $\mathbb{Z}$ acts on $\mathbb{Z}$ as

$n\cdot k = n+k$.





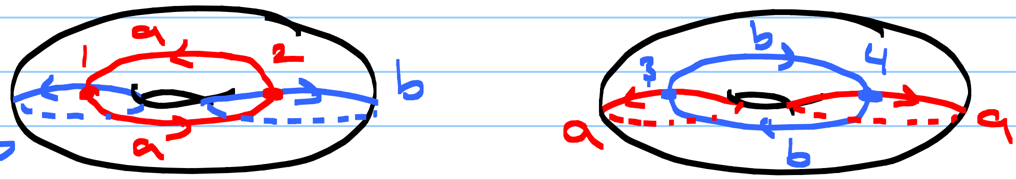
3) $X = T^2 = S^1 \times S^1$, $\pi_1(X) \simeq \mathbb{Z} \times \mathbb{Z}$



i) 4-fold cover of T^2:

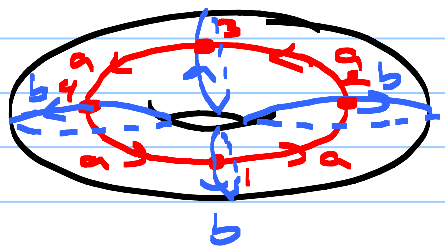
$\pi_1(T^2) \simeq \mathbb{Z} \times \mathbb{Z} \longrightarrow S_4$, $a \longmapsto (12)$, $b \longmapsto (34)$

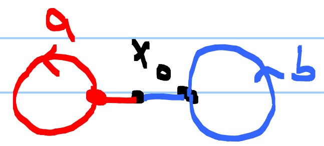
$\tilde{X}$:



ii) $\pi_1(T^2) \simeq \mathbb{Z} \times \mathbb{Z} \longrightarrow S_4$ $a \mapsto (1234)$, $b \mapsto e$

$\tilde{X}$

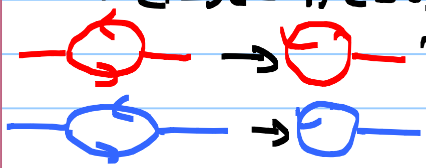


4) A normal S_3-covering of $X =$ 

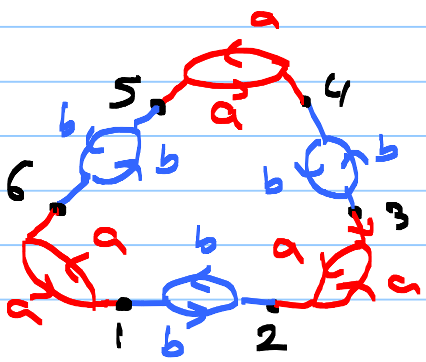
$\pi_1(X) \simeq F_2 = \langle a, b \rangle$ let the action be given by

$a \mapsto (16)(23)(45)$

$b \mapsto (12)(34)(56)$



} double covers



$\tilde{X} \to X$
6-fold
regular
covering.

__Fact:__ For a transport simple space B the equivalence $COV_B \to \pi_b\text{-}SET$ show that every covering is determined by an action of $\pi_b = \pi(B, b)$ on some set. Connected covers corresponds to transitive actions.

Finally, it is clear that every transitive action is of the form

$$G \times X \longrightarrow X, \text{ where } X = \{Hg \mid g \in G\} \text{ for some}$$

subgroup $H \leq G$. Moreover, such actions, up to isomorphism, are determined by the conjugacy class of H in G.

Any transitive action of a group G is determined by the conjugacy class of the stabilizer of any element:

$$G \times X \longrightarrow X, \qquad H = \mathrm{Stab}_G(x).$$

Moreover, in this case we may take $X = \{Hg \mid g \in G\}$ the set of all right cosets of H and the action of G on X is short right multiplication.

Note that for any points $x, y \in X$ if $y = gx$ then

$$\mathrm{Stab}_G(x) = g\, \mathrm{Stab}_G(y)\, g^{-1}$$

For a connected covering $p: E \to B$, fix some $x \in E$, $b = p(x)$. Then this cover is uniquely determined by the action of $G = \pi_1(B, b)$ on the fiber $F_x = p^{-1}(b)$, namely the $\mathrm{Stab}_G(x) = H = p_* \pi_1(E, x)$.

The equivalence of categories implies that any automorphism of $p: E \to B$ (i.e. a deck transformation) is an automorphism of the group action $G \times X \to X$.

It is easy to see that automorphism group is N_H/H, where N is the normalizer of H in G (note that as in the case of coverings, any automorphism is determined by its action on a single element).

Automorphism of Group Actions

Let $\varphi: G \times X \to X$ be a transitive action. Let $H = \mathrm{Stab}_G(x_0)$ of some point $x_0 \in X$. Then $\mathrm{Aut}(\varphi) \simeq N/H$, where N is the normalizer of H in G.

Proof: Let $\phi: X \to X$ be an automorphism of φ. Then $\varphi(g, \phi(x)) = \phi(\varphi(g,x))$, for all $(g,x) \in G \times X$.

In particular, $g\,\phi(x_0) = \phi(g x_0)$, for all $g \in G$.

Let $g \in H = \mathrm{Stab}_G(x_0)$. Then $g\,\phi(x_0) = \phi(g x_0) = \phi(x_0)$ so that $g \in \mathrm{Stab}_G(\phi(x_0)) = g_1 H g_1^{-1}$, where $\phi(x_0) = g_1 x_0$. This shows that $H \subseteq g_1 H g_1^{-1}$.

Similarly, if $g \in \mathrm{Stab}_G(\phi(x_0)) = g_1 H g_1^{-1}$, then $g\phi(x_0) = \phi(x_0)$.

So, $\phi(x_0) = g\phi(x_0) = \phi(gx_0)$ and hence $x_0 = gx_0$ because ϕ is a bijection. So, $g \in Stab_G(x_0) = H$.

Hence, $H = g_1 H g_1^{-1}$ or equivalently, $g_1 \in N_G(H)$.

Let $X = \{Hg \mid g \in G\}$ be the set of right cosets of H, and $\varphi(g_1, Hg_2) = Hg_2 g_1$, for any $g_1, g_2 \in G$, the usual action.

For any $g \in N_G(H)$, define $\phi_g : X \to X$, $Hg_1 \mapsto g^{-1}Hg_1$.

<u>Claim:</u> ϕ_g is an automorphism of the action φ.

$$\begin{aligned}\varphi(g_2, \phi_g(Hg_1)) &= \varphi(g_2, g^{-1}Hg_1) \\ &= g^{-1}Hg_1g_2 \\ &= g^{-1}\varphi(g_2, Hg_1) \\ &= \phi_g(\varphi(g_2, Hg_1)).\end{aligned}$$

(Of course, one has to first notice that ϕ_g is a well-defined map of X to itself, because

$$\phi_g(Hg_1) = g^{-1}Hg_1 = Hg^{-1}g_1 \in X$$

is a right coset, for any $g_1 \in G$ and $g \in N$.

The above arguments prove that

$$Aut(\varphi) \cong N/H = N_G(H)/H.$$

Monodromy and Regular Coverings:

Let $P: \tilde{X} \to X$ be a (connected) covering map. Fix some $x_0 \in X$ and let $F_{x_0} = P^{-1}(x_0)$ be the fiber over x_0.

For any choice $\tilde{x}_0 \in F_{x_0}$ consider the monodromy

$\varphi_{\tilde{x}_0}: \pi_1(X, x_0) \to F_{x_0}$, given by $\varphi([\gamma]) = \tilde{\gamma}(1)$, for any $[\gamma] \in \pi_1(X, x_0)$, where $\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}_0$.

This defines a homomorphism

$$\varphi: \pi_1(X, x_0) \to \mathrm{Aut}(F_{x_0}) = \mathrm{Per}(F_{x_0})$$
$$\gamma \longmapsto (\varphi_{\tilde{x}}[\gamma])_{\tilde{x} \in F_{x_0}}$$

whose kernel is $\bigcap_{g \in \pi_1(X, x_0)} H^g$. Hence, it follows that

if the covering is normal then $H \triangleleft \pi_1(X, x_0)$ and hence $\ker \varphi = H$.

Clearly, the converse is also true: If $\ker \varphi = H$ then the covering is normal.

The arguments on page 9 show that

$\mathrm{Aut}(P: \tilde{X} \to X) = N/H$, when $N = N_G(H)$.

In particular, if the covering is normal then

$\mathrm{Aut}(P: \tilde{X} \to X) \simeq \pi_1(X, x_0)/H \simeq \mathrm{Im}\,\varphi$, where φ is the monodromy homomorphism.

Note that there is a also natural injective map $\psi : Aut(P: \tilde{X} \to X) \longrightarrow Per(F_{x_0})$. For a general covering $Im\,\psi \simeq NH/H \leq Per(F_{x_0})$ and $Im\,\varphi \simeq \pi_1(X, x_0) / \bigcap_{\gamma \in \pi_1(X,x_0)} H^{\gamma}$ and thus, in general, $Im\,\varphi$ is "bigger" than $Im\,\psi \simeq Aut(P: \tilde{X} \to X)$.

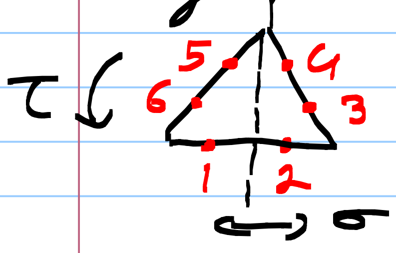
If the covering is normal then $NH = \pi_1(X, x_0)$ and $\bigcap_{g \in \pi_1(X,x_0)} H^{g} = H$ so that $Im\,\varphi \simeq Im\,\psi$.

However, even in this case as subgroups of $Per(F_{x_0})$ $Im\,\varphi$ and $Im\,\psi$ need not to be the same.

Example: The Example 4 exhibits the 6-fold regular cover $P: \tilde{X} \to X$, where $\pi_1(X, x_0) \simeq F_2$ and $Aut(P) \simeq S_3$.

Recall that $\varphi : \pi_1(X, x_0) \longrightarrow S_6$ has image $\langle \varphi(a), \varphi(b) \rangle \simeq S_3 \leq S_6$, where $\varphi(a) = (16)(23)(45)$ and $\varphi(b) = (12)(34)(56)$.

On the other hand, $\psi : Aut(P) \longrightarrow S_6$ has image $\langle \psi(\tau), \psi(\sigma) \rangle = \langle (135)(246), (12)(36)(45) \rangle$.



The images $\varphi(a)$ and $\varphi(b)$ are both non-geometric and thus $Im(\varphi)$ and $Im\,\psi$ are not the same subgroups.

Here τ is the $2\pi/3$-radian counterclockwise rotation and σ is the reflection with respect to the y-axis.

Outer Automorphisms of F_n.

Let K be a finite group of order n, generated by r elements. Then there is an exact sequence of groups:

$$0 \to F_s \to F_r \to K \to 0,$$ where

$$s-1 = k(r-1) \text{ and hence } s = 1+k(r-1).$$

Example: If $r=2$ and $K = D_{2n}$, the dihedral group of order $2n$, then $s = 1+2n$.

Claim: The canonical map $K \to Aut(F_s) \to Out(F_s)$

is injective provided that $r \geq 2$.

Proof: If some $g \in G$ acts trivial on F_s (by conjugation) then $g \in C_{F_r}(F_s)$, the centralizer of F_s in F_r.

In a free group two elements commute if and only if they are powers of a common element. This implies that F_s is cyclic, which is not possible if $r \geq 2$.
This finishes the proof. ∎

Example: Both $\mathbb{Z}_6$ and $D_6 = S_3$ are groups of order 6 generated by two elements. Hence, $Out(F_7)$ contains $\mathbb{Z}_6$ and S_3 as subgroups.