Milnor's Exotic 7-Spheres

Note Title 7.12.2014

$M = M^7$ smooth oriented closed manifold $B = B^8$ is a smooth oriented compact manifold so that $\partial B = M$ as oriented manifolds.

<u>Lemma 6.4.1.</u> Assume that M and B are as above and $H^3_{DR}(M) = 0 = H^4_{DR}(M)$. Then the quadratic form

$$H^4_{DR}(B) \to \mathbb{R}, \; [\alpha] \mapsto \int_B \alpha^2$$

is well defined.

<u>Proof</u>: <u>Step 1</u>: Let $[\alpha] = [\alpha']$ then $\alpha - \alpha' = d\beta$ for some $\beta \in \Omega^3(B)$. Then we must show $\int_B \alpha^2 = \int_B \alpha'^2$.

$\alpha = \alpha' + d\beta \Rightarrow \alpha^2 = \alpha'^2 + 2\alpha' \wedge d\beta + d\beta \wedge d\beta$

and thus

$$\int_B \alpha^2 = \int_B \alpha'^2 + \int_B 2\alpha' \wedge d\beta + d\beta \wedge d\beta$$

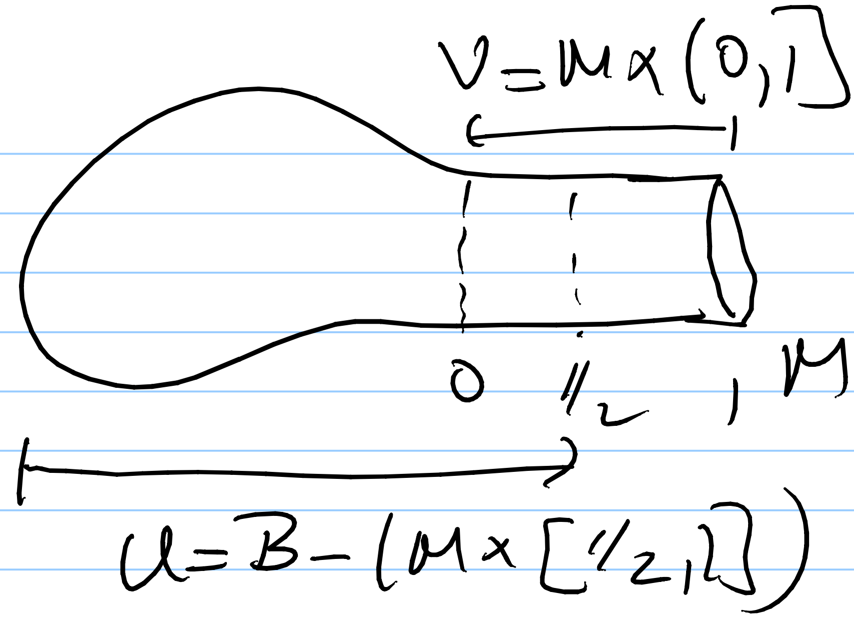
$$\int_B \alpha^2 - \int_B \alpha'^2 = \int_B (2\alpha' + d\beta) \wedge d\beta$$

$$= \int_B d\left[(2\alpha' + d\beta) \wedge \beta\right]$$

$$= \int_{M = \partial B} (2\alpha' + d\beta) \wedge \beta .$$

Hence to finish the proof of the lemma it is enough to show that β can be chosen so that $\beta|_M = 0$.

Step 2:
Let



Then $U \cap V = M \times (0, 1/2)$.

Thm 4.3.14 $H_c^{k+1}(M \times \mathbb{R}) = H_c^k(M)$

So $\boxed{\begin{aligned} H_c^k(U \cap V) &= H_c^k(M \times (0, 1/2)) \\ &\simeq H_c^{k-1}(M) \\ &\simeq H_{DR}^{k-1}(M). \end{aligned}}$

Also $H_c^k(U \cup V) = H_c^k(B) = H_{DR}^k(B)$
and $H_c^k(U) \simeq H_c^k(B - M)$
(U and $B-M$ are diffeomorphic)
Now use Local Cohomology Sequence for compactly supported

cohomology: $U \subseteq M$ open subset

$\Rightarrow \Omega_c^k(U) \hookrightarrow \Omega_c^k(M)$

$\Rightarrow 0 \to \Omega_c^k(M) \to \Omega_c^k(M) \to \Omega_c^k(M) / \Omega_c^k(U) \to 0$

This gives a long exact seq.

$\cdots \xrightarrow{\delta} H^n_c(U) \to H^n_c(M) \to H^n_c(M,U) \to H^{n+1}_c(U) \to \cdots$

Special case $L \subseteq M$ closed submanifold. Let $U = M \setminus L$.

Then one obtains

Thm 4.3.18.

$$H_c^k(M, M\setminus L) \cong H_c^k(L).$$

Thm 4.3.17 now implies that

$$\boxed{H_c^k(V, V\setminus M) \cong H_c^k(M) = H^k_{DR}(M)}$$

Claim: $H^k_c(V)=0$.

Proof: local cohomology sequence for $(V, V-M)$ gives

$$\dashrightarrow H^n_c(V-M) \to H^n_c(V) \to H^n_c(V, V-M)$$

$$\to H_c^{n+1}(V-M) \to \cdots .$$

Now use $H^k_c(V-M) = H^k_c(M\times(0,1))$

$$\simeq H_c^{k-1}(M)$$

to get

$$\simeq H^{k-1}_{DR}(M)$$

$$\dashrightarrow H^{n-1}_{DR}(M) \to H^n_c(V) \to H^n_{DR}(M) \xrightarrow{\sim}$$

$$H^n_{DR}(M) \to \cdots$$

$$\Rightarrow H^n_c(V) = 0.$$

Step 3 Mayer-Vietoris Exact sequence for locally supported coh. sequence for $B = U \cup V$.

Then

$$\to H^k_c(U\cap V) \to H^k_c(U)\oplus H^k_c(V) \to H^k_c(B)$$

$$\to H^{k+1}_c(U\cap V) \to \cdots$$ Implies that

$$\dashrightarrow H^3_{DR}(M) \to H^4_c(B-M) \xrightarrow{\sim} H^4_{DR}(B)$$

$$\to H^4_{DR}(M) \to \cdots$$

Since by assumption $H^3_{DR}(M) = 0$ $= H^4_{DR}(M)$ we see that

$H^4_{DR}(B) = H^4_c(B-M)$. Hence, the form β in Step 1 can be chosen so that $\beta|_M = 0$. This finishes the proof of the lemma. ∎

Let's denote the index of the quadratic form by $I(B)$.

- Note that $\tau(-B) = -\tau(B)$.
- Similarly, we define the first Pontryagin number of B as follows:

$$q(B) \doteq Q(P_1(B)) = \int_B P_1^2(B),$$

which is again well defined.

- We'll see later that $q(B)$ is an integer.
- Let $\lambda(M) \doteq 2q(B) - \tau(B) \pmod 7$.

Theorem 6.4.2. $\lambda(M)$ is independent of the choice of B and determined only by M.

Corollary 6.4.3. If $\lambda(M) \neq 0$ then

M cannot be the boundary of a compact manifold B with $H^4_{DR}(B) = 0$.

- Note that $\lambda(-M) = -\lambda(M)$ and thus we get

<u>Corollary 6.4.4.</u> If $\lambda(M^7) \neq 0$ then M does not admit an orientation reversing diffeomorphism.

<u>Proof of Theorem 6.4.2</u>:

<u>Step 1)</u> Let B_1^8 and B_2^8 be two compact oriented manifolds with $\partial B_i^8 = M$. Let $C^8 = B_1^8 \cup_\partial -B_2^8$. Then by Thom / Hirzebruch signature

Formula implies that

$$\tau(C) = \frac{1}{45}\int_C 7p_2(C) - p_1^2(C)$$

$$\Rightarrow 45\tau(C) + q(C) = 45\tau(C) + \int_C p_1^2(C)$$

$$= 7\int_C p_2(C) \equiv 0 \pmod 7.$$

$$\Rightarrow 2q(C) - \tau(C) \equiv 0 \pmod 7.$$

<u>Step 2</u> We'll prove that

$\tau(C) = \tau(B_1) - \tau(B_2)$ and

$q(C) = q(B_1) - q(B_2)$.

<u>proof</u>: Using similar ideas as in the proof of Lemma 6.4.1. we get a commutative diagram where each arrow is an isomorphism

$$\begin{array}{ccc} H^4_c(C) & \leftarrow & H^4_c(B_1 - M) \oplus H^4_c(B_2 - M) \\ \downarrow & & \downarrow \qquad\qquad \downarrow \\ H^4_{DR}(M) & \longrightarrow & H^4_{DR}(B_1) \oplus H^4_{DR}(B_2) \end{array}$$

So for any $\alpha \in H^4_{DR}(M)$ we can write $\alpha = \beta_1 + \beta_2$ when $\beta_1 \in H^4_{DR}(B_1)$, $\beta_1|_{\partial B_1 = M} = 0$ and $\beta_2 \in H^4_{DR}(B_2)$, $\beta_2|_{\partial B_2 = M} = 0$.

Hence $\alpha^2 = \beta_1^2 + \beta_2^2$ and thus

$$\tau(C) = \tau(B_1) - \tau(B_2)$$

and using the naturality of the pontryagin classes we get $q(C) = q(B_1) - q(B_2)$.

This finishes the proof. ∎

$\mathbb{R}^4$-bundles over S^4:

$$S^2 = \mathbb{CP}^1 = \mathbb{C} \dot\cup \mathbb{C} / z \sim \phi(z) = \frac{1}{z}$$

$$T_* S^2 = T_* \mathbb{C} \dot\cup T_* \mathbb{C} / (z,w) \sim \left(\frac{1}{z}, -\frac{w}{z^2}\right)$$

$$S^4 = \mathbb{H} \dot\cup \mathbb{H} / p \sim \frac{1}{p} = \frac{\bar{p}}{\|p\|^2} = \phi(p)$$

$$T_* S^4 = T_* \mathbb{H} \dot\cup T_* \mathbb{H} / (p,v) \sim \left(\frac{1}{p}, \phi'(p)(v)\right)$$

$$\phi'(p)(v) = \lim_{h\to 0} \frac{\phi(p+hv) - \phi(p)}{h}$$

$$= -\frac{1}{p} v \frac{1}{p}$$

However, there is a path in $\mathbb{H}^*$ going -1 to 1 and thus $(p,v) \longmapsto -\frac{1}{p} v \frac{1}{p}$ is homotopy to $(p,v) \longmapsto \frac{1}{p} v \frac{1}{p}$.

$$\Rightarrow T_* S^4 = T_* \mathbb{H} \dot\cup T_* \mathbb{H} / (p,v) \sim \left(\frac{1}{p}, \frac{1}{p} v \frac{1}{p}\right)$$

For any pair of integers (h,j)

Let $\xi_{h,j} \to S^4$ be the bundle defined by

$$\xi_{h,j} = \mathbb{H}\times\mathbb{H} \,\dot\cup\, \mathbb{H}\times\mathbb{H} \Big/ (p,v)\sim(p^h v p^j) .$$

So $\xi_{-1,-1} = T_* S^4$.

<u>Lemma:</u> $p_1(\xi_{h,j}) = 2(h-j)\,v$

and $e(\xi_{h,j}) = -(h+j)\,v$,

where $v \in H^4_{DR}(S^4)$ with $\int_{S^4} v = 1$.

<u>Proof:</u> $e(\xi_{h,j}) = -(h+j)v$:

If $h+j \leq 0$ then

$s_i : \mathbb{H} \to T_*\mathbb{H},\ p \mapsto (p, 1+p^{-h-j})$

$i=1,2$, satisfy

$$p^j s_1(p)\, p^h = s_2(1/p)$$

and thus they define a section
So the Euler number is the
number of zeros of the
function $p \mapsto 1 + p^{-h-j}$,
which is $-h-j$ (Eilenberg-Niven 1944 Bull. Amer. Math. Soc.)

If $h+j > 0$ then note that
change the orientation of the
fiber by the coordinate
change $v \longmapsto u = \bar{v}$. This
new bundle has gluing
function

$$(p,u) \longmapsto \left(\frac{1}{p}, p^{-j} u p^{-h}\right).$$

So $-\xi_{h,j} = \xi_{-j,-h}$

changing the orientation
of the fibers.

So $e(\xi_{h,j}) = -e(-\xi_{h,j})$

$$= -e\left(\xi_{-j,-h}\right)$$

$$= -(-(-j-h))v$$

$$= -(h+j)v$$

For the calculation of the $p_1(\xi_{h,j})$ consider changing the orientation of both the fiber and the base.

$$\mathbb{H}^* \times \mathbb{H} \longleftrightarrow \mathbb{H}^* \times \mathbb{H}$$

$$(p, v) \longleftrightarrow (q, \tilde{v}) = (\bar{p}, \bar{v})$$

Then the gluing function of $\xi_{h,j}$ transforms to

$$(q, \tilde{v}) \longrightarrow \left(\frac{1}{q}, q^{j} \tilde{v} q^{h}\right).$$

So $\xi_{h,j} \longrightarrow \xi_{j,h}$

However, changing the orientation of the fiber does not change p_1 (since the complexified spaces have still the same orientation). On the other hand, changing the orientation of the base multiplies p_1 by -1. So $p_1(\xi_{j,h}) = -p_1(\xi_{h,j})$.

Cor: $p_1(S^4) = p_1(\xi_{-1,-1}) = 0$.

We need one more trick:

For any $p, q \in \mathbb{H}$, $\overline{pq} = \bar{q}\,\bar{p}$.

Although the maps

$\psi_1 : \mathbb{R}^4 = \mathbb{H} \to \mathbb{H} = \mathbb{R}^4, \; v \mapsto vp$

$\psi_2 : \mathbb{R}^4 = \mathbb{H} \to \mathbb{H} = \mathbb{R}^4, \; v \mapsto \overline{vp} = \bar{p}\bar{v}$

are <u>not</u> homotopic then complexifications

$\psi_1 \otimes id_{\mathbb{C}} : \mathbb{C}^4 = \mathbb{H}^4 \otimes \mathbb{C} \to \mathbb{H}^4 \otimes \mathbb{C} = \mathbb{C}^4$

$\psi_2 \otimes id_{\mathbb{C}} : \mathbb{C}^4 = \mathbb{H}^4 \otimes \mathbb{C} \to \mathbb{H}^4 \otimes \mathbb{C} = \mathbb{C}^4$

are homotopic.

Hence the clutching maps

$(p,v) \mapsto ((1/p, p^h v p^j)$ and

$(p,v) \mapsto (1/p, p^h \bar{p}\, \bar{v}\, p^{j-1}))$

$\qquad\qquad = \|p\|^2 p^{h-1} \bar{v} p^{j-1}$

are homotopic once complexified. However, changing the orientation of the fiber does <u>not</u> effect the orientation of the complexification and thus $p_1(\xi_{h,j}) = p_1(\xi_{h-1,j-1})$

So $p_1(\xi_{h,j}) = p_1(\xi_{h-j,0})$.

Also, $\exists\, g_k : S^4 \to S^4,\ p \mapsto p^k$

then $g_k^*(\xi_{1,0}) = \xi_{k,0}$.

So $p_1(\xi_{h,j}) = p_1(\xi_{h-j,0})$

$$= (h-j)\, p_1(\xi_{1,0}).$$

Finally, $\xi_{1,0}$ is a complex vector bundle and thus

$p_1(\xi_{1,0}) = -2e(\xi_{1,0}) = 2v$.

So $p_1(\xi_{h,j}) = 2(h-j)v$. $\blacksquare$

$\xi_{1,0} = \mathbb{H}\times\mathbb{H} \cup \mathbb{H}\times\mathbb{H} \,/\, (p,v)\sim\left(\frac{1}{p}, pv\right)$

Milnor's Spheres:

For any odd Integer k choose $h, j \in \mathbb{Z}$ with
$h + j = -1$ and $h - j = k$ and let B_k be the unit disk bundle of $\xi_{h,j}$. Also let

$M_k = \partial B_k$ - the unit sphere bundle.

Lemma $\lambda(M_k) = k^2 - 1 \pmod 7$.

Proof Let $\pi : B_k \to S^4$ be the restriction of the bundle projection $\xi_{h,j} \to S^4$. Then

$$T_* B_k = \pi^* (T_* S^4) \oplus \pi^* (\xi_{h,j})$$

Let $\alpha = \pi^* (\nu)$. Then

$$\begin{aligned} p_1 (B_k) &= \pi^* (p_1 (T_* S^4)) \\ &\quad + \pi (p_1 (\xi_{h,j})) \\ &= 0 + 2(h-j)\alpha = 2k\alpha \end{aligned}$$

Also note that since $h+j=-1$ the bundle $\xi_{h,j}$ has Euler class $-(h+j)\,v = v$.

<u>Claim</u> $\int_B \alpha^2 = 1$.

<u>proof</u>: Note that

$\pi^*: H^4_{DR}(S^4) \longrightarrow H^4_{DR}(B_k)$ is an isomorphism and thus $\alpha \in H^4_{DR}(B_k)$ is a generator.

Let, denote the zero section of this bundle by S^4 also.

Let $\beta \in H^4_{DR}(B_k)$ be the Poincare dual of the submanifold $S^4 \subseteq B_k$.

Then $\beta = a\alpha$ for some $a \in \mathbb{R}$. Since the bundle has Euler number 1 we have

$$1 = S^4 \mp S^4 = \int_{S^4} \beta = \int_{B_k} \beta^2.$$

However, by its definition

$$\int_{S^4} \alpha = 1 \quad \text{and thus } \alpha = \beta$$

and $\int_{B_k} \alpha^2 = 1.$

So $\tau(B_k) = 1$ and

$$q(B_k) = \int_{B_k} p_1^2(B_k) = 4k^2.$$

$$\Rightarrow \lambda(M_k) = 2q(B_k) - \tau(B_k)$$

$$= 8k^2 - 1$$

$$= k^2 - 1.$$

Theorem For any odd integer k M_k is homeomorphic to S^7. On the other hand, if $k^2-1 \not\equiv 0$ then M_k is not diffeomorphic to S^7.

Proof: Note that $\lambda(S^7)=0$ and thus if $\lambda(M_k)=k^2-1 \not\equiv 0$ then M_k is not diffeomorphic to S^7.

For the first part we need a result due to Reeb: If a closed manifold M^n admits a Morse function with only two critical points then the manifold is homeomorphic to S^n.

Now we define $F: M_k \to \mathbb{R}$ as follows: (ω, u)

$$M_k = \mathbb{H} \times S^3 \cup \mathbb{H} \times S^3$$

(p, v)

$$(p,v) \sim \left(\frac{1}{p}, \frac{1}{\|p\|^{h+j}} p^h v p^j\right)$$

$$\parallel$$

$$(q, u) = \left(\frac{1}{p}, \|p\| p^h v p^j\right) \text{ since } h+j=-1.$$

Also let $\omega = q\frac{1}{u} = \frac{1}{p}\|p\| \bar{p}^j \bar{v} \bar{p}^h$

then $\|\omega\| = \frac{1}{\|p\|}$.

Then check that

$$\frac{Re(\omega)}{\sqrt{1+\|\omega\|^2}} = \frac{Re(v)}{\sqrt{1+\|q\|^2}}$$

and the function F defined with these formulas is a Morse function with exactly two critical points. ∎

The book referred in the notes is the author's book "Türevlenebilir Manifoldlara Giriş". which contains a section on Milnor's work on exotic 7-spheres. We've just followed Milnor's Annals paper, replacing singular cohomology with De Rham cohomology. since the book did not have a section on singular (co)homology.