TRACE HOMOMORPHISM FOR SMOOTH MANIFOLDS

YILDIRAY OZAN

ABSTRACT. In this note we prove the following result: Let G be a group acting on a smooth manifold M and let $E \to M$ be a G-equivariant bundle. If $k \geq 1$ then the image of the trace map

$$H_k(G) \times H_l(X) \stackrel{tr_*}{\to} H_{k+l}(X)$$

annihilates any characteristic class of the bundle E. We also give several corollaries of the theorem.

1. Introduction and the results

Let G be any topological group acting on a topological space X and R any commutative ring. We define the trace homomorphism,

$$H_k(G) \times H_l(X) \stackrel{tr_*}{\to} H_{k+l}(X),$$

corresponding to this action as follows: if $\phi : U \to G$ and $\sigma : A \to X$ are cycles in G and X of degrees k and l representing classes υ, α, respectively, let $tr_*(\upsilon, \alpha)$ be the class represented by the homology cycle $(u, a) \mapsto \phi(u)(a)$, $(u, a) \in U \times A$.

In 2003, it is proved in [1, 2] that the trace homomorphism of the Hamiltonian group of a closed symplectic manifold (M, ω) on the rational homology of M,

$$H_k(\mathrm{Ham}(M, \omega), \mathbb{Q}) \times H_l(M, \mathbb{Q}) \stackrel{tr_*}{\to} H_{k+l}(M, \mathbb{Q}),$$

is trivial, for $k \geq 1$. Inspired by this result we prove the following

Theorem 1.1. *Let G be a group acting on a smooth manifold M and let $E \to M$ be a G-equivariant bundle. If $k \geq 1$ then the image of the trace map*

$$H_k(G) \times H_l(X) \stackrel{tr_*}{\to} H_{k+l}(X)$$

annihilates any characteristic class of the bundle E.

Some immediate corollaries of the above result are as follows.

1991 *Mathematics Subject Classification.* Primary 57S05. Secondary 53D35.

Key words and phrases. Trace homomorphism, smooth manifold.

The author is partially supported by the Turkish Academy of Sciences (TUBA-GEBIP-2004-17).

Corollary 1.2. Let G be any subgroup of the group of all diffeomorphisms of M, Diff(M). Then the frame bundle $P \to M$ is G-equivariant and hence every characteristic class annihilates the trace homomorphism.

For the next corollary, let Γ be a geometric structure on M, i.e., a reduction of the frame bundle $P \to M$. For example, Γ can be induced by a foliation (suggested by Kotschich), a volume form, a symplectic form, an almost complex structure or a Riemannian metric. Then, we have

Corollary 1.3. Let Γ be as above and G the subgroup of Diff(M) preserving the geometric structure Γ. Then Γ is clearly G-equivariant and hence every characteristic class of Γ annihilates the trace homomorphism.

We will finish this section with the following corollary: Let G be any subgroup of Diff(M). Since Diff(M) acts transitively on M we have so called the evaluation bundle

$$\mathrm{Diff}(M, x) \to \mathrm{Diff}(M) \to M,$$

which is clearly G-equivariant. Let $f : M \to B\mathrm{Diff}(M, x)$ be a classifying map of this bundle. Since any class in the image of $f^* : H^*(B\mathrm{Diff}(M, x)) \to H^*(M)$ is a characteristic class of this bundle, we obtain

Corollary 1.4. Let M and G be as in the above paragraph. Then every class in the image of $f^* : H^*(B\mathrm{Diff}(M, x)) \to H^*(M)$ annihilates the trace homomorphism.

2. Proof of the Theorem

To prove the above theorem we need to recall the definition and some basic properties of equivariant bundles: Let G be any Lie group and $F \to E \xrightarrow{\pi} B$ a fiber bundle. If G acts on both E and B such that the projection map π is G-equivariant; i.e., $\pi(v \cdot g) = \pi(v) \cdot g$, for all $g \in G$ and $v \in E$, we say that the bundle is G-equivariant. Note that if X is also a G-space and $f : X \to B$ is a G-equivariant map then the pullback bundle has an induced G-equivariant structure.

Example 2.1. **i)** Let $F \to E \xrightarrow{\pi} B$ be a G-equivariant fiber bundle, where the action of G on B, and hence on E, is free. Taking quotients of both the total space and the base by G, we get another fiber bundle $F \to E/G \xrightarrow{\tilde{\pi}} B/G$, whose pullback via the quotient map $p : B \to B/G$ is isomorphic to the bundle $F \to E \xrightarrow{\pi} B$.

ii) Let M be a smooth manifold and G be any subgroup of the group of all diffeomorphisms of M, Diff(M). Since any diffeomorphism of M, $\phi : M \to M$, extends to the tangent bundle $\phi_* : T_*M \to T_*M$, we see that the tangent bundle is G-equivariant.

Proof of Theorem 1.1. Consider the trace map

$$tr : G \times M \to M, \ \ (g, x) \mapsto x \cdot g, \ \text{for all } (g, x) \in G \times M.$$

To prove the theorem it suffices to show that $tr^*(c(E)) = 0$, for any characteristic class $c(E)$, of degree $l + k$, of the bundle $E \to M$.

Note that G acts on $M \times G$ by right multiplication on the second factor, which makes the trace map G-equivariant. By assumption the bundle $E \to M$ is G-equivariant and hence the pullback bundle $tr^*(E) \to M \times G$ is G-equivariant. Since the G-action on the base space $M \times G$ is free, this bundle is induced from the quotient bundle $tr^*(E)/G \to (M \times G)/G$, which is isomorphic to $\pi^*(E)$. Hence, $tr^*(E) = \pi^*(E)$.

Now for any classes $\alpha \in H_k(G)$ and $\beta \in H_l(M)$, we have $c(E)(tr_*(\alpha, \beta)) = tr^*(c(E))(\alpha, \beta) = \pi^*(c(E))(\alpha, \beta) = c(E)(\pi_*(\alpha, \beta)) = 0$, since $k \geq 1$. □

3. Applications

3.1. **The tori T^n.** The tangent bundle and hence the frame bundle of the n-tori T^n is trivial. So there is no characteristic class which will annihilate the trace homomorphism. Indeed, since T^n is naturally contained in Diff(M), by the Künneth formula the trace homomorphism is nontrivial.

3.2. **Triviality of the trace homomorphism.** Let M be a closed connected smooth manifold, $G = \text{Diff}(M)$ and R denote the either field $\mathbb{Z}_2$ or $\mathbb{Q}$. Also let P denote the subalgebra of the cohomology algebra $H^*(M, R)$, generated by the Stiefel-Whitney classes $w_i(M)$, if $R = \mathbb{Z}_2$, and the subalgebra generated by the Pontryagin classes $p_i(M)$ and the Euler class $e(M)$, if $R = \mathbb{Q}$. If M is such that P is the whole of the cohomology algebra $H^*(M, R)$ then the trace homomorphism must be trivial for the group G.

Note that even dimensional spheres and some real projective spaces satisfies these conditions.

The analogous result holds if M has an almost complex structure, G is the subgroup of Diff(M) that consists of those diffeomorphisms preserving the almost complex structure and P is the subalgebra of the rational cohomology of M generated by the Chern classes of the complex structure. For example, the rational cohomology of the complex projective space $\mathbb{C}P^n$ is generated by its Chern classes and hence the trace homomorphism must be trivial.

3.3. **Trace homomorphism on cohomology.** For $R = \mathbb{Z}_2$ or $\mathbb{Q}$ we have $H^p(M, R) = \text{Hom}(H_p(M, R), R)$ and using this duality we may define trace homomorphism in cohomology: Let $u \in H_k(\text{Diff}_0(M), R)$ and define

$$tr_u^* : H^p(M, R) \to H^{p-k}(M, R)$$

by the formula $a \mapsto tr_u^*(a)$, $a \in H^p(M,R)$, where

$$tr_u^*(a) : H_{p-k}(M,R) \to R,\ tr_u^*(a)(\alpha) = a(tr_*(u,\alpha)),\ \alpha \in H_{p-k}(M,R).$$

Hence, the conclusion of Theorem 1.1 can be written as $tr_u^*(P) = 0$, for all $u \in H_k(\mathrm{Diff}_0(M), R)$, $k \geq 1$.

Suppose that $u \in H_k(\mathrm{Diff}_0(M), R)$ is of the form $u = \psi_*([S^k])$, where $\psi : S^k \to \mathrm{Diff}_0(M)$ is a smooth map. Using this cycle representing u we can build a fiber bundle $M \to E \to S^{k+1}$, such that the connecting homomorphism in the Wang sequence corresponding to this bundle is nothing but the trace homomorphism:

$$\to H^{p-1}(E,R) \to H^{p-1}(M,R) \stackrel{tr_u^*}{\to} H^{p-k-1}(M,R) \to H^p(E,R) \to$$

It is well known that the connecting homomorphism in the Wang sequence is a derivation of degree k ([4]). In other words, for any $x, y \in H^*(M,R)$,

$$tr_u^*(xy) = tr_u^*(x)\ y + (-1)^{k\deg(x)} x\ tr_u^*(y).$$

On the other hand, for general u, since $\mathrm{Diff}_0(M)$ is an H-space any rational homology class is a product of rational homotopy classes (cf. see Section 5 of [1]) and therefore tr_u is the composition of the trace homomorphisms corresponding to the factors of u, in the factorization of u as a product of rational homotopy classes. Hence, we obtain the following result:

Proposition 3.1. Let $u \in H_k(\mathrm{Diff}_0(M), \mathbb{Q})$, $k > 0$. For any cohomology classes $x, y \in H^*(M,\mathbb{Q})$ such that $y \in P$ (hence $tr_u^*(y) = 0$) we have $tr_u^*(xy) = tr_u^*(x)y$. Moreover, if $\deg(x) < k$ then $tr_u^*(xy) = 0$.

The above proposition yields the following corollary:

Corollary 3.2. The natural map

$$tr^* : H_k(\mathrm{Diff}_0(M), \mathbb{Q}) \to \hom_P(H^*(M,\mathbb{Q}), H^{*-k}(M,\mathbb{Q}))$$

is a homomorphism, where we regard $H^*(M,\mathbb{Q})$ as a right module over its subalgebra P and $\hom_P(H^*(M,\mathbb{Q}), H^{*-k}(M,\mathbb{Q}))$ denotes the group of P-module homomorphisms.

Example 3.3. Let u be as in the complex analog of the above proposition, where P is generated by the Chern classes of the almost complex manifold (M,J) and u belongs to $H_k(\mathrm{Diff}_0(M,J), R)$. Assume that (M,ω) is a monotone closed symplectic manifold of dimension $2n$. So $[\omega]$ is a multiple of $c_1(M)$ and hence it is in P. Assume further that M has the Hard Lefschetz Property, i.e.,

$$\cup\ [\omega]^r : H^{n-r}(M,\mathbb{C}) \to H^{n+r}(M,\mathbb{C})$$

is an isomorphism for any $r \geq 0$. So, if $b \in H^{n+r}(M,\mathbb{C})$ then $b = a\ [\omega]^r$ for some $a \in H^{n-r}(M,\mathbb{C})$ and hence

$$tr_u^*(b) = tr_u^*(a\ [\omega]^r) = tr_u^*(a)\ [\omega]^r.$$

In particular, $tr_u^*([\omega]^r) = 0$. It follows that, if $k > n$ then $tr_u^* = 0$.

Acknowledgment. The author would like to thank Dusa McDuff for pointing out some errors in the earlier version of the article and to Dieter Kotschich for his remarks and suggestions. Finally, I would like to thank the anonymous referee whose suggestions improved the paper substantially.

References

1. F. Lalonde, D. McDuff, *Symplectic structures on fiber bundles*, Topology **42** (2003), 309-347.
2. F. Lalonde, D. McDuff, L. Polterovich, *Topological rigidity of Hamiltonian loops and Quantum homology*, Invent. Math. **135** (1999), 369-385.
3. R. Thom, *Sous-variétés et classes d'homologie des variétés différentiables*, Séminaire Bourbaki, Vol. 2, Exp. No. 78, (Soc. Math. France, Paris, 1995) pp. 271–277.
4. G. W. Whitehead, *Elements of Homotpy Theory*, 3rd Edition, (Springer-Verlag, 1995) pp. 319-320.

Department of Mathematics, Middle East Technical University, 06531 Ankara TURKEY

E-mail address: ozan@metu.edu.tr