Video 1

Math 371 Differential Geometry

Note Title 24.03.2020

Chapter 4 : Calculus on a Surface

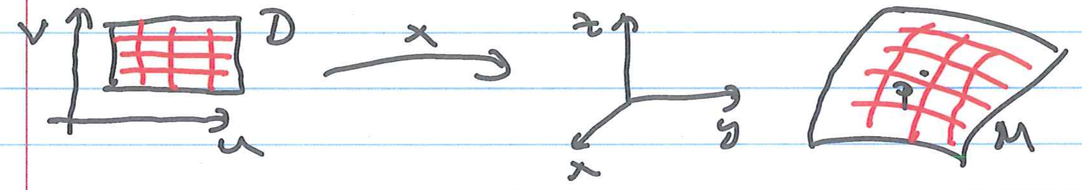
§4.1. Surfaces in $\mathbb{R}^3$:

A surface in $\mathbb{R}^3$ is a subset of $\mathbb{R}^3$ that looks like the plane locally.

Definition: A coordinate patch $x: D \to \mathbb{R}^3$ is a one-to-one regular mapping of an open subset D of $\mathbb{R}^2$ into $\mathbb{R}^3$.

regular means that x is differentiable and its derivative $x_*: T_pD \to T_{x(p)}\mathbb{R}^3$ is injective at all $p \in D$.

We'll ask also that x is a proper map, meaning its inverse $x^{-1}: x(D) \to D$ is also continuous.



Definition A surface in $\mathbb{R}^3$ is a subset M of $\mathbb{R}^3$ such that for each point $p \in M$ there is a proper coordinate patch in M whose image contains a neighborhood of p in M.

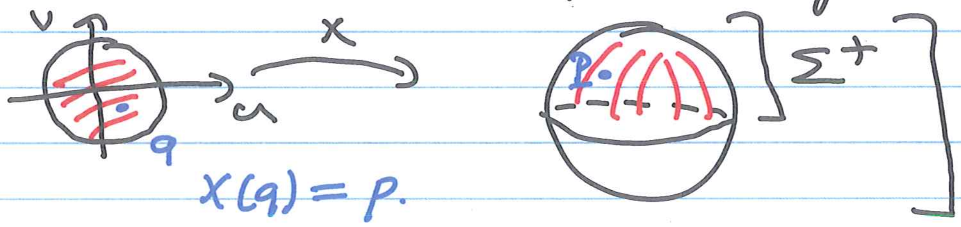
Example: The unit sphere Σ in $\mathbb{R}^3$

$\Sigma = \{p=(p_1,p_2,p_3)\in\mathbb{R}^3 \mid p_1^2+p_2^2+p_3^2=1\} \subseteq \mathbb{R}^3$.

let $p=(p_1,p_2,p_3)\in M$ be any point. Since $p_1^2+p_2^2+p_3^2=1$ some $p_i \neq 0$. Say $p_3>0$. Then consider the function $x : D \to \mathbb{R}^3$, $x(u,v)=(u,v,\sqrt{1-u^2-v^2})$, where $D=\{(u,v)\in\mathbb{R}^2 \mid u^2+v^2<1\}$. Note that since $p_3>0$ and $\underbrace{p_1^2+p_2^2}+p_3^2=1$, $(p_1,p_2)\in D$ so that

$$x(p_1,p_2)=(p_1,p_2,\underset{\sqrt{p_3^2}=|p_3|=p_3}{\underset{\shortparallel}{\sqrt{1-p_1^2-p_2^2}}})=(p_1,p_2,p_3).$$

Clearly, $x(D)$ is the Northern hemisphere of Σ and thus it contains a neighborhood of p.



must check that x is a proper coordinate patch.

1) x is one to one: $\Sigma^+=\{(x,y,z)\in\Sigma \mid z>0\}$. Then $x : D \to \Sigma^+$ is a bijection with inverse $x^{-1} : \Sigma^+ \to D$, $x^{-1}(x,y,z)=(x,y)$.

2) x is proper: x^{-1} is just the restriction of the continuous map $\mathbb{R}^3 \to \mathbb{R}^2$, $(x,y,z) \longmapsto (x,y)$,

which is continuous, and thus x^{-1} is continuous.

3) x is regular: To see this we just compute the Jacobian matrix of x: $x(u,v) = (u, v, \underbrace{\sqrt{1-u^2-v^2}}_{\overset{\shortparallel}{f}})$

$$J = \begin{pmatrix} \frac{\partial u}{\partial u} & \frac{\partial u}{\partial v} \\ \frac{\partial v}{\partial u} & \frac{\partial v}{\partial v} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}_{3\times 2}$$

rank $J = 2$, thus $x_* : T_pD \longrightarrow T_{x(p)}\mathbb{R}^3$ is injective.

$$x_*(w_p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u}(p) & \frac{\partial f}{\partial v}(p) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_1 \frac{\partial f}{\partial u}(p) + w_2 \frac{\partial f}{\partial v}(p) \end{pmatrix}_{x(p)}$$

Hence, x is regular.
So x is a coordinate at $p = (p_1, p_2, p_3)$.

It is clear that at any point $p \in \Sigma$ we can find a similar coordinate patch. For example, at the point, say $p = (-1, 0, 0)$ we may use the coordinate patch

$$x(u,v) = (-\sqrt{1-u^2-v^2}, u, v).$$

Hence, Σ is a surface in $\mathbb{R}^3$.

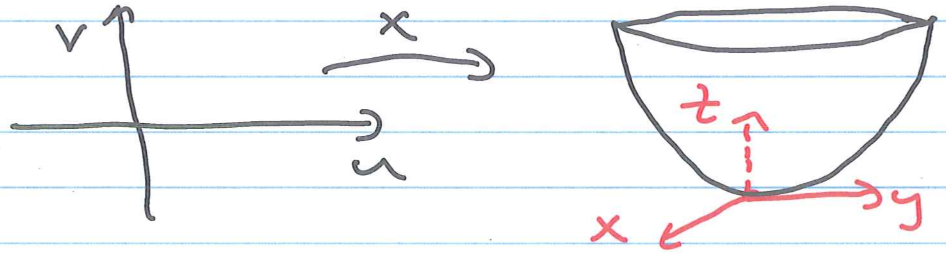
Coordinate patches of the type $(u,v) \mapsto (u, v, f(u,v))$ are called Monge patches.

Example: Let $f:\mathbb{R}^2\to\mathbb{R}$ be a differentiable function and $\Sigma=\{(x,y,z)\in\mathbb{R}^3 \mid z=f(x,y)\}$. In this case, the function $x:\mathbb{R}^2\to\mathbb{R}^3$, given by

$x(u,v)=(u,v,f(u,v))$ is a proper surface patch covering all Σ. Hence, Σ is a surface. (Details are left as an exercise!)

Ex: $f(x,y)=x^2+y^2$

$\Sigma=\{(x,y,z)\in\mathbb{R}^3 \mid z=f(x,y)=x^2+y^2\}\subseteq\mathbb{R}^3$



Theorem: Let $g:\mathbb{R}^3\to\mathbb{R}$ be a differentiable function and $c\in\mathbb{R}$ be a real number. Set

$$M=g^{-1}(c)=\{(x,y,z)\in\mathbb{R}^3 \mid g(x,y,z)=c\}.$$

Then M is a surface provided that $dg(p)\neq 0$ for all $p\in M$.

Proof: Let $\overset{(p_1, p_2, p_3)}{\overset{\shortparallel}{p}} \in M$, then by the assumption $dg(p) = \frac{\partial g}{\partial x}(p)\,dx + \frac{\partial g}{\partial y}(p)\,dy + \frac{\partial g}{\partial z}(p)\,dz \neq 0$. So at least one of the partial derivatives is not zero, say $\frac{\partial g}{\partial z}(p) \neq 0$. Now by the Implicit Function Theorem there is a differentiable function $h : D \to \mathbb{R}$, where D is an open subset of $\mathbb{R}^2$ containing (p_1, p_2), where $p = (p_1, p_2, p_3)$, satisfying

1) for each $(u,v) \in D$, $g(u, v, h(u,v)) = c$ $(\Rightarrow (u, v, h(u,v)) \in M)$,

2) Points of the form $(u, v, h(u,v))$ with $(u,v) \in D$ "fill a neighborhood of p in M".

In particular, the map $(u,v) \mapsto (u, v, \overset{\text{diff.}}{h}(u,v))$ is a Monge patch around p, and thus M is a surface in $\mathbb{R}^3$.

Last time: Theorem: let $g: \mathbb{R}^3 \to \mathbb{R}$ diff. function and $c \in \mathbb{R}$. Let $M = g^{-1}(c) = \{(x,y,z) \in \mathbb{R}^3 \mid g(x,y,z) = c\}$. If $dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz \neq 0$ at all points of M then M is a surface in $\mathbb{R}^3$.

Example: Let $g: \mathbb{R}^3 \to \mathbb{R}$, $g(x,y,z) = x^2+y^2+z^2$ and $c=1$. Then $M = g^{-1}(1) = S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$.

Since $dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz$

$= 2x\,dx + 2y\,dy + 2z\,dz$, and if

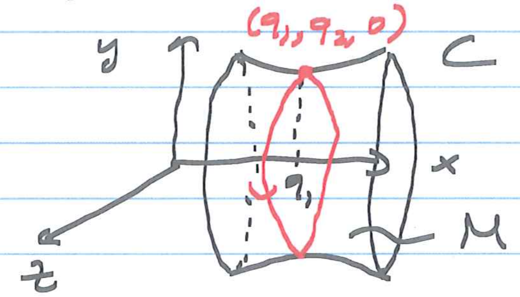
$p = (x_0, y_0, z_0) \in S^2$, then $x_0^2 + y_0^2 + z_0^2 = 1 \neq 0$ so that at least one of x_0, y_0, z_0 is not zero.

$dg(p) = 2x_0\,dx + 2y_0\,dy + 2z_0\,dz \neq 0$. Hence, by the theorem $S^2 = g^{-1}(1)$ is a surface in $\mathbb{R}^3$.

Example: Surfaces of revolutions

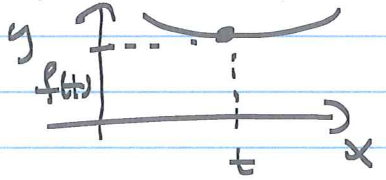
Let C be a curve in the xy-plane. If (q_1, q_2, q_3) is a point on C, then rotating this point about the x-axis we obtain the curve

$(q_1, q_2 \cos v, q_2 \sin v)$, $v \in [0, 2\pi]$.



Let M be the surface obtained by revolving C about the x-axis. If C is given by $y = f(x)$,

$f : I \to \mathbb{R}^2$, $I = (a,b)$.



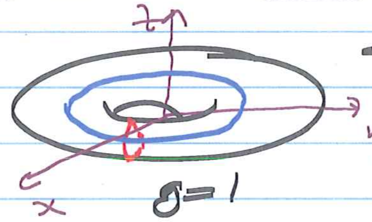
$(q_1, q_2, 0) = (t, f(t), 0)$

$$x(t,v) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos t \end{bmatrix} \begin{bmatrix} t \\ f(t) \\ 0 \end{bmatrix} = (t, f(t)\cos v, f(t) \sin v).$$

We'll see that $x(t,v)$ is a proper surface patch in the next section.

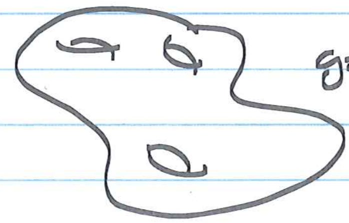
Some Examples of Surfaces:

1)



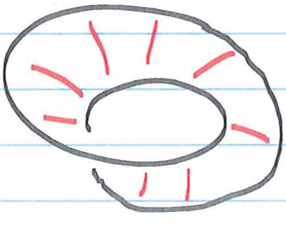
T^2: Surface of revolution, just rotate the red circle about the z-axis.

2)



$g=3$ (g: # of holes, called the genus of the surface)

3)



Möbius Band
(Non orientable surface)

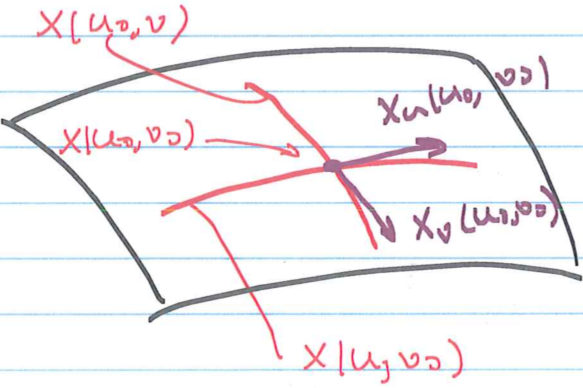
§4.2. Patch Computations:

Let $x : D \longrightarrow \mathbb{R}^3$ be surface patch and $(u_0, v_0) \in D$. Then the functions $u \longmapsto x(u, v_0)$ and $v \longmapsto x(u_0, v)$ define two curves on the surface passing through the point $x(u_0, v_0)$.

Definition: The velocity vector of the above parametric curves are denoted by

$$\frac{d}{du}(x(u,v_0))\Big|_{u=u_0} = x_u(u_0, v_0) \quad \text{and}$$

$$\frac{d}{dv}(x(u_0,v))\Big|_{v=v_0} = x_v(u_0, v_0).$$

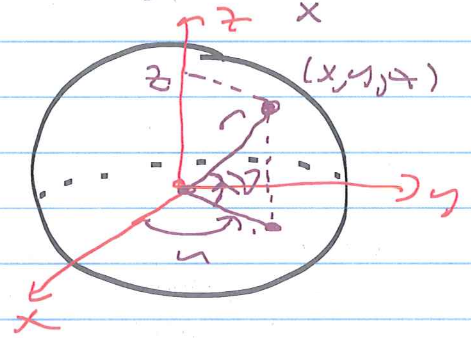


In the coordinates, if $x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ then $x_u = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right)$ and $x_v = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right)$.

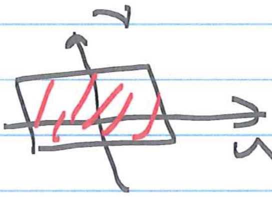
Example: $\Sigma = S^2$ the sphere with center at the origin and radius r.

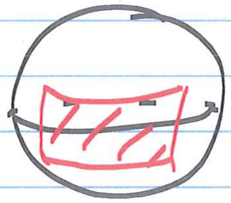
Using spherical coordinates we get the coordinate patch

$$x(u,v)=(\underset{x}{r\cos v\cos u},\ \underset{y}{r\cos v\sin u},\ \underset{z}{r\sin v})$$



$u\in(-\pi,\pi)$, $v\in(-\pi/2,\pi/2)$





$$x_u=(-r\cos v\sin u,\ r\cos v\cos u,\ 0)$$

$$x_v=(-r\sin v\cos u,\ -r\sin v\sin u,\ r\cos v).$$

Definition: A regular mapping $x:D\to\mathbb{R}^2$ whose image lies in a surface M is called a parametrization of the region x(D) of M.

(So a coordinate patch is a one to one proper parametrization.)

Back to above example: The spherical parametrization defined above is not one-to-one in general, since $x(u+2\pi,v)=x(u,v)$

Why is x regular?

$$X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$$

$$X_* = \begin{bmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} \end{bmatrix} : T_p D \longrightarrow T_{X(p)} S^2$$

$$\begin{array}{cc} \underbrace{\qquad}_{X_u} & \underbrace{\qquad}_{X_v} \end{array}$$

$p = (u_0, v_0)$

X_* must be 1-1. In other words, the colums X_u, X_v must be linearly independent. However, this is equivalent to showing that $X_u \times X_v \neq 0$.

$$X_u \times X_v = r^2 \begin{vmatrix} u_1 & u_2 & u_3 \\ -\cos v \sin u & \cos v \cos u & 0 \\ -\sin v \cos u & -\sin v \sin u & \cos v \end{vmatrix}$$

$$= r^2(\cos^2 v \cos u, \cos^2 v \sin u, \cos v \sin v)$$

$\Rightarrow \|X_u \times X_v\| = r^2 \cos^2 v \neq 0$ provided that

$v \in (-\pi/2, \pi/2)$.

So X is regular and hence a parametrization.

Video 3

Example: Surface of a revolution.

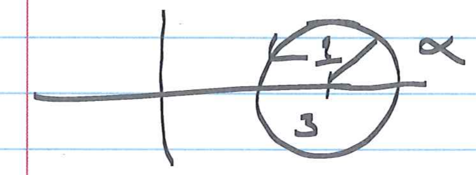
Let $\alpha(u)=(g(u),h(u),0)$, $\alpha: I \Longrightarrow \mathbb{R}^2 \subseteq \mathbb{R}^3$, a curve in the xy-plane. Then the surface obtained by revolving α about the x axis has a parametrization given by

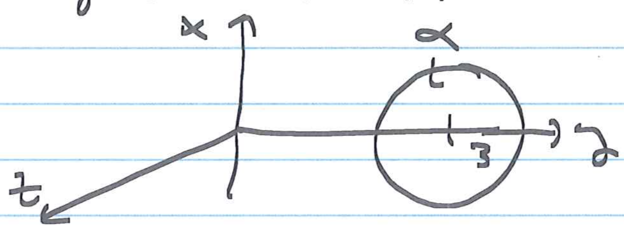
$$x(u,v)=(g(u),\, h(u)\cos v,\, h(u)\sin v), \quad v\in[0,2\pi),\ u\in I.$$

Then $x_u=(g',\, h'(u)\cos v,\, h'(u)\sin v)$ and

$$x_v=(0,\, -h(u)\sin v,\, h(u)\cos v).$$

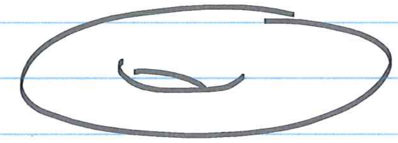
As a concrete example take $\alpha(u)=(3+\cos u, \sin u, 0)$





$g(u)=3+\cos u$

$h(u)=\sin u$



So, $x_u\times x_v = \begin{vmatrix} u_1 & u_2 & u_3 \\ g' & h'\cos v & h'\sin v \\ 0 & -h\sin v & h\cos v \end{vmatrix}$

$$=(hh',\, -g'h\cos v,\, -g'h\sin v)$$

$$\|x_u\times x_v\| = h^2h'^2+g'^2h^2(\cos^2 v+\sin^2 v)$$
$$=h^2(h'^2+g'^2)$$
$$=h^2\|\alpha'(u)\|^2>0$$ provided that

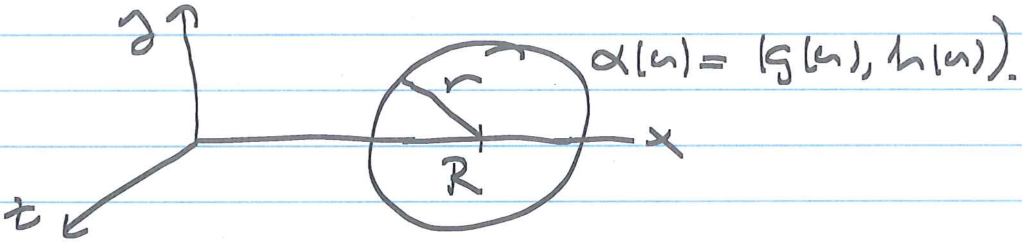
α is a regular curve ($\alpha'(u) \neq 0 \ \forall u$) and α never touches the x-axis.

Hence, x is a surface patch.

By the way the general torus has parametrization given by $\alpha(u) = (R + r\cos u, r\sin u, 0)$

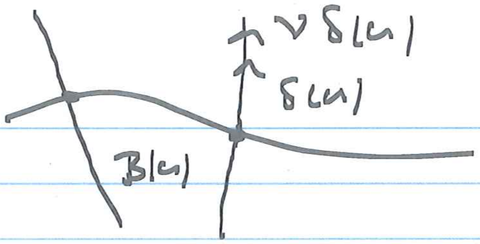
$x(u,v) = ((R + r\cos u), r\sin u \cos v, r \sin u \sin v)$

$g(u) = R + r\cos u$, $h(u) = r\sin u$

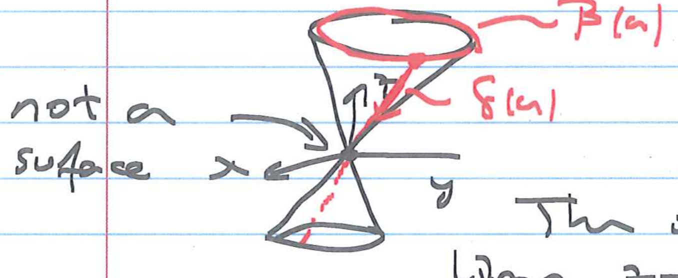


Definition: A ruled surface is a surface swept out by a straight line l along a curve β. The various positions of lines generating the surface are called the rulings of the surface. Such a surface has parametrization given by

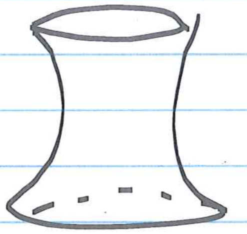
$x(u,v) = \beta(u) + v\,\delta(u)$, $\beta(u)$ the base curve, $\delta(u)$ the director.



Example: Consider the cone given by $z^2 = x^2+y^2$, which is a ruled surface.



Let $y=0$. Then $z^2 = x^2 \Rightarrow$ $(z-x)(z+x)=0$. The surface contains the lines $z=\pm x$.



Example $z^2 = x^2+y^2-1$.

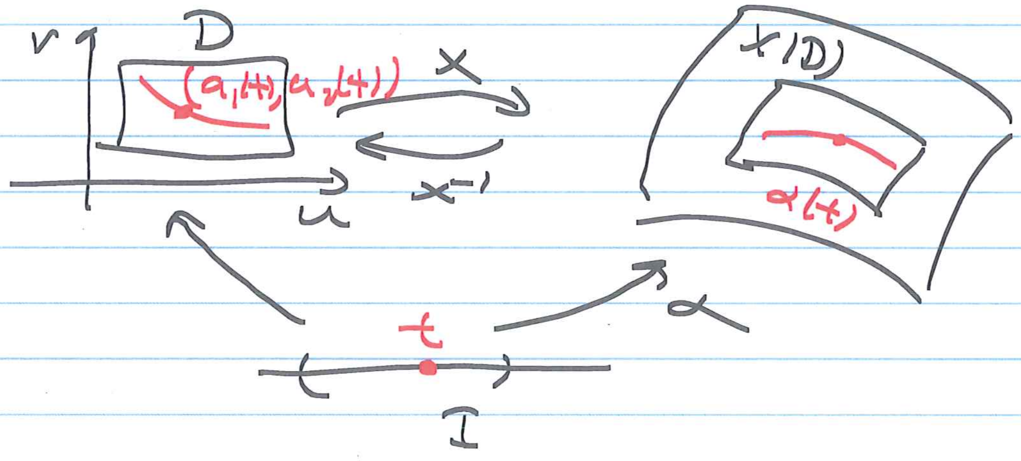
If $y=0$, then $z^2=x^2-1$

$\Rightarrow z^2=(x-1)(x+1)$ and then the surface contain the lines $\begin{pmatrix} z = x-1 \\ y=0 \end{pmatrix}$ and $\begin{pmatrix} z = x+1 \\ y=0 \end{pmatrix}$.

§4.3. Differentiable Functions and Tangent Vectors:

Lemma: Let $\alpha : I \to M$ be a (differentiable) curve on a surface M so that $\alpha(I)$ lies inside the image $x(D)$ of a surface patch $x : D \to M$. Then there exists unique differentiable functions a_1 and a_2 on I so that $\alpha(t) = x(a_1(t), a_2(t))$, for all $t \in I$.

Proof:



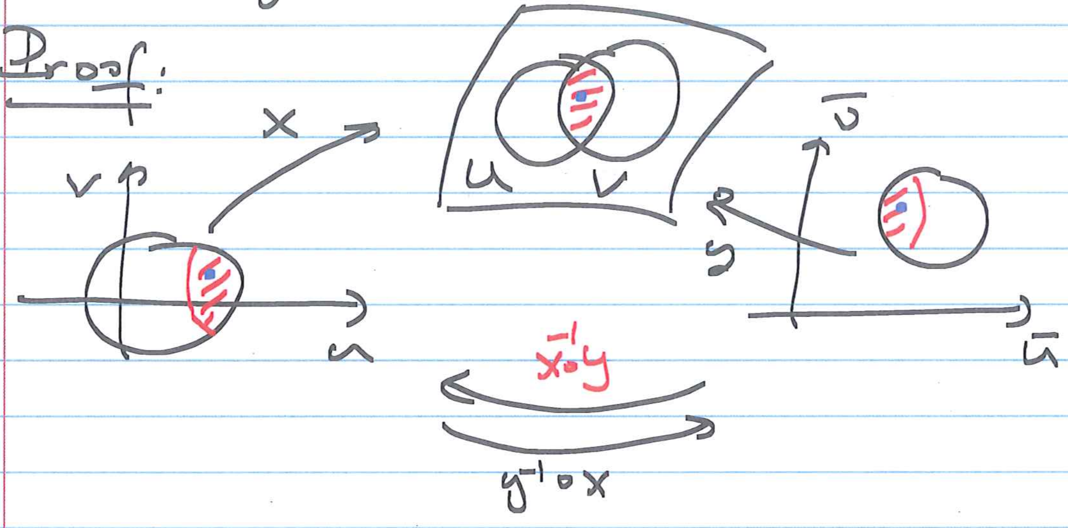
Consider the composition $x^{-1} \circ \alpha : I \to \mathbb{R}^2_{uv}$.

Since α and x^{-1} are differentiable we see that $(x^{-1} \circ \alpha)(t) = (a_1(t), a_2(t))$ for some differentiable functions $a_i : I \to \mathbb{R}$.

Hence, $\alpha(t) = x((x^{-1} \circ \alpha)(t)) = x(a_1(t), a_2(t))$. ∎

Theorem: If x and y are overlapping surface patches in M, then there exist unique differentiable functions $\bar{u}$ and $\bar{v}$ such that $y(u,v) = x(\bar{u}(u,v), \bar{v}(u,v))$, for all (u,v) in the domain of $x^{-1}y$.

Proof:

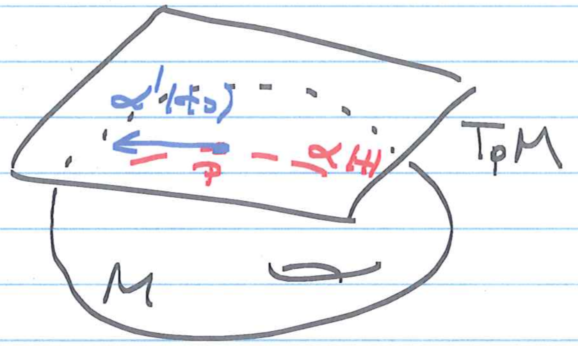


So, $(y^{-1} \circ x)(u,v) = (\bar{u}(u,v), \bar{v}(u,v))$. Apply y to both sides to get $x(u,v) = y(\bar{u}(u,v), \bar{v}(u,v))$.

Uniqueness part follows from the fact that x and y are homeomorphisms. ∎

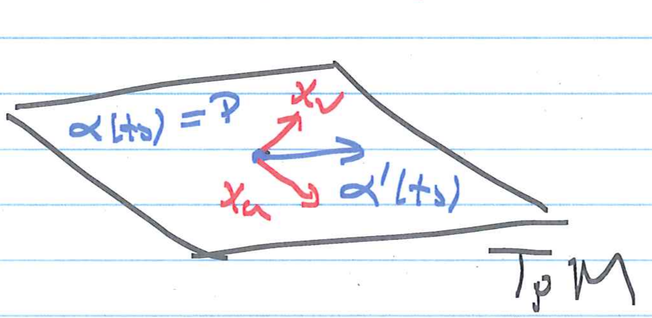
Definition: A tangent vector to a surface M at a point $p \in M$ is the velocity vector $\alpha'(t_0)$

of a curve $\alpha : I \to M$, where $p = \alpha(t_0)$. The set of all tangent vectors to M at the point p is called the tangent space to M at p and denoted as T_pM.



<u>Lemma</u>: Any tangent vector in T_pM can be written as a linear combination of $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$, where $x : D \to M$ is a surface patch and $x(u_0, v_0) = p$.

<u>Proof</u>:



Let's take a tangent vector $\alpha'(t_0)$ in T_pM.

$\alpha : I \to M$, $\alpha(t_0) = p$. Then $\alpha(t) = x(a_1(t), a_2(t))$ for some differentiable functions

$a_1 : I \to \mathbb{R}$ and $a_2 : I \to \mathbb{R}$, by a previous lemma. So taking $\frac{d}{dt}$ of both sides at t_0 we get

$$\alpha'(t_0) = x_u(u_0,v_0)\,\overbrace{a_1'(t_0)}^{=c_1} + x_v(u_0,v_0)\,\overbrace{a_2'(t_0)}^{=c_2},$$

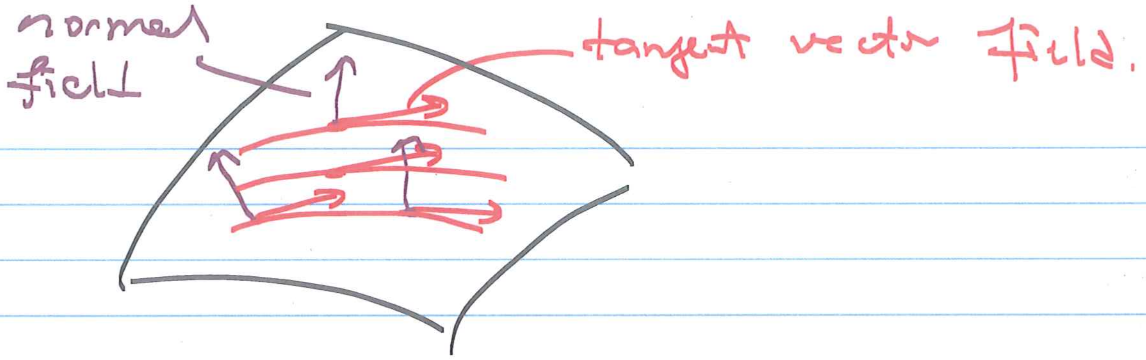
$x(u_0,v_0) = p = \alpha(t_0)$. Hence

$\alpha'(t_0) = c_1\, x_u(u_0,v_0) + c_2\, x_v(u_0,v_0)$, where $c_i = a_i'(t_0)$. This finishes the proof. ∎

<u>Definition</u>: A Euclidean vector field Z on a surface M in $\mathbb{R}^3$ is a function that assigns to each point p of M a tangent vector $Z(p)$ to $\mathbb{R}^3$ at p.

If $Z(p) \in T_pM$ for all $p \in M$ then we say that Z is a tangent vector field to M.

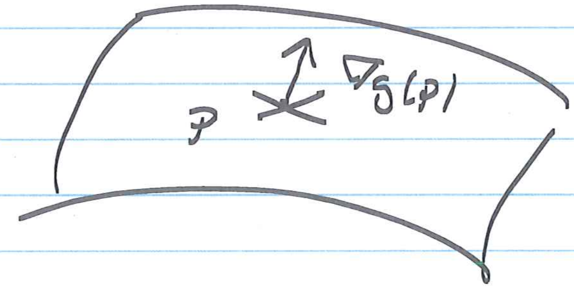
If $Z(p) \perp T_pM$ for all $p \in M$ then we say that Z is a normal vector field on M.



__Lemma:__ If $M: g=c$ is a level surface in $\mathbb{R}^3$ for some differentiable function $g: \mathbb{R}^3 \to \mathbb{R}$, then the gradient vector field

$\nabla g = \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} U_i$ is a normal vector field on M.

__Proof__ $M = g^{-1}(c) = \{p=(p_1, p_2, p_3) \in \mathbb{R}^3 \mid g(p)=c\}$



$M = g^{-1}(c)$

$g: \mathbb{R}^3 \to \mathbb{R}$

__Need to prove:__ $\nabla g(p) \perp \alpha'(t_0)$ for all tangent vectors $\alpha'(t_0) \in T_pM$, $p = \alpha(t_0)$.

Let $\alpha : I \to M$, $\alpha(t_0) = p$. Since $\alpha(t) \in M$ we see that $g(\alpha(t)) = c$ for all $t \in I$.

Say $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$, then

$g(\alpha_1(t), \alpha_2(t), \alpha_3(t)) = c$ $\forall t \in I$. Take $\frac{d}{dt}$ of both sides to get

$$0 = \frac{\partial g}{\partial x_1}\alpha_1'(t_0) + \frac{\partial g}{\partial x_2}\alpha_2'(t_0) + \frac{\partial g}{\partial x_3}\alpha_3'(t_0)$$

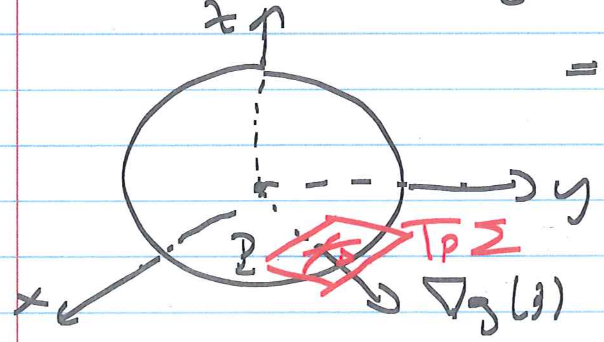
$$= \left(\frac{\partial g}{\partial x_1}(p), \frac{\partial g}{\partial x_2}(p), \frac{\partial g}{\partial x_3}(p)\right) \cdot (\alpha_1'(t_0), \alpha_2'(t_0), \alpha_3'(t_0))$$

$$= \nabla g(p) \cdot \alpha'(t_0) \Rightarrow \nabla g(p) \perp \alpha'(t_0).$$

Since $\alpha(t)$ was arbitrary we see that $\nabla g(p) \perp T_pM$. ∎

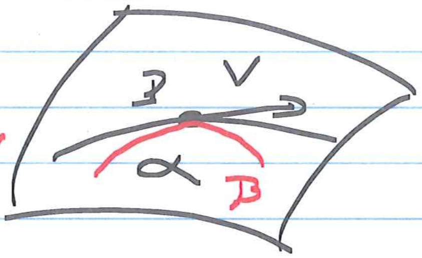
Example Let $g : \mathbb{R}^3 \to \mathbb{R}$, $g(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$. $\Sigma = g^{-1}(c)$, $c = R^2 > 0$, hence Σ is the sphere centered at the origin with radius $R > 0$. Then $\nabla g(p) = (2x_1, 2x_2, 2x_3)_p$

$$= (2p_1, 2p_2, 2p_3), \quad p = (p_1, p_2, p_3).$$



Definition: Let v be a tangent vector to M at p, and let f be a differentiable function on M. The derivative v[f] of f with respect to v is the *common value* of $\frac{d}{dt}(f(\alpha(t))(0)$, for all curves $\alpha : I \to M$ with $\alpha(0) = p$.

$\alpha'(0) = \beta'(0) = v$



$$v[f] = \frac{d}{dt}\left(f(\alpha(t))\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(f(\beta(t))\right)\Big|_{t=0}$$

∎

Video 4

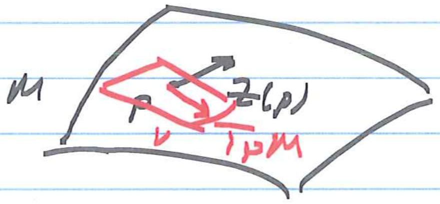
CHAPTER 5: Shape Operator

Note Title 6.04.2020

§5.1. The Shape Operator of $M \subseteq \mathbb{R}^3$.

Let Z be a Euclidean vector field on a surface $M \subseteq \mathbb{R}^3$:

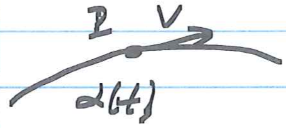
$$Z : M \longrightarrow \mathbb{R}^3, \quad Z(p) \in T_p\mathbb{R}^3.$$



We define the covariant derivative of Z at p along a vector $v \in T_pM$ as

$$\nabla_v Z = (Z_\alpha)'(0), \text{ where } Z_\alpha(t) = Z(\alpha(t)) \text{ and}$$

$\alpha : I \to M$ curve with $\alpha(0) = p$ and $\alpha'(0) = v$.



Lemma: Let $Z = \sum_{i=1}^{3} z_i U_i$, $z_i : M \to \mathbb{R}$. Then

$$\nabla_v Z = \sum_{i=1}^{3} v[z_i]\, U_i, \quad U_i : \mathbb{R}^3 \to \mathbb{R}^3$$

$U_1(p) = (1,0,0)$, $U_2(p) = (0,1,0)$ and $U_3(p) = (0,0,1)$.

Proof: $Z_\alpha(t) = Z(\alpha(t))$

$$= \sum_{i=1}^{3} z_i(\alpha(t))\, U_i .$$

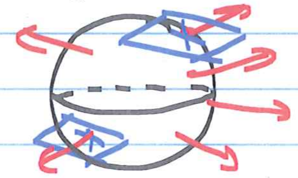
So, $$(Z_\alpha)'(0) = \sum_{i=1}^{3} \frac{d}{dt}(z_i(\alpha(t)))\Big|_{t=0} U_i$$

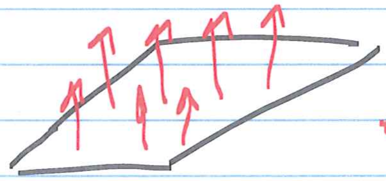
$$= \sum_{i=1}^{3} v[z_i]\, U_i . \qquad \blacksquare$$

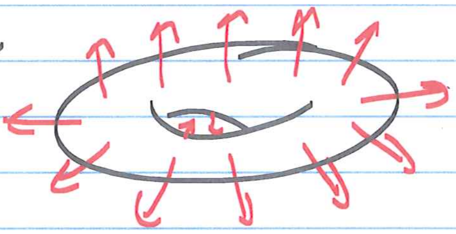
This lemma shows that $\nabla_v Z = (Z_\alpha)'(0)$ is independent of α and thus it is well-defined.

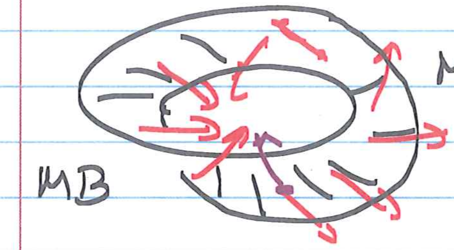
Definition: A surface $M \subseteq \mathbb{R}^3$ is called orientable if there is a continuous function $U: M \to \mathbb{R}^3$ so that $U(p) \neq 0$ and $U(p) \perp T_pM$, for all $p \in M$. Each choice of such function U is called an orientation on M.

Examples Sphere, plane, torus are all orientable surfaces.





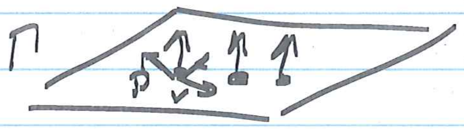




Möbius Band is non-orientable

Definition: If $p \in M$ is a point, then for each tangent vector $v \in T_pM$, $S_p(v) \doteq -\nabla_v U$, where U is a unit normal vector field defined in a neighborhood p on M. S_p will be called the shape operator of M.

Note that $S_p(v)$ measures "how fast" the normal vector (hence the tangent plane T_pM) varies as we move from the point in the direction of v.



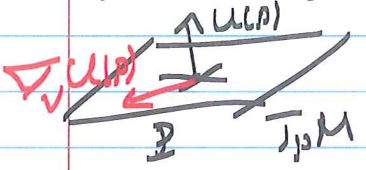
Plane $S_p(v) = 0$, for all $p \in \Pi$ and $v \in T_p\Pi$.

Lemma: For each $p \in M \subseteq \mathbb{R}^3$, the shape operator is a linear operator

$$S_p : T_pM \to T_pM, \text{ for all } p \in M.$$

Proof: Since U is a unit vector field on M we have $U(p) \cdot U(p) = 1$ for all $p \in M$.

$0 = \nabla_v (1) = \nabla_v (U(p) \cdot U(p)) = 2\, U(p) \cdot \nabla_v U(p)$, for all $p \in M$ and $v \in T_pM$. Hence, $\nabla_v U(p) \perp U(p)$ for all $p \in M$, $v \in T_pM$. It follows that



$$\nabla_v U(p) \in T_pM, \text{ for all } p \in M, v \in T_pM.$$

For linearity, note that, for any $a, b \in \mathbb{R}$, $v, w \in T_pM$,

$$S_p(av+bw) = -\nabla_{av+bw} U = -(a\nabla_v U + b\nabla_w U)$$

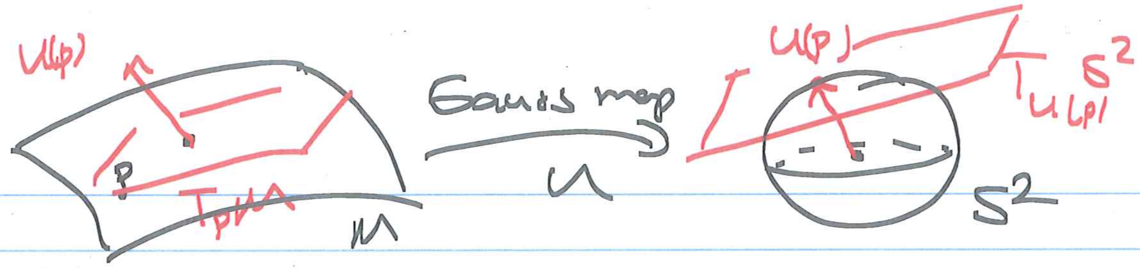
$$= -a\nabla_v U - b\nabla_w U$$

$$= a\, S_p(v) + b\, S_p(w).$$

Hence, S_p is linear on T_pM. ■

Remark: The map $U : M \to S^2$, $p \mapsto U(p)$, the unit normal vector to M at p, is called the Gauss map of M. (If M is orientable then each orientation determines one Gauss map).

For any $p \in M$ both T_pM and $T_{U(p)}S^2$ have $U(p)$ as their normal vectors and hence they are parallel.



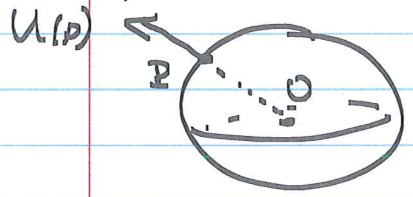
$\|U(p)\| = 1$ $\quad U_*(p): T_pM \longrightarrow T_{U(p)}S^2 = T_pM$

Is given by $U_*(p)(v) = \frac{d}{dt}(U(\alpha(t)))\Big|_{t=0}$, where $\alpha: I \longrightarrow M$, $\alpha(0) = p$, $\alpha'(0) = v$. Hence, by its definition

$$U_*(p)(v) = \nabla_v U = -S_p(v).$$

<u>Conclusion</u>: The shape operator of an oriented surface is nothing but the derivative of the Gauss map U.

<u>Examples</u> 1) $\Sigma = \{(x_1, x_2, x_3) / \sum_{i=1}^{3} x_i^2 = r^2\}$, the sphere in $\mathbb{R}^3$ with radius r.



$$U(p) = \frac{P}{r} = \frac{1}{r}\sum_{i=1}^{3} x_i U_i, \quad P = (x_1, x_2, x_3).$$

$$\text{Then } \nabla_v U = \nabla_v\left(\frac{P}{r}\right) = \frac{1}{r}\nabla_v P$$

$$= \frac{1}{r}\nabla_v \sum_i x_i U_i \Big|$$

$v = (v_1, v_2, v_3)$

$v[f] = \sum \frac{\partial f}{\partial x_i} v_i$

$$= \frac{1}{r}\sum_i v[x_i]\, U_i$$

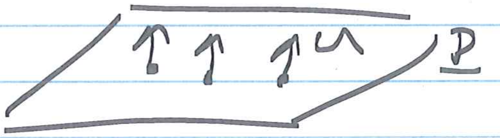
$$= \frac{1}{r}\sum_i \sum_j \overset{= \delta_{ij}}{\boxed{\frac{\partial x_i}{\partial x_j}}} v_j U_i$$

$$= \frac{1}{r}\sum_i v_i U_i = \frac{v}{r}.$$

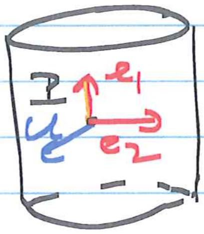
Hence, $S_p(v) = -\nabla_v U = -\frac{v}{r}$, so that S_p is the linear map given by scalar multiplication by the constant $-1/r$.

2) Let P be a plane in $\mathbb{R}^3$. Then $S_p(v) = 0$ for all $p \in P$ and $v \in T_pP$, since the unit normal vector field to P is a constant vector field and thus

$$S_p(v) = -\nabla_v U = 0, \text{ (since } U \text{ is constant).}$$



3)

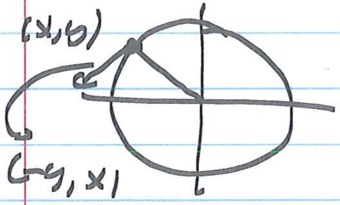


$p \in C$, $p = (x,y,z)$

$T_pC = \text{span}\{e_1, e_2\}$

$e_1 = (0,0,1)_p$, $e_2 = (-y, x, 0)$

$C: x^2+y^2=r^2$



$$U(p) = U(x,y,z) = (xU_1 + yU_2)\frac{1}{r}$$

$$\nabla_{e_1} U = \nabla_{U_3}(xU_1 + yU_2)\frac{1}{r}$$

$$= \frac{1}{r}\cdot 0 = 0.$$

On the other hand,

$$\nabla_{e_2} U(p) = \nabla_{(-y,x,0)}(xU_1 + yU_2)\frac{1}{r}$$

$$= \frac{1}{r}\nabla_{-yU_1}(xU_1 + yU_2) + \frac{1}{r}\nabla_{xU_2}(xU_1 + yU_2)$$

$$= -\frac{y}{r}\nabla_{U_1}(xU_1 + yU_2) + \frac{x}{r}\nabla_{U_2}(xU_1 + yU_2)$$

$$= -\frac{y}{r}(1\cdot u_1 + 0\cdot u_2) + \frac{x}{r}(0\cdot u_1 + 1\cdot u_2)$$

$$= \frac{1}{r}(-y u_1 + x u_2)$$

$= \frac{e_2}{r}$. Hence, $S_p(e_2) = -\frac{e_2}{r}$

4) The Saddle Surface $M: z = xy$.

Exercise: Compute $S_p(v)$, where $p = (0, 0, 0)$.

$$S_p(a u_1 + b u_2) = b u_1 + a_2 u_2.$$

Video 5

Lemma: For any $p \in M$ the shape operator

$$S: T_pM \longrightarrow T_pM \text{ is symmetric, that is}$$

$S_p(v) \cdot w = S_p(w) \cdot v$, for any $v, w \in T_pM$.

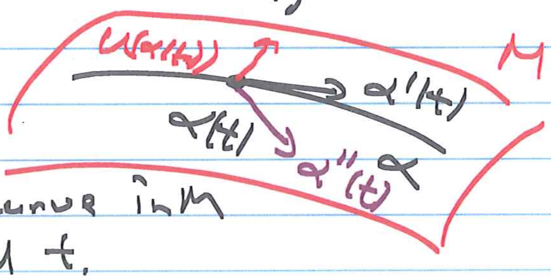
Proof will be given later.

§ 5.2. Normal Curvature:

Let M be a surface and U an orientation on M (i.e. a choice of unit normal vector field on M.)

Lemma: If α is a curve in $M \subseteq \mathbb{R}^3$, then

$$\alpha'' \cdot U = S_{\alpha(t)}(\alpha') \cdot \alpha'$$



Proof: Since $\alpha(t)$ is a curve in M
$\alpha'(t) \in T_{\alpha(t)}M$ for all t.

Hence, $\alpha'(t) \cdot U(\alpha(t)) = 0$, for all t. Take derivative of both sides:

$$0 = \frac{d}{dt}(\alpha'(t) \cdot U(\alpha(t))) = \alpha''(t) \cdot U(\alpha(t)) + \alpha'(t) \cdot (U(\alpha(t)))'$$

$$0 = \alpha''(t) \cdot U(\alpha(t)) - \alpha'(t) \cdot S_{\alpha(t)}(\alpha'(t))$$

$$\Rightarrow S_{\alpha(t)}(\alpha'(t)) \cdot \alpha'(t) = \alpha''(t) \cdot U(\alpha(t)). \quad \blacksquare$$

Remark $\alpha''(t) \cdot U(\alpha(t))$ is nothing but the normal component of the acceleration vector $\alpha''(t)$.

Definition: Let u be a unit tangent vector to $M \subseteq \mathbb{R}^3$ at a point p. Then the number $k(u) \doteq S(u) \cdot u$ is called the **normal curvature** of M in the direction $u \in T_pM$.

Remark: 1) $k(u) = S(u) \cdot u = (-S(u)) \cdot (-u)$

$$= S(-u) \cdot (-u)$$
$$= k(-u)$$

2) Let $u = \alpha'(0)$ for some curve α in M. Then

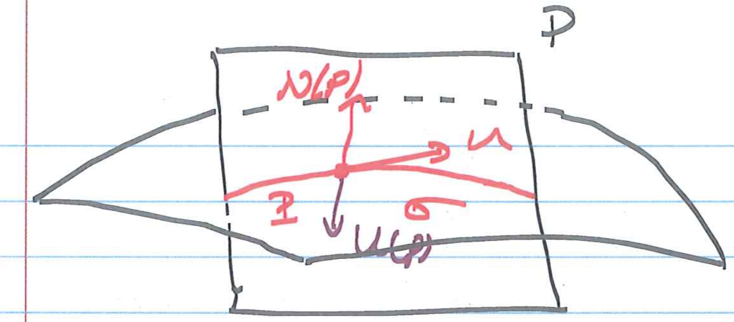
$$k(u) = S(u) \cdot u = S(\alpha'(0)) \cdot \alpha'(0) = \alpha''(0) \cdot U(\alpha(0))$$

$$= \kappa(0)\, N(0) \cdot U(\alpha(0))$$
$= \kappa(0) \cos\vartheta$, where κ is curvature of

$\alpha(t)$ at $p = \alpha(0)$ and ϑ is the angle between the vector $N(0)$ and $U(p)$.

$N(0)$: principal normal vector of α, at $p = \alpha(0)$
$U(p)$: unit normal vector to the surface M at $p \in M$.

3) Let $u \in T_pM$ be a unit tangent vector to M at p and let $\mathbb{P}$ be the plane through the point p containing the vectors u and $U(p)$. Let $\sigma(s)$ be a unit parametrization for the intersection curve $M \cap \mathbb{P}$. Since $\sigma(s) \in \mathbb{P}$ for all s, the principal normal vector of $\sigma(s)$ lies in $\mathbb{P}$ and thus
$N(p) = \pm U(p)$.



$U(p) \perp u$ since $u \in T_pM$
$N(p) \perp u$

$\sigma''(0) = \kappa_\sigma(0)\, N(0)$

So $k(u) = \sigma''(0) \cdot U(p)$

$= \kappa_\sigma(0)\, \underbrace{N(0) \cdot U(p)}_{= \pm 1}$

$= \pm \kappa_\sigma(0)$

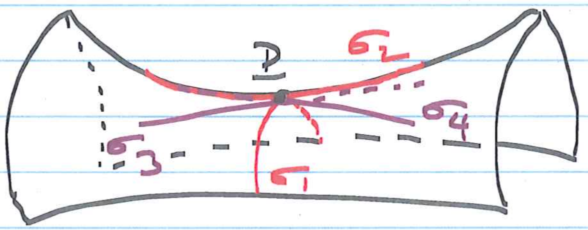
Some observations

1) If $k(u) > 0$, $N(0) = U(p)$ so the normal section σ is bending toward $U(p)$ at p. In other words, in the direction u the surface M is bending toward $U(p)$.

2) If $k(u) < 0$, $N(0) = -U(p)$, so the normal section σ is bending away from $U(p)$ at p. Thus in the direction u, M is bending away from $U(p)$.

3) If $k(p) = 0$, then $\kappa_\sigma(0) = 0$ and $N(0)$ is *not* defined. Hence the normal section σ is not bending at $\sigma(0) = p$.

Example $M: z = xy$



i) $\sigma_1 = M \cap P_1$

$P_1 : y = -x$

$\sigma_1 : z = -x^2$
$y = -x$

The normal curvature is negative.

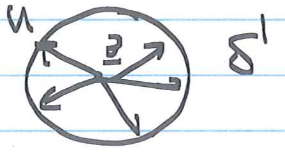
ii) $\sigma_1 = M \cap P_2$, $P_2 : y = x$, $\sigma_2 : \begin{matrix} z = x^2 \\ y = x \end{matrix}$

The normal curvature is positive

iii) σ_3 : line $z = 0, x = 0$,
σ_4 : line $z = 0, y = 0$

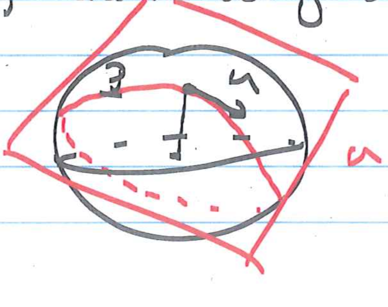
In these case, the normal curvatures are zero.

Definition: Let p be a point of a surface M in $\mathbb{R}^3$. The maximum and minimum values of the normal curvature $k(u)$ of M at p are called the principal curvatures of M at p, and are denoted k_1 and k_2. The directions in which these extreme values are obtained are called the principal vectors of M at p.



Definition: A point p of M is called umbilic provided that the normal curvature $k(u)$ is constant on all unit tangent vectors u at p.

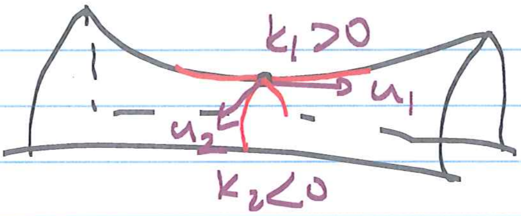
Examples 1) Σ : sphere of radius r. We've seen that the normal curvature at any point p and along any direction is $-1/r$.

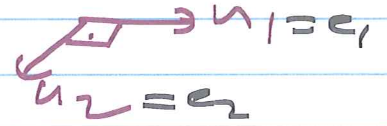


$$u \cdot S_\Sigma(u) = -\nabla_u U \cdot u$$
$$= -1/r.$$

Theorems 1) If p is an umbilic point of $M \subset \mathbb{R}^3$, then the shape operator S at p is just the scaler multiplication by the constant $k = k_1 = k_2$.

2) If p is a unumbilic point, then $k_1 \neq k_2$ and there are exactly two principal directions, and these are orthogonal.





Furthermore, if e_1 and e_2 are principal vectors in these directions, then

$$S(e_1) = k_1 e_1 \text{ and } S(e_2) = k_2 e_2$$

Video 6

Theorem: 1) If $p \in M$ is an umbilic point $M \subseteq \mathbb{R}^3$, then the shape operator S at p is just scalar multiplication by $k = k_1 = k_2$.

2) If $p \in M$ is a nonumbilic point, $k_1 \neq k_2$, then there are exactly two principal directions, and these are orthogonal. Furthermore if e_1 and e_2 are principal vectors in these directions, then $S(e_1) = k_1 e_1$ and $S(e_2) = k_2 e_2$.

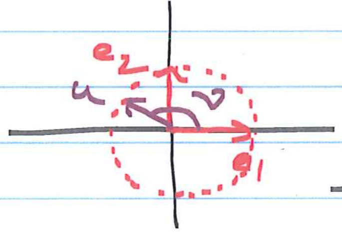
Proof: By the lemma which states that the shape operator $S_p : T_pM \to T_pM$ is symmetric, there is an orthonormal basis $\beta = \{e_1, e_2\}$ of T_pM in which the matrix representation of S_p is diagonal. So

$$[S_p]_\beta = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \text{ for some real } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Now any unit vector $u \in T_pM$ can be written as

$$u = c e_1 + s e_2 \text{ for some } c, s \in \mathbb{R} \text{ with}$$

$c^2 + s^2 = 1.$



$c = \cos\nu, \ s = \sin\nu$

$T_p\mathbb{R}^2$ $\qquad e_i \cdot e_j = \delta_{ij}$

Then
$$\begin{aligned} k(u) &= S(u) \cdot u \\ &= S(ce_1 + se_2) \cdot (ce_1 + se_2) \\ &= (c\,S(e_1) + s\,S(e_2)) \cdot (ce_1 + se_2) \\ &= (c\lambda_1 e_1 + s\lambda_2 e_2) \cdot (ce_1 + se_2) \\ &= \lambda_1 c^2 + \lambda_2 s^2 \end{aligned}$$

without loss of generality assume that $\lambda_1 \geq \lambda_2$.

$$\begin{aligned} k(u) &= \lambda_1 c^2 + \lambda_2 s^2 \\ &= \lambda_1 \cos^2 v + \lambda_2 \sin^2 v \\ &= (\lambda_1 - \lambda_2) \cos^2 v + (\lambda_2 \cos^2 v + \lambda_2 \sin^2 v) \\ &= \lambda_2 + \underbrace{(\lambda_1 - \lambda_2)}_{\geq 0} \cos^2 v \end{aligned}$$

The maximum of $k(u)$ is obtained when $\cos v = \pm 1. \Rightarrow v = 0, \pi$. Hence, $k(u)$ becomes maximal if $u = \pm e_1$ and in that case, the maximal value is

$k_1 = k(\pm e_1) = \lambda_2 + (\lambda_1 - \lambda_2) \cdot 1 = \lambda_1 \Rightarrow \lambda_1 = k_1$.

Similarly, $k(u) = \lambda_2 + (\lambda_1 - \lambda_2) \cos^2 v$ takes its minimal value if $\cos v = 0 \Leftrightarrow u = \pm e_2$. Moreover, in this case, the minimal value is $k_2 = k(\pm e_2) = \lambda_2 + (\lambda_1 - \lambda_2) \cdot 0 = \lambda_2 \Rightarrow \lambda_2 = k_2$.

Note that, if $\lambda_1 = \lambda_2$, then

$$[S]_{\beta} = \begin{bmatrix} \lambda_1 & \\ & \lambda_1 \end{bmatrix} \Rightarrow S(u) = \lambda u$$

$$k(u) = S(u) \cdot u = \lambda \underbrace{u \cdot u}_{=1} = \lambda \text{ constant function}$$

$\Rightarrow k_1 = k_2 = \lambda$. ∎

<u>Summary</u> e_1, e_2, k_1 and k_2 are as above. Then for any unit vector $u \in T_pM$, we have

$u = \cos v \, e_1 + \sin v \, e_2$ and

$k(u) = S(u) \cdot u = k_1 \cos^2\gamma + k_2 \sin^2\gamma.$

§5.3. Gaussian Curvature:

Definition: Let $p \in M \subseteq \mathbb{R}^3$ and $S_p : T_pM \to T_pM$ be the shape operator at $p \in M$. The determinant of S_p is called the Gaussian curvature of the surface M at p, and the half of the trace of S_p is called the scalar curvature of M at p.

Notation: $K(p) \doteq \det(S_p)$, $H(p) \doteq \frac{1}{2} \operatorname{tr}(S_p)$.

Lemma: If k_1 and k_2 are the principal curvatures of M at p, then

$$K(p) = k_1 k_2 \quad \text{and} \quad H(p) = \frac{1}{2}(k_1 + k_2).$$

Proof: $[S_p]_\beta = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$, $\beta = \{e_1, e_2\}$, where e_1 and e_2 are the principal directions:

$$S_p(e_1) = k_1 e_1 \quad \text{and} \quad S_p(e_2) = k_2 e_2$$

$$K(p) = \det(S_p) = k_1 k_2, \quad H(p) = \frac{1}{2} \operatorname{tr}(S_p) = \frac{1}{2}(k_1 + k_2).$$

Remark: If β' is any other basis for T_pM and $A = [S_p]_{\beta'}$, then $\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = P^{-1} A P$ where $P = [I]_{\beta}^{\beta'}$, the base change matrix.

Hence, $k_1k_2 = \det(P^{-1}AP) = \overbrace{\det(P^{-1})\det A(\det P)}^{=1}$

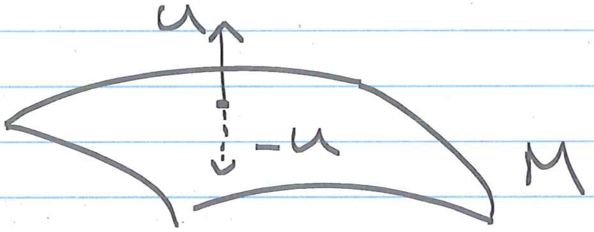
$= \det A$

and similarly $k_1+k_2 = tr(P^{-1}AP)$

$= tr(AP.P^{-1})$

$= tr(A)$ ∎

Remark:



Let S_p be the shape operator determined by the normal field U. Then we know that the shape operator for the field $-U$ is $-S_p$. Hence, replacing U with $-U$ replaces the principle curvature k_1 and k_2 by $-k_1$ and $-k_2$. Thus, the Gaussian curvature

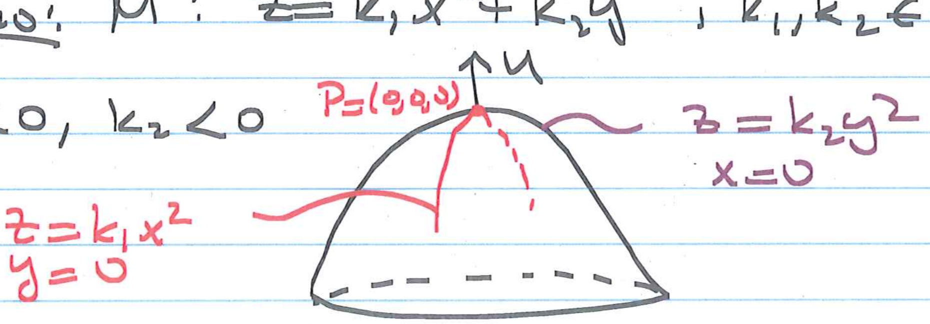
$K = k_1k_2 = (-k_1)(-k_2)$ is not altered, but

the scalar curvature $H = \frac{1}{2}(k_1+k_2)$ becomes

$-H = \frac{1}{2}(-k_1-k_2)$.

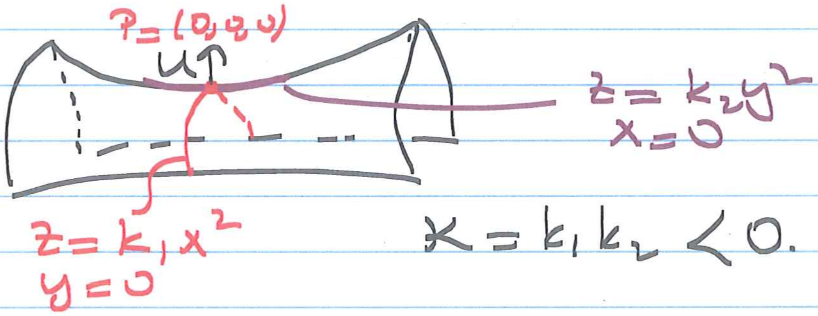
Examples: $M: z = k_1x^2 + k_2y^2$, $k_1, k_2 \in \mathbb{R}$

1) $k_1 < 0$, $k_2 < 0$



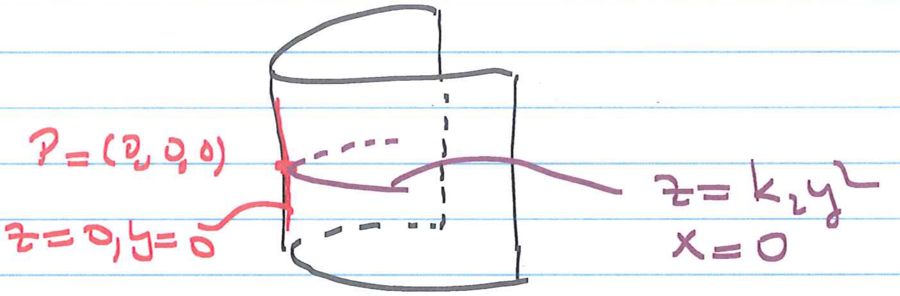
$K = k_1k_2 > 0$, $H = \frac{1}{2}(k_1+k_2)$

2) $z = k_1 x^2 + k_2 y^2$, $k_1 < 0$, $k_2 > 0$



$K = k_1 k_2 < 0.$

3) $k_1 = 0$, $k_2 < 0$ $\quad z = k_1 x^2 + k_2 y^2 = k_2 y^2$



$K = k_1 k_2 = 0.$

Lemma: If v and w are linearly independent vectors T_pM, $M \subseteq \mathbb{R}^3$, then

$$S_p(v) \times S_p(w) = K(p)\, v \times w, \text{ and}$$

$$S_p(v) \times w + v \times S_p(w) = 2H(p)\, v \times w.$$

Proof: Since v and w are linearly independent then $B = \{v, w\}$ is a basis for T_pM. Let

$$A = (S_p)_B, \text{ where } A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Then $S_p(v) = av + bw$, $S_p(w) = cv + dw$.

Now, $S_p(v) \times S_p(w) = (av+bw) \times (cv+dw)$

$$= (ad-bc) \quad v \times w$$

$$= \det A \; v \times w$$

$$= \det(S_p) \; v \times w$$

$$= K(p) \; v \times w.$$

Similarly,

$$S_p(v) \times w + v \times S_p(w) = (av+bw) \times w + v \times (cv+dw)$$

$$= a \, v \times w + d \, v \times w$$

$$= (a+d) \; v \times w$$

$$= tr(A) \; v \times w$$

$$= tr(S_p) \; v \times w$$

$$= 2H(p) \; v \times w.$$

▬

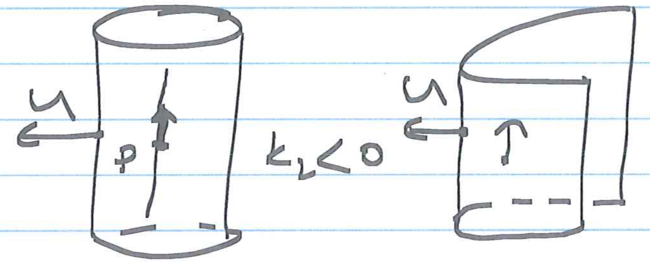
<u>Remark:</u> $k_1 k_2 = K$ and $k_1 + k_2 = 2H$ and then k_1 and k_2 are the roots of $\lambda^2 - 2H\lambda + K = 0$.

Hence, $k_1, k_2 = \dfrac{-(-2H) \pm \sqrt{4H^2 - 4K}}{2}$

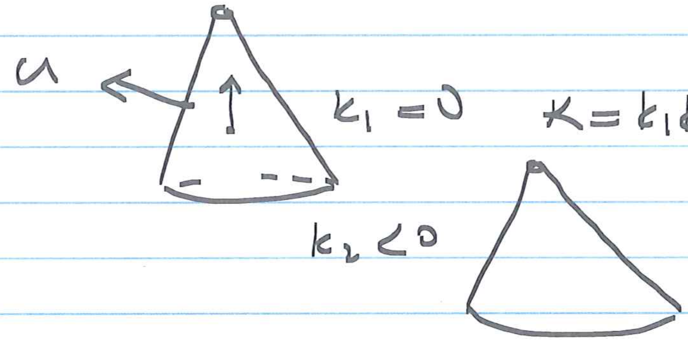
$$= H \pm \sqrt{H^2 - K}.$$

<u>Definition:</u> A surface $M \subseteq \mathbb{R}^3$ is called flat if $K(p) = 0$ for all $p \in M$ and is called minimal if $H(p) = 0$, for all $p \in M$.

Example: Any cylinder or a cone is a flat surface.

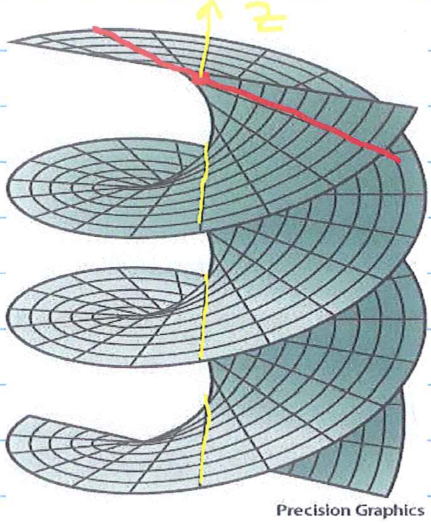


In both case $k_1=0$. Hence, $K=k_1k_2=0$



$K=k_1k_2=0.$

Video 7



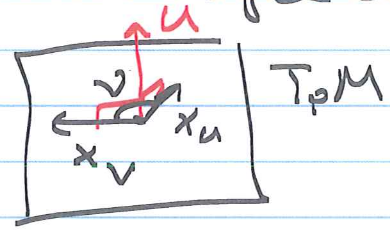
Precision Graphics

§5.4. Computational Techniques:

$M \subseteq \mathbb{R}^3$ a surface, $x : D \to M$ a surfa patch.

$E = x_u \cdot x_u$, $F = x_u \cdot x_v$, $G = x_v \cdot x_v$.

Remark: 1) $F = x_u \cdot x_v = \|x_u\| \cdot \|x_v\| \cos\vartheta$, where ϑ is the angle between x_u and x_v in T_pM.



2) $$\|x_u \times x_v\| = \|x_u\|^2 \|x_v\|^2 - (x_u \cdot x_v)^2$$
$$= EG - F^2$$

3) If $v = v_1 x_u + v_2 x_v$ and $w = w_1 x_u + w_2 x_v$, then

$$v \cdot w = (v_1 x_u + v_2 x_v) \cdot (w_1 x_u + w_2 x_v)$$
$$= v_1 w_1 E + v_2 w_2 G + (v_1 w_2 + v_2 w_1) F$$

4) The normal vector field U to M can be written as

$$U = \frac{x_u \times x_v}{\|x_u \times x_v\|}.$$

* * *

Let $x = x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$, then

$$x_{uu} = \left(\frac{\partial^2 x_1}{\partial u^2}, \frac{\partial^2 x_2}{\partial u^2}, \frac{\partial^2 x_3}{\partial u^2}\right), \quad x_{uv} = \left(\frac{\partial^2 x_1}{\partial u \partial v}, \frac{\partial^2 x_2}{\partial u \partial v}, \frac{\partial^2 x_3}{\partial u \partial v}\right)$$

and $x_{vv} = \left(\frac{\partial^2 x_1}{\partial v^2}, \frac{\partial^2 x_2}{\partial v^2}, \frac{\partial^2 x_3}{\partial v^2}\right).$

We also define quantities:

$L \doteq S(x_u) \cdot x_u$
$M \doteq S(x_u) \cdot x_v = S(x_v) \cdot x_u$
$N \doteq S(x_v) \cdot x_v$

<u>Corollary</u> If x is a surface patch in $M \subseteq \mathbb{R}^3$, then

$$K(x) = \frac{LN - M^2}{EG - F^2}, \quad H(x) = \frac{GL + EN - 2FM}{2(EG - F^2)}.$$

<u>Proof</u>: In the last lecture we've seen that

1) $S(v) \times S(w) = K(x)\, v \times w$

2) $S(v) \times w + v \times S(w) = 2H(x)\, v \times w$.

We'll make use of the so called the

"Lagrange Identity": For any vectors x, y, v, w in $\mathbb{R}^3$ we have

$$(x \times y) \cdot (v \times w) = \begin{vmatrix} x \cdot v & x \cdot w \\ y \cdot v & y \cdot w \end{vmatrix}$$ (Exercise 6 of § 6.3)

Now take the dot product of the terms in the equation (1) above with $\underline{v \times w}$ to get

$$\begin{vmatrix} S(v) \cdot v & S(v) \cdot w \\ S(w) \cdot v & S(w) \cdot w \end{vmatrix} = K(x) \begin{vmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{vmatrix}$$ and hence

$$K(x) = \frac{\begin{vmatrix} S(v) \cdot v & S(v) \cdot w \\ S(w) \cdot v & S(w) \cdot w \end{vmatrix}}{\begin{vmatrix} v \cdot v & v \cdot w \\ w \cdot v & w \cdot w \end{vmatrix}}$$ (but $v = x_u$, $w = x_v$)

$$= \frac{LN - M^2}{EG - F^2}$$

Similarly using the second equation we get

$$H(x) = \frac{GL + EN - 2FM}{2(EG - F^2)}$$

This finishes the proof. ▬

<u>Remark</u>: Since $x_u, x_v \in T_pM$ and $U \perp T_pM$

we get $U \cdot x_u = 0$.

$$0 = \frac{\partial}{\partial x_v}(U \cdot x_u) = U_v \cdot x_u + U \cdot x_{uv}$$

Since, $S_p(v) = -\nabla_v U = -U_v$ we get

$S(x_v) = -\nabla_{x_v} U = -U_v$.

Here, $S_p(x_v)\cdot x_u = -U_v \cdot x_u = -(-U\cdot x_{uv})$

$\Rightarrow S_p(x_v)\cdot x_u = U\cdot x_{uv}$.

Hence, $S_p(x_u)\cdot x_v = U\cdot x_{vu} = U\cdot x_{uv} = S_p(x_v)\cdot x_u$, which proves that S is symmetric.

Now we have a

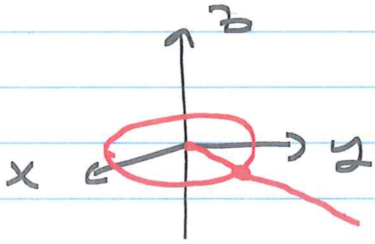
Lemma: If x is a surface patch for $M\subseteq\mathbb{R}^3$, then

1) $L = S(x_u)\cdot x_u = U\cdot x_{uu}$
2) $M = S(x_u)\cdot x_v = U\cdot x_{uv}$
3) $N = S(x_v)\cdot x_v = U\cdot x_{vv}$.

Proof: (1) and (3) follows from the first Lemma of Section 5.2. which states that " $\alpha''\cdot U = S(\alpha')\cdot\alpha'$ ".

The middle one is proved above. ■

Examples: 1) Helicoid.



$x(u,v) = (u\cos v, u\sin v, bv)$, $b \neq 0$

$x_u = (\cos v, \sin v, 0)$, $x_v = (-u\sin v, u\cos v, b)$

$E = x_u \cdot x_u = 1$, $F = x_u \cdot x_v = 0$, $G = x_v \cdot x_v = u^2 + b^2$

$$U = \frac{\begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & b \end{vmatrix}}{\| \downarrow \|} = \frac{(b\sin v, -b\cos v, u)}{\sqrt{b^2+u^2}}.$$

$x_{uu} = (0,0,0)$, $x_{uv} = (-\sin v, \cos v, 0)$, $x_{vv} = (-u\cos v, -u\sin v, 0)$

Hence, $L = x_{uu} \cdot U = 0$, $N = x_{vv} \cdot U = 0$

$$M = x_{uv} \cdot U = \frac{-b}{\sqrt{b^2+u^2}}$$

Hence, $$K = \frac{LN - M^2}{EG - F^2} = \frac{-\left(-b/\sqrt{b^2+u^2}\right)^2}{(b^2+u^2)} = \frac{-b^2}{(b^2+u^2)^2}$$

and $$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = 0.$$

Example 2: The Saddle Surface: $M: z = xy$

$x(u,v) = (u, v, uv)$.

$x_u = (1, 0, v)$, $x_v = (0\ 1, u)$

$E = x_u \cdot x_u = 1+v^2$, $F = x_u \cdot x_v = uv$, $G = x_v \cdot x_v = 1+u^2$

$$U = \frac{x_u \times x_v}{\| x_u \times x_v \|} = \frac{(-v, -u, 1)}{\sqrt{1+u^2+v^2}}$$

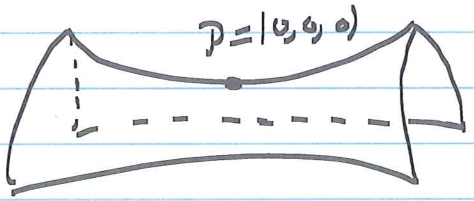
$x_{uu} = 0$, $x_{uv} = (0,0,1)$, $x_{vv} = 0$.

$L = x_{uu} \cdot U = 0$, $N = x_{vv} \cdot U = 0$

$$M = x_{uv} \cdot U = \frac{1}{\sqrt{1+u^2+v^2}}$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-1}{(1+u^2+v^2)^2}$$ and

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{-uv}{(1+u^2+v^2)^{3/2}}.$$



$K(p) = -1$, $H(p) = 0$.

§5.5. The Implicit Case:

$M \subseteq \mathbb{R}^3$ surface, given by $M: g=0$, for some smooth function $g:\mathbb{R}^3 \to \mathbb{R}$.

We assume that $0 \in \mathbb{R}$ is a regular value for g and thus

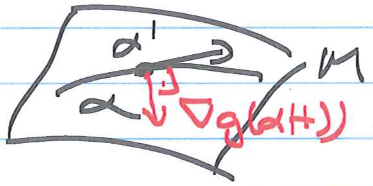
$g_* : T_p\mathbb{R}^3 \to T_0\mathbb{R} \simeq \mathbb{R}$, for any

$p \in M$ ($p \in M \iff g(p)=0$), is surjective.

g_* is given by $\nabla g = \sum_{i=1}^{3} \frac{\partial g}{\partial x_i} U_i$ where

$$g_*(v) = \nabla g(p) \cdot v = \sum_{i=1}^{3} \frac{\partial g}{\partial x_i}(p)\, v_i \quad , \quad v = \sum_{i=1}^{3} v_i U_i .$$

Hence, we see that $\nabla g(p) \neq 0$ for all $p \in M$.



Let $\alpha : I \to M \subseteq \mathbb{R}^3$ be any curve. Then

$g(\alpha(t)) = 0$ for all t, and $\alpha'(t) \in T_{\alpha(t)} M$.

$0 = \frac{d}{dt}(g(\alpha(t))) = \nabla g(\alpha(t)) \cdot \alpha'(t)$, for all t.

Hence, $\nabla g(\alpha(t)) \perp T_{\alpha(t)} M$.

Let $Z \doteq \nabla g$ and $U = \frac{Z}{\|Z\|}$ be the unit normal vector field on M determined by Z.

Let S be the shape operator of M corresponding to the normal field $U = \frac{Z}{\|Z\|}$.

Then $S_p(v) = -\nabla_v U$, for any $v \in T_pM$.

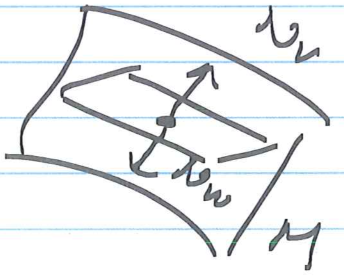
Let's write $Z = \sum_{i=1}^{3} z_i U_i$, then

$$\nabla_v Z = \sum_{i=1}^{3} v[z_i] U_i \text{, and}$$

$$\nabla_v U = \nabla_v \left(\frac{Z}{\|Z\|}\right) = \frac{\nabla_v Z}{\|Z\|} + \nabla_v \left(\frac{1}{\|Z\|}\right) Z$$

$$= \frac{1}{\|Z\|} \nabla_v Z + v\left(\frac{1}{\|Z\|}\right) Z.$$

Note that $v\left(\frac{1}{\|Z\|}\right) Z$ is normal to the surface and we denote it by $-N_v$.



So, $S(v) = -\nabla_v U = \frac{-\nabla_v Z}{\|Z\|} + N_v$.

<u>Remark</u> 1) If $W \in T_pM$, then N_W is normal to M and thus parallel to N_v. Hence, $N_v \times N_w = 0$.

2) Also if Y is any tangent vector field to M, then $Y \times N_v$ is perpendicular to N_v, and thus $Y \times N$ is another tangent vector field.

Lemma: Let Z be a nowhere zero normal field on M. If V and W are tangent vector fields on M such that $V \times W = Z$, then

$$K = \frac{Z \cdot (\nabla_V Z \times \nabla_W Z)}{\|Z\|^4} \quad \text{and}$$

$$H = - Z \cdot \frac{\nabla_V Z \times W + V \times \nabla_W Z}{2\|Z\|^3}.$$

Proof: We know from Lemma 3.4. that

1) $S(V) \times S(W) = K(p)\, \underline{V \times W}$

2) $S(V) \times W + V \times S(W) = 2H(p)\, V \times W.$

From the above computation we have

$$S(V) = -\frac{\nabla_V Z}{\|Z\|} + N_V \quad \text{and} \quad S(W) = -\frac{\nabla_W Z}{\|Z\|} + N_W$$

By the assumption $V \times W = Z$. Taking dot product of (1) by $Z = V \times W$ we get

$$K(p) = \frac{\boxed{Z} \cdot (S(V) \times S(W))}{\|Z\|^2}$$

$$\text{Now, } S(V) \times S(W) = \frac{\nabla_V Z \times \nabla_W Z}{\|Z\|^2} - \overbrace{\frac{\nabla_V Z \times N_W}{\|Z\|}}^{\perp Z} - \underbrace{\frac{N_V \times \nabla_W Z}{\|Z\|}}_{\perp Z} + \underbrace{N_V \times N_W}_{=0}$$

Hence, we see that

$$K(p) = \frac{Z\cdot(\nabla_V Z \times \nabla_W Z)}{\| Z\|^4}.$$

Similar arguments prove the second statement. ▬

<u>Example</u>: let $M: g = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$, be an ellipsoid. The the normal field $Z = \frac{1}{2}\nabla g$ is given by $Z = \sum_{i=1}^{3} \frac{x_i}{a_i^2} U_i$.

let $V = \sum_{i=1}^{3} v_i U_i$ be tangent field on M.

Then $\nabla_V Z = \sum_{i=1}^{3} \frac{V[x_i]}{a_i^2} U_i = \sum_{i=1}^{3} \frac{v_i}{a_i^2} U_i$,

since $V[x_i] = dx_i(V) = v_i$.

Now, $Z\cdot(\nabla_V Z \times \nabla_W Z) = \begin{vmatrix} \frac{x_1}{a_1^2} & \frac{x_2}{a_2^2} & \frac{x_3}{a_3^2} \\ \frac{v_1}{a_1^2} & \frac{v_2}{a_2^2} & \frac{v_3}{a_3^2} \\ \frac{w_1}{a_1^2} & \frac{w_2}{a_2^2} & \frac{w_3}{a_3^2} \end{vmatrix}$

$$\Rightarrow Z\cdot(\nabla_V Z \times \nabla_W Z) = \frac{1}{a_1^2 a_2^2 a_3^2}(X\cdot(V\times W))$$

$X = \sum x_i U_i$, $V = \sum v_i U_i$, $W = \sum w_i U_i$.

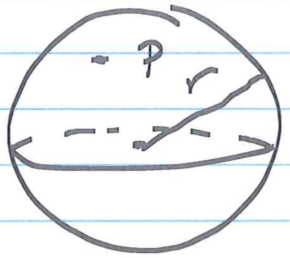
Moreover, $X \cdot (V \times W) = X \cdot Z = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = 1.$

Hence, $K(p) = \frac{1}{a_1^2 a_2^2 a_3^3 \|Z\|^4}$, $\|Z\|^4 = \left(\sum_{i=1}^{3} \frac{x_i^2}{a_i^2} \right)^2.$

$p = (x_1, x_2, x_3).$

If $a_1 = a_2 = a_3 = r > 0$ (i.e. the ellipsoid is a sphere of radius r), then

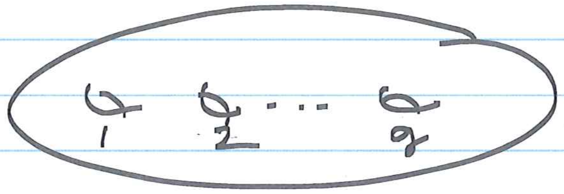
$$K(p) = \frac{1}{r^6 (1/r^4)} = \frac{1}{r^2}.$$



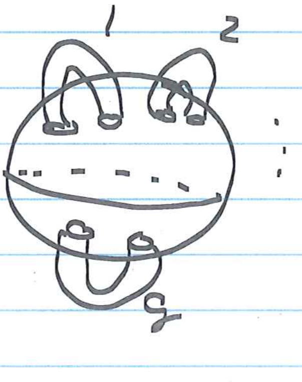
Remark: $\int_{S_r^2} K(p)\, dA = \frac{1}{r^2}(4\pi r^2) = \underline{4\pi}$

This is a special instance of the Gauss-Bonnet Theorem.

Description of Gauss-Bonnet Theorem:



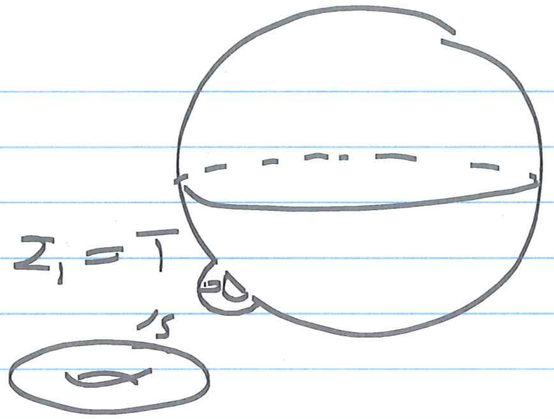
$M = \Sigma_g$



$K(p)$: Gaussian Curvature function

$$\int_{\Sigma_g} K(p)\, dA = 4\pi(1-g).$$

$\left(g = 0,\ \Sigma_0 = S^2 \Rightarrow 4\pi(1-0) = 4\pi \right)$



$$\int_{S^2} \kappa(p) = 4\pi$$

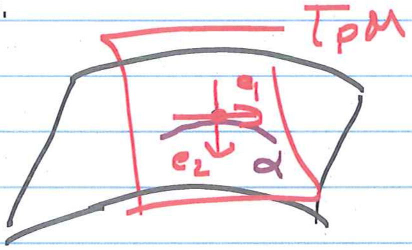
$$\int_{T^2} \tilde{\kappa}(p) = 4\pi(1-\overset{1}{\overset{\shortparallel}{g}}) = 0.$$

Video 9

Note Title 24.04.2020

§5.6. Special Curves in a Surface

Definition: A regular curve α in a surface $M\subseteq\mathbb{R}^3$ is a principal curve provided that the velocity vector α' of α always points in a principal direction.



Remark: Principal curves moves in directions in which the surface bends in extereme values. Through an non umbilic point there passes exactly two principal curves.

Lemma: Let α be a regular curve in $M\subseteq\mathbb{R}^3$, and let U be a unit normal field along α. Then

1) The curve α is principal if and only if U' and α' are collinear at each point.

2) If α is a principal curve, then the principal curvature along α' is $\alpha''\cdot U/(\alpha'\cdot\alpha')$

Proof: Claim $S(\alpha')=-U'$.

Proof of the claim: We know that $S_p(v)=-D_pU(v)$ and thus

$$S_p(\alpha')=-\frac{d}{dt}(U(\alpha(t))$$
$$=-U'(t).$$

Hence, U' and α' are collinear if and only if

$S(\alpha')$ and α' are collinear. However this means that $S(\alpha') = \lambda \alpha'$ and thus α' points in a principal direction, that α is a principal curve.

2) If α is a principal curve, $\frac{\alpha'}{\|\alpha'\|}$ is a principal direction. Hence, if k_i is the principal curvature in the direction of $\frac{\alpha'}{\|\alpha'\|}$ then

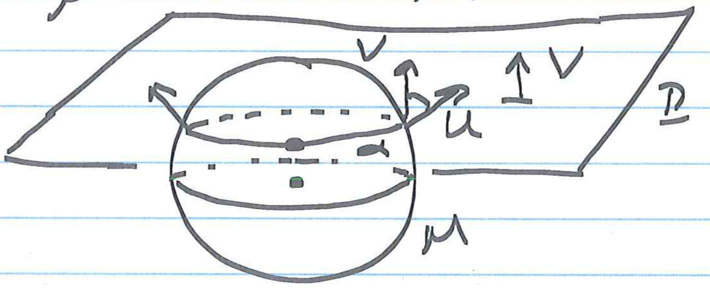
$$S\left(\frac{\alpha'}{\|\alpha'\|}\right) = k_i \frac{\alpha'}{\|\alpha'\|}. \quad \text{So}$$

$$k_i = \frac{\alpha'}{\|\alpha'\|} \cdot \frac{\alpha'}{\|\alpha'\|} = S\left(\frac{\alpha'}{\|\alpha'\|}\right) \cdot \frac{\alpha'}{\|\alpha'\|}$$

$$= \frac{S(\alpha') \cdot \alpha'}{\alpha' \cdot \alpha'}. \quad \blacksquare$$

Lemma: Let α be a curve cut from a surface $M \subset \mathbb{R}^3$ by a plane P. If the angle between M and P is constant along α, then α is a principal curve in M.

Ex



Proof: Let U and V be unit normals to M and P respectively. Thus $V' = 0$ since P is a plane and hence V is constant on P. On the other hand, by the assumption

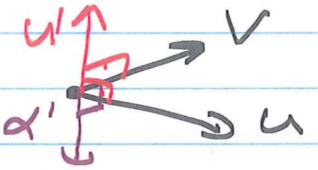
$$U \cdot V = \|U\| \, \|V\| \cos \nu = 1 . 1 . \cos \nu = C \text{ constant}$$

Take derivative w.r.t. t to get

$$u'\cdot V + u\cdot \overset{\overset{0}{\shortparallel}}{V'} = 0 \Rightarrow u'\cdot V = 0.$$

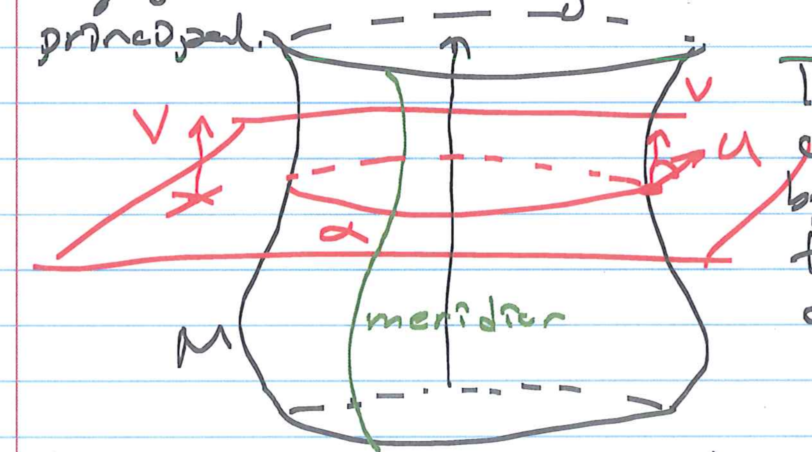
Hence $u' \perp V$ along α. We also know that since α is a unit speed curve in both M and P, α' is perpendicular to both u and V.

Since u is a unit vector field u' is perpendicular to u also. It follows that α' and u' are collinear. Hence, by the previous lemma α is a principal curve. Of course, implicitly we assumed that u and V are linearly independent.



If u and V are linearly dependent then $u = \pm V$. Hence, $u' = \pm V' = 0$. In this case, the α is clearly principal.

Examples Let M be a surface of revolution. Then any parallel and any meridian curve is principal.



To get a parallel cut the surface by a plane having the revolution axis as its normal.

A meridian is obtained by cutting the surface M by a plane containing the axis of revolution. In this case the angle, along the meridian, between u and v is $\pi/2$.

Definition: Directions tangent to $M \subseteq \mathbb{R}^3$ in which the normal curvature is zero are called asymptotic directions. So a tangent vector v is asymptotic if $k(v) = S(v) \cdot v = 0$.

Lemma: Let $p \in M \subseteq \mathbb{R}^3$ be a point.

1) If $K(p) > 0$, then there are no asymptotic directions at p.

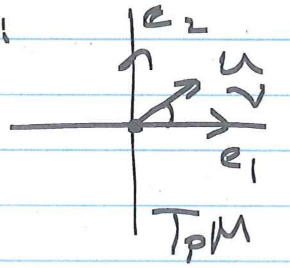
2) If $K(p) < 0$, then there are exactly two asymptotic directions at p, and they are bisected by the principal direction at angle ν such that

$$\tan^2 \nu = - \frac{k_1(p)}{k_2(p)}.$$

3) If $K(p) = 0$, then every direction is asymptotic is p is a planar point, otherwise there is exactly one asymptotic direction and it is also principal.

Proof: Recall Euler's formula:

$$k(u) = k_1(p) \cos^2 \nu + k_2(p) \sin^2 \nu$$



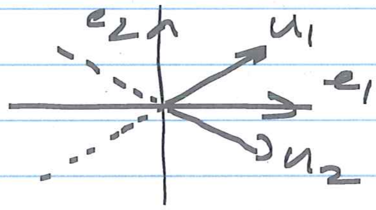
1) If $K(p) = k_1(p) k_2(p) > 0$ then $k_1(p)$ and $k_2(p)$ have the same sign and thus $k(u)$ is never zero.

2) If $K(p) < 0$ then $k_1(p) k_2(p) < 0$. Then

for a vector u with angle ν satisfying

$$\tan^2\gamma = -\frac{k_1(p)}{k_2(p)} > 0 \text{ (such } \nu \text{ exists)}$$

$$\tan\nu_1 = \sqrt{-k_1/k_2}\,, \quad \tan\nu_2 = -\sqrt{\frac{-k_1}{k_2}}$$

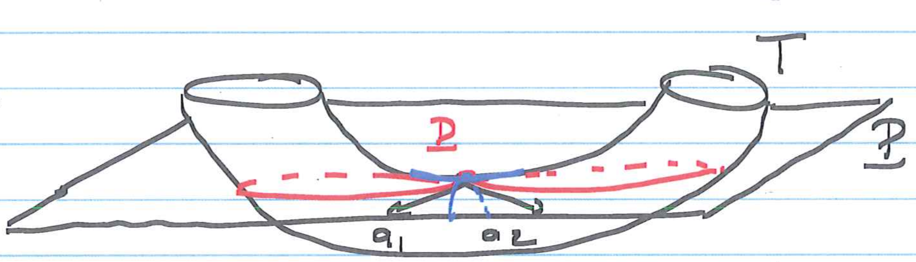


3) Now assume $K(p)=0$. Then either $k_1=k_2=0$ which means p is a planar point and in this case every direction is asymptotion.
Or only one of k_1 or k_2 is zero. Say $k_1=0$ then

$$k(u) = k_2 \sin^2\nu \text{ and } k(u)=0 \text{ exactly if}$$

$\nu = 0, \pi. \Longrightarrow u = e_1.$

Example



a_1, a_2 asymptotic directions.

Definition: A regular curve α in $M \subseteq \mathbb{R}^3$ is an asymptotic curve provided its velocity α' always points in an asymptotic direction.
Then α is asymptotic if and only if

$$k(\alpha') = S(\alpha')\cdot\alpha' = 0.$$

Remark: Since we know that $S(\alpha') = -U'$, α is asymptotic if and only if $\alpha' \cdot U' = 0$.

On the other hand, α is a curve in M and thus $\alpha' \cdot U = 0$. So taking derivative we get

$$\underbrace{\alpha' \cdot U'}_{\overset{\shortparallel}{0}} + \alpha'' \cdot U = 0 \Rightarrow \alpha'' \cdot U = 0.$$

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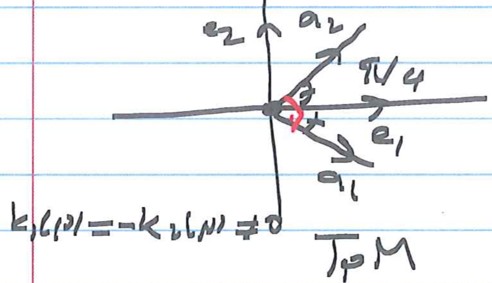
§5.6. Special Curves:

An Application to Minimal and Flat Surfaces

Recall that a surface $M \subseteq \mathbb{R}^3$ is minimal if $H(p) = k_1(p) + k_2(p) = 0$, for all $p \in M$. So by the previous lemma either both $k_i(p) = 0$ for $i = 1, 2$ or $k_1(p) = -k_2(p) \neq 0$ ($\Rightarrow K(p) < 0$) and there are exactly two asymptotic direction with $v = \pm \pi/2$.



$$\tan^2 v = -\frac{k_1(p)}{k_2(p)} = 1$$

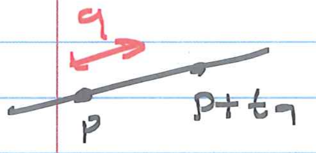
$$\tan v = \pm 1 \Rightarrow v = \pm \pi/2$$

In particular, the asymptotic directions a_1 and a_2 are orthogonal either. Thus a surface with $K < 0$ is minimal if and only if through every point there are exactly two asymptotic curves passes and they are orthogonal.

Recall that for the Helicoid we had $K < 0$ and $H = 0$ at all points and thus the Helicoid is a minimal surface. Saddle surface had also $K < 0$, and both surfaces were ruled.

Lemma: A ruled surface M has Gaussian curvature $K \leq 0$. Furthermore, $K = 0$ if and only if the unit normal U is parallel along each ruling of M (so all the points on a ruling have the same Euclidean tangent plane.)

Proof A ruling in a ruled surface is a straight line $t \mapsto p+tq$ and it is clearly asymptotic because its acceleration is zero. Hence, $K(p) \leq 0$ for all $p \in M$



Pf If U is parallel along $\alpha(t)=p+tq$, then $S(\alpha')=-U'=0 = 0\cdot\alpha'$. Thus α is a principal curve with principal curvature $k(\alpha')=0$. Hence, $K=k_1k_2=0$.

Conversely, if $K=0$ we see from the previous lemma that asymptotic directions (and thus curves) are also principal. Therefore, each ruling is principal ($S(\alpha')=k(\alpha')\alpha'$) as well as asymptotic ($k(\alpha')=0$), hence $U'=-S(\alpha')=0$, and thus U is parallel along each ruling of M. ■

Definition: A curve α in a surface $M \subseteq \mathbb{R}^3$ is a geodesic of M if its acceleration α'' is always normal to the surface.

Proposition: A geodesic has constant speed.

Proof: $\frac{d}{dt}(\|\alpha'\|^2) = \frac{d}{dt}(\alpha'\cdot\alpha') = 2\alpha'\cdot\alpha'' = 0$ if $\alpha(t)$ is a geodesic. Hence, $\|\alpha'\|$ is constant. ■

Proposition: Any line in a surface is a geodesic.

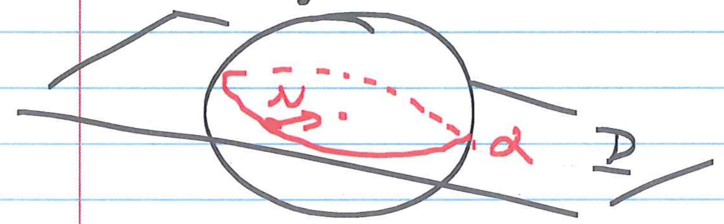
Proof: Any line have the form $\alpha(t)=p+tq$ and thus $\alpha''=0 \Rightarrow \alpha''$ is normal to the surface. Hence, α is a geodesic. ■

Examples 1) Let P be a plane in $\mathbb{R}^3$. If α is any curve in P, then $\alpha' \cdot U = 0$ and by taking derivative we obtain $\alpha'' \cdot U + \alpha' \cdot \underbrace{U'}_{=0} = 0 \Rightarrow \alpha'' \cdot U = 0$.

Here, if α is a geodesic then $\alpha'' = \lambda U$ and thus $\alpha'' \cdot U = 0 \Rightarrow \lambda \|U\|^2 = \lambda = 0 \Rightarrow \alpha'' = 0 \Rightarrow$ $\alpha(t) = p + tq$, i.e., it is a line.

Hence, on a plane $P \subseteq \mathbb{R}^3$ a curve on P is a geodesic if and only if it is a line.

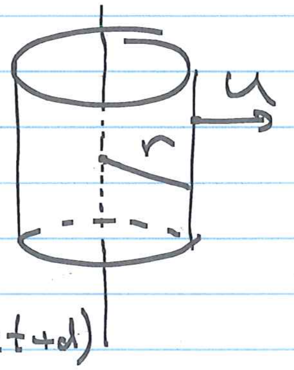
2) Let $\Sigma \subseteq \mathbb{R}^3$ be a sphere. Let α be a unit speed parametrization of the circle cut out by a plane P passing through the center of the sphere Σ.



$\alpha'' \parallel N \parallel U \Rightarrow \alpha'' \perp \Sigma$

$\Rightarrow \alpha$ is a geodesic.

Remark: Indeed, great circles are the only geodesics of a sphere.



3) **Cylinders** $M \subseteq \mathbb{R}^3$, $x^2 + y^2 = r^2$

Claim: Any geodesic on M has the form

$$\alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d)$$

for some constant a, b, c, d.

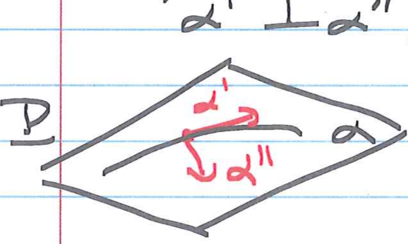
Proof: Let $\alpha(t) = (r\cos\vartheta(t), r\sin\vartheta(t), h(t))$ be any geodesic on M. Since $\alpha'' \parallel U = (*, *, 0)$ we see that $h''(t) = 0$. Hence, $h(t) = ct + d$ for some $c, d \in \mathbb{R}$. Also the speed of α must be constant

$$\|\alpha'\| = \sqrt{r^2(\gamma'(t))^2 + \underbrace{(h'(t))^2}_{\text{constant}}} = \text{Constant}$$

$\Rightarrow \gamma'(t)$ is constant $\Rightarrow \gamma(t) = at + b$ for some $a, b \in \mathbb{R}$. ■

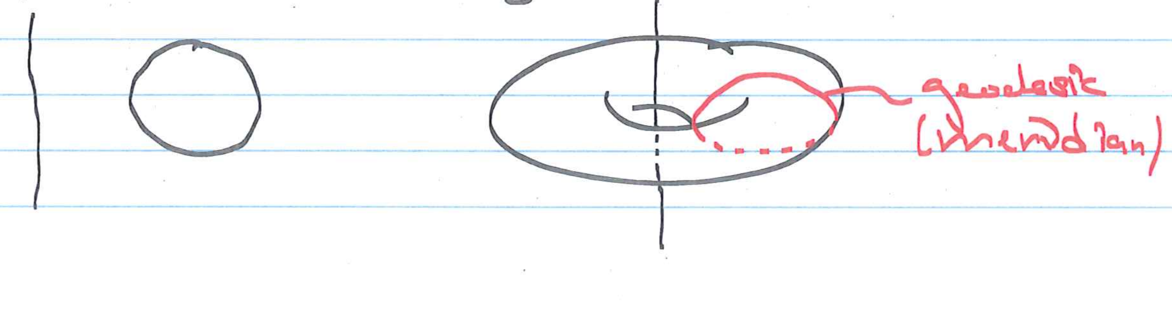
Proposition: Let P be a plane orthogonal to a surface M at any point of intersection curve, say α. Then (assuming α has unit speed) the curve α is a geodesic.

Proof Since α is a unit speed curve in P, $\alpha' \perp \alpha''$, where both vector lie in P.



However, U is in P and $U \perp \alpha'$ because α is in M. Hence, U and α'' are collinear. Thus, α is a geodesic. ■

Example: Let Σ be a surface of revolution and P a plane containing the axis of revolution. Hence, P is normal to the surface and thus the intersection curve $P \cap \Sigma$ is a geodesic. In other words, meridians on a surface of revolution are geodesics.



Summary

Principal curves	$k(\alpha') = k_1$ or k_2	$S(\alpha') // \alpha'$
Asymptotic curves	$k(\alpha') = 0$	$S'(\alpha) \perp \alpha'$ $\alpha'' \in T_pM.$
Geodesics		$\alpha'' \perp T_pM.$

§2.7. Connection Forms:

Lemma: Let E_1, E_2, E_3 be a frame field for $\mathbb{R}^3$. For each tangent vector v to $\mathbb{R}^3$ at a point $p \in \mathbb{R}^3$, let

$$\omega_{ij}(v) = (\nabla_v E_i) \cdot E_j(p), \text{ for all } i,j=1,2,3.$$

Then each ω_{ij} is a 1-form and $\omega_{ij} = -\omega_{ji}$. These 1-forms are called the connection 1-forms for the frame field E_1, E_2, E_3.

Proof: $\omega_{ij}(av+bw) = (\nabla_{av+bw} E_i) \cdot E_j(p)$, $v, w \in T_p\mathbb{R}^3$.

$$= (a\nabla_v E_i + b\nabla_w E_i) \cdot E_j(p)$$

$$= a(\nabla_v E_i) \cdot E_j(p) + b(\nabla_w E_i) \cdot E_j(p)$$

$= a\,\omega_{ij}(v) + b\,\omega_{ij}(w)$, and thus ω_{ij} is a 1-form.

For the second statement, note that

$$0 = v(\delta_{ij}) = v(E_i \cdot E_j)$$
$$= (\nabla_v E_i) \cdot E_j + (\nabla_v E_j) \cdot E_i$$

$= \omega_{ij}(v) + \omega_{ji}(v)$, for all $v \in T_pM$

$\Rightarrow \omega_{ij} = -\omega_{ji}$. ▬

Theorem: For any vector field V on $\mathbb{R}^3$, we have

$$\nabla_V E_i = \sum_j \omega_{ij}(V) E_j.$$

Proof: By definition $\omega_{ij}(v) = (\nabla_v E_i)\cdot E_j(p)$, for any $p \in \mathbb{R}^3$ and $v \in T_p\mathbb{R}^3$. Since E_1, E_2, E_3 is a frame we see that

$$\nabla_v E_i = \sum_j \omega_{ij}(v)\, E_j(p), \text{ for all } p, v.$$

$$\Rightarrow \nabla_V E_i = \sum_j \omega_{ij}(V)\, E_j.$$

$\blacksquare$

Remark: Since $\omega_{ji} = -\omega_{ij}$, we have $\omega_{ii} = 0$.

Thus the matrix (ω_{ij}) of connection 1-forms is

$$\omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

Given a frame field E_1, E_2, E_3 on $\mathbb{R}^3$, write

$$E_i = \sum_{j=1}^{3} a_{ij} U_j, \quad a_{ij} = E_i \cdot U_j, \text{ where}$$

$$U_1 = (1,0,0),\ U_2 = (0,1,0),\ U_3 = (0,0,1).$$

The matrix $A = (a_{ij})$ is called the attitude matrix of the field E_1, E_2, E_3.

$$a_{ij} : \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad a_{ij}(p) = E_i(p)\cdot U_j(p).$$

We define the differential of A as the matrix of 1-forms, da_{ij}'s:

$$dA = (da_{ij}).$$

Theorem Assume the above set up. Then

$$\omega = dA \cdot A^t \quad \text{or equivalently}$$

$$\omega_{ij} = \sum_{k=1}^{3} da_{ik} \, a_{jk} = \sum_{k=1}^{3} a_{jk} \, da_{ik}.$$

Proof: $E_i = \sum_{k=1}^{3} a_{ik} U_k.$

$$\omega_{ij}(v) = (\nabla_v E_i) \cdot E_j(p), \quad v \in T_p\mathbb{R}^3, \; p \in \mathbb{R}^3.$$

$$= \left(\nabla_v \sum_{k=1}^{3} a_{ik} U_k\right) \cdot E_j(p)$$

$$= \left[\sum_{k=1}^{3} (\nabla_v a_{ik} U_k)\right] \cdot E_j(p)$$

$$= \left[\sum_{k=1}^{3} (\nabla_v a_{ik}) U_k + a_{ik} \sum_{k=1}^{3} a_{ij} \overbrace{\nabla_v U_k}^{\overset{0}{\shortparallel}}\right] \cdot E_j(p)$$

$$= \left(\sum_{k=1}^{3} da_{ik}(v) U_k\right) \cdot E_j(p)$$

$$= \left(\sum_{k=1}^{3} da_{ik}(v) U_k\right) \cdot \left(\sum_{l=1}^{3} a_{jl} U_l\right)$$

$$= \sum_{k,l} da_{ik}(v) \, a_{jl} \boxed{U_k \cdot U_l} = \delta_{kl}$$

$$= \sum_{k=1}^{3} da_{ik}(v) \, a_{jk}, \quad \text{for all } v.$$

Hence, $\omega_{ij} = \sum_{k=1}^{3} da_{ik} \, a_{jk}.$ ∎

§ 2.8. The Structural Equations:

Definition: For any frame field E_1, E_2, E_3 on $\mathbb{R}^3$, we define the dual 1-forms $\theta_1, \theta_2, \theta_3$ as

$$\theta_i(V) = V \cdot E_i(p), \text{ for any } V \in T_p\mathbb{R}^3,\ p \in \mathbb{R}^3.$$

Example: Consider the natural frame U_1, U_2, U_3. Then $\theta_i(v) = v \cdot U_i(p) = (v_1, v_2, v_3) \cdot U_i(p) = v_i$.

Since $dx_i(v) = v_i$, for any $v \in T_p\mathbb{R}^3$, we see that $\theta_i = dx_i$.

Lemma: Let E_i and θ_i, $i = 1, 2, 3$, be as above. Then for any 1-form ϕ on $\mathbb{R}^3$ has a unique expansion

$$\phi = \sum_{i=1}^{3} \phi(E_i)\, \theta_i.$$

Proof:

$$\Big(\sum_i \phi(E_i)\theta_i\Big)(V) = \sum_i \phi(E_i)\,\theta_i(V)$$

$$= \phi\Big(\sum \theta_i(V) E_i\Big)$$

$$= \phi(V), \text{ for any}$$

vector field V. Hence,

$$\phi = \sum_{i=1}^{3} \phi(E_i)\,\theta_i.$$

$\blacksquare$

Recall that the matrix $A=(a_{ij})$ is defined by the equation

$$E_i = \sum_j a_{ij} U_j.$$

Hence, $\theta_i(v) = v \cdot E_i(p)$

$$= v \cdot \left(\sum_j a_{ij} U_j\right)$$

$$= \sum_j a_{ij} (v \cdot U_j)$$

$$= \sum_j a_{ij}\, dx_j(v),$$ so that

we obtain

$$\theta_i = \sum a_{ij}\, dx_j.$$

Moreover, since $a_{ij} = E_i \cdot U_j = \theta_i(U_j)$ we have $\theta_i = \sum_j \theta_i(U_j)\, dx_j$.

Theorem (Cartan Structural Equations)

Let E_i, θ_i and ω_{ij} as above. Then we have

1) The first structural equations:

$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j.$$

2) The second structural equations:

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} \quad (d\omega = \omega\omega^t)$$

Proof 1) Recall that we have $\theta_i = \sum_j a_{ij}\, dx_j$.

This can be written as $\theta = A\, d\xi$, where

$A = (a_{ij})$, $\theta = \begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}$ and $d\xi = \begin{pmatrix}dx_1\\ dx_2\\ dx_3\end{pmatrix}$.

Now, $d\theta = d(A\,d\xi) = dA \wedge d\xi + A\,\underbrace{d(d\xi)}_{d^2\xi = 0.}$

$$= dA \wedge d\xi$$

or $= dA\, d\xi$

$$= dA\,(A^t A)\, d\xi$$

$$= (dA\, A^t)(A\, d\xi)$$

$= \omega\,\theta$, because the attitude matrix A is orthogonal so that $A^tA = Id$. Since $E_i = a_{ij} U_j$ we see that $A = (a_{ij})$ is the base change matrix from the orthonormal basis $\{U_i\}$ to the orthonormal basis $\{E_i\}$.

2) $d\omega = d(dA\, A^t)$

$$= \underbrace{d^2A}_{= 0}\, A^t - dA\, dA^t$$

$$= -\, dA\, dA^t$$

$$= -dA\,\underbrace{A^t\,A}_{Id}\, dA^t \qquad \omega = dA\, A^t$$

$$= -\,(dA\, A^t)(A\, dA^t)$$

$$= -\,\omega\, (dA\, A^t)^t = -\omega\,\omega^t = \omega\,\omega$$

because $\omega^t = -\omega$. ∎

Note Title 5.05.2020

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Example: Consider the spherical coordinates given by

$$\begin{pmatrix} \rho & \varphi & \nu \\ 1 & 2 & 3 \end{pmatrix}$$

$x_1 = \rho \cos\varphi \cos\nu$
$x_2 = \rho \sin\varphi \cos\nu$
$x_3 = \rho \sin\nu$

Since $\theta_i = \sum a_{ij}\, dx_j$, where $a_{ij} = E_i \cdot U_j$

$$E_1 = \frac{\frac{\partial}{\partial\rho}(x_1, x_2, x_3)}{\|\frac{\partial}{\partial\rho}(x_1, x_2, x_3)\|} = \frac{(\cos\varphi\cos\nu, \sin\varphi\cos\nu, \sin\nu)}{1}$$

$$E_1 = (\overset{a_{11}}{\cos\varphi\cos\nu}, \overset{a_{12}}{\sin\varphi\cos\nu}, \overset{a_{13}}{\sin\nu})$$

$$E_2 = \frac{(-\rho\sin\varphi\cos\nu, \rho\cos\varphi\cos\nu, 0)}{\rho\cos\nu}$$

$$E_2 = (\overset{a_{21}}{-\sin\varphi}, \overset{a_{22}}{\cos\varphi}, \overset{a_{23}}{0})$$

$$E_3 = \frac{(-\rho\cos\varphi\sin\nu, -\rho\sin\varphi\sin\nu, \rho\cos\nu)}{\rho}$$

$$E_3 = (\overset{a_{31}}{-\cos\varphi\sin\nu}, \overset{a_{32}}{-\sin\varphi\sin\nu}, \overset{a_{33}}{\cos\nu})$$

Hence, $\theta_1 = \sum a_{1j}\, dx_j = a_{11}\,dx_1 + a_{12}\,dx_2 + a_{13}\,dx_3$

$$\Rightarrow \theta_1 = \cos\varphi\cos\nu(\cos\varphi\cos\nu\, d\rho - \rho\sin\varphi\cos\nu\, d\varphi - \rho\cos\varphi\sin\nu\, d\nu) + (\sin\varphi\cos\nu)(\sin\varphi\cos\nu\, d\varphi - \rho\cos\varphi\cos\nu\, d\varphi - \rho\sin\varphi\sin\nu\, d\nu) + \sin\nu(\sin\nu\, d\rho + \rho\cos\nu\, d\nu)$$

$$= d\rho$$

Similarly, $\theta_2 = \rho\cos\varphi\, d\vartheta$ and $\theta_3 = \rho\, d\varphi$.

We know that $\omega = dA\, A^t$ or equivalently,
$\omega_{ij} = \sum_k a_{jk}\, da_{ik}$ and hence,

$\omega_{12} = \cos\varphi\, d\vartheta$, $\omega_{13} = d\varphi$ and $\omega_{23} = \sin\varphi\, d\vartheta$.

Now from the first structural equation

$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$ and thus

$d\theta_3 = \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2$
$= d\rho \wedge d\varphi$, which is the case since $\theta_3 = \rho\, d\varphi$.

Similarly,
$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$ and say

$$d\omega_{12} = \sum_k \omega_{1k} \wedge \omega_{k2}$$

$$= \omega_{13} \wedge \omega_{32}$$

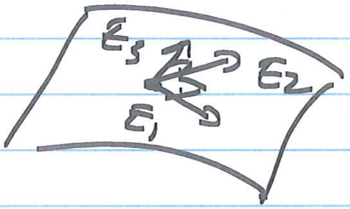
$= -\sin\varphi\, d\varphi \wedge d\vartheta$, which is the case

since $d\omega_{12} = d(\cos\varphi\, d\vartheta)$.

CHAPTER 6: Geometry of Surfaces in $\mathbb{R}^3$

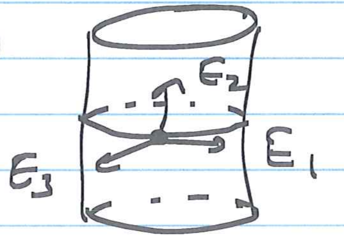
§6.1. The Fundamental Equations:

Definition: An adapted frame field E_1, E_2, E_3 on a region $\mathcal{O}$ in $M \subseteq \mathbb{R}^3$ is a Euclidean frame such that E_3 always normal to the surface.

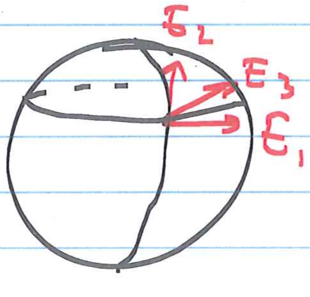


Lemma: There is an adapted frame field on a region $\mathcal{O}$ in M if and only if $\mathcal{O}$ is orientable.

Examples 1)



2)



Definition The 1-forms ω_{ij} defined by the equation

$$\nabla_V E_i = \sum_{j=1}^{3} \omega_{ij}(V) E_j(p),$$

are called the connection 1-forms on M.

Lemma: $S(V) = \omega_{13} E_1(p) + \omega_{23}(V) E_2(p)$.

Proof $S(V) = -\nabla_V E_3 = -\omega_{31}(V) E_1(p) - \omega_{32}(V) E_2(p)$
$= \omega_{13}(V) E_1(p) + \omega_{23}(V) E_2(p)$. ■

Definition: Given an adapted frame E_1, E_2, E_3 for a region U in M we define the dual 1-forms $\theta_1, \theta_2, \theta_3$ by

$$\theta_i(v) \doteq v \cdot E_i(p).$$

Note that $\theta_3(v) = v \cdot E_3(p) = 0$ for all $v \in T_pM$, and θ_3 is identically zero on M.

Theorem: If E_1, E_2, E_3 is an adapted frame for M, then we have

1) $\begin{cases} d\theta_1 = \omega_{12} \wedge \theta_2 \\ d\theta_2 = \omega_{21} \wedge \theta_1 \end{cases}$ First Structural Equations

2) $\omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 = 0$ Symmetry equation

3) $d\omega_{12} = \omega_{13} \wedge \omega_{32}$ Gauss Equation

4) $\begin{cases} d\omega_{13} = \omega_{12} \wedge \omega_{23} \\ d\omega_{23} = \omega_{21} \wedge \omega_{13} \end{cases}$ Codazzi Equations.

Proof: All these formulas follow from Cartan's Structural equations of §2.8.:

1) $d\theta_i = \sum_j \omega_{ij} \wedge \theta_j$

2) $d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$.

Hence, for example, $d\theta_1 = \omega_{12} \wedge \theta_2 + \omega_{13} \wedge \overset{0}{\theta_3}$
$= \omega_{12} \wedge \theta_2.$

Similarly, $d\omega_{13} = \omega_{12} \wedge \omega_{23}$.
(The rest is left as an exercise!)

§ 6.2. Form Computations:

Lemma: (The Basis Formula)
If ϕ is a 1-form on M and μ is a 2-form on M, then

1) $\phi = \phi(E_1)\theta_1 + \phi(E_2)\theta_2$ and

2) $\mu = \mu(E_1, E_2)\,\theta_1 \wedge \theta_2$.

Proof: Since E_1, E_2 form a basis for T_pM at any point $p \in M$ it is enough to check these on these vectors:

$$1)\ (\phi(E_1)\theta_1 + \phi(E_2)\theta_2)(E_1) = \phi(E_1)\overbrace{\theta_1(E_1)}^{=1} + \phi(E_2)\overbrace{\theta_2(E_1)}^{=0}$$
$$= \phi(E_1)$$

Similarly, $(\phi(E_1)\theta_1 + \phi(E_2)\theta_2)(E_2) = \phi(E_2)$

$$\Rightarrow \phi = \phi(E_1)\theta_1 + \phi(E_2)\theta_2.$$

$$2)\ \mu(E_1,E_2)\,\theta_1\wedge\theta_2(E_1,E_2) = \mu(E_1,E_2)\det\begin{bmatrix} \overset{=1}{\theta_1(E_1)} & \overset{=0}{\theta_1(E_2)} \\ \underset{=0}{\theta_2(E_1)} & \underset{=1}{\theta_2(E_2)} \end{bmatrix}$$
$$= \mu(E_1, E_2)$$

$$\Rightarrow \mu = \mu(E_1,E_2)\,\theta_1\wedge\theta_2. \quad ■$$

Lemma 1) $\omega_{13} \wedge \omega_{23} = K\,\theta_1 \wedge \theta_2$

2) $\omega_{13} \wedge \theta_2 + \theta_1 \wedge \omega_{23} = 2H\,\theta_1 \wedge \theta_2$.

Proof By definition $S(E_1) = -\nabla_{E_1} E_3$

$\Rightarrow S(E_1) = -\omega_{31}(E_1)E_1 - \omega_{32}(E_1)E_2$ and similarly,

$S(E_2) = -\nabla_{E_2}E_3 = -\omega_{31}(E_2)E_1 - \omega_{32}(E_2)E_2.$

Therefore, the matrix representation of the operator S in the basis $\{E_1, E_2\}$ of T_pM is

$$[S]_{\{E_1,E_2\}} = \begin{pmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{pmatrix}$$

Now by the previous lemma,

$$\underline{K\theta_1\wedge\theta_2}(E_1,E_2) = K = \det(S)$$

$$= \omega_{13}(E_1)\omega_{23}(E_2) - \omega_{13}(E_2)\omega_{23}(E_1)$$

$$= \underline{\omega_{13}\wedge\omega_{23}}(E_1,E_2)$$

$\Rightarrow K\theta_1\wedge\theta_2 = \omega_{13}\wedge\omega_{23}.$

The second identity is left as an exercise!

By the second structural equation (Gauss Equation) we have

$d\omega_{12} = -\omega_{13}\wedge\omega_{23}$ and thus we have

Corollary $d\omega_{12} = -K\theta_1\wedge\theta_2.$

Remark: In the previous section (§6.1) we had computed θ_i and ω_{ij} for the spherical coordinates on a sphere of radius r. Hence,

$$\theta_1 \wedge \theta_2 = r^2 \cos\varphi \, d\vartheta \wedge d\varphi$$

$$= -r \cos\varphi \, d\varphi \wedge d\vartheta$$

$$d\omega_{12} = d(\sin\varphi \, d\vartheta) = \cos\vartheta \, d\varphi \wedge d\vartheta.$$

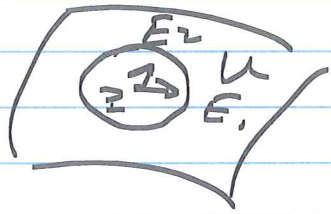
So, by the Corollary $K = \frac{1}{r^2}$, as expected.

Video 13

Definition: A principal frame on a surface M is an adapted frame E_1, E_2, E_3 so that E_1 and E_2 are principal vectors.

Lemma: If p is a non-umbilic point of M, then there is a principal frame in a neighborhood of p in M.

Proof



By hypothesis $k_1(p) \neq k_2(p)$ and thus since k_i's are continuous functions of p, $k_1 \neq k_2$ on a neighborhood of p. Let $\{F_1, F_2, F_3\}$ be any adapted frame in this neighborhood, and let

$$S = (S_{ij}) = \begin{pmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{pmatrix}$$ be the matrix representation of S in this basis. The characteristic equation is

$$0 = \det(S - \lambda I) = \lambda^2 - tr(S)\,\lambda + \det(S)$$

$$0 = \lambda^2 - 2H\lambda + K$$

$$\Rightarrow \lambda_{1,2} = \frac{2H \pm \sqrt{4H^2 - 4K}}{2} = H \pm \sqrt{H^2 - K}.$$

So $k_1 = H + \sqrt{H^2 - K}$ and $k_2 = H - \sqrt{H^2 - K}$.

$H = \frac{1}{2}(S_{11} + S_{22})$, $K = S_{11}S_{22} - S_{12}^2$ both continuous and hence k_1 and k_2 are continuous.

Principal directions are the eigenvectors of S.

$S e_1 = k_1 e_1$, $e_1 = ?$ $\quad (S - k_1 I) e_1 = 0$.

$(S_{11} - k_1) a + S_{12} b = 0$, $\quad e_1 = \begin{pmatrix} a \\ b \end{pmatrix}$.

$\Rightarrow e_1 = (S_{12}, k_1 - S_{11})$.

Similarly, $e_2 = (k_2 - S_{22}, S_{12})$.

Hence, $E_1 = \dfrac{S_{12} F_1 + (k_1 - S_{11}) F_2}{\| S_{12} F_1 + (k_1 - S_{11}) F_2 \|}$ and

$E_2 = \dfrac{(k_2 - S_{22}) F_1 + S_{12} F_2}{\| (k_2 - S_{22}) F_1 + S_{12} F_2 \|}$.

E_1 and E_2 are clearly continuous and just let $E_3 = E_1 \times E_2$. Hence, $\{E_1, E_2, E_3\}$ is a principal frame. $\blacksquare$

Now let $\{E_1, E_2, E_3\}$ be a principal frame with

$$S(E_1) = k_1 E_1 \quad \text{and} \quad S(E_2) = k_2 E_2.$$

By a Corollary from §6.1. we know that

$S(V) = \omega_{13}(V) E_1 + \omega_{23}(V) E_2$ and thus

$\omega_{13}(E_1) = k_1$, $\omega_{23}(E_1) = 0$ and
$\omega_{13}(E_2) = 0$, $\omega_{23}(E_2) = k_2$.

Now it follows that $\omega_{13} = k_1 \theta_1$ and $\omega_{23} = k_2 \theta_2$.

<u>Theorem</u>: If E_1, E_2, E_3 is a principal frame on $M \subseteq \mathbb{R}^3$, then

$$E_1[k_2] = (k_1 - k_2)\, \omega_{12}(E_2), \text{ and}$$

$$E_2[k_1] = (k_1 - k_2)\, \omega_{12}(E_1).$$

<u>Proof</u>: From the Codazzi Equations we have

$$d\omega_{13} = \omega_{12} \wedge \omega_{23} \text{ and } d\omega_{23} = \omega_{21} \wedge \omega_{13}.$$

Since by the line above the theorem $\omega_{13} = k_1 \theta_1$ and $\omega_{23} = k_2 \theta_2$ we get

$$d(k_1 \theta_1) = d\omega_{13} = \omega_{12} \wedge k_2 \theta_2$$

$$\Rightarrow dk_1 \wedge \theta_1 + k_1\, d\theta_1 = k_2\, \omega_{12} \wedge \theta_2.$$

From the structural equations we have $d\theta_1 = \omega_{12} \wedge \theta_2$ and thus

$$dk_1 \wedge \theta_1 + k_1\, \omega_{12} \wedge \theta_2 = k_2\, \omega_{12} \wedge \theta_2$$

$$\Rightarrow dk_1 \wedge \theta_1 = (k_2 - k_1)\, \omega_{12} \wedge \theta_2.$$

Let's compute both sides on the pair of vectors E_1, E_2:

$$(dk_1 \wedge \theta_1)(E_1, E_2) = dk_1(E)\, \overset{\overset{0}{\shortparallel}}{\theta_1(E_2)} - dk_1(E_2)\, \overset{\overset{1}{\shortparallel}}{\theta_2(E_2)}$$
$$= -dk_1(E_2).$$

Similarly, $(k_2-k_1)\omega_{12}\wedge\theta_2(E_1,E_2) = (k_2-k_1)\,\omega_{12}(E_1)$.

Then $E_2[k_1] = dk_1(E_2) = (k_1-k_2)\,\omega_{12}(E_1)$.

The second statement is similar. ∎

§6.3. Some Global Theorems:

Theorem: If the shape operator is identically zero, then M is a part of plane in $\mathbb{R}^3$, provided that M is connected.

Proof $S=0$ implies that for any unit normal field U, $0=S_v(U)=-U'_v$, so that U is parallel (constant) along any vector V.

Choose any point $p\in M$. For any other point $q\in M$ let $\alpha(t)$ be a path with
$\alpha(0)=p$ and $\alpha(1)=q$ (since M is connected)

Now consider the function $f(t)=(\alpha(t)-p)\cdot U$, $t\in[0,1]$. Now

$$f'(t) = \alpha'(t)\cdot U + \alpha(t)\cdot \overset{''0}{\overline{U'}} = \alpha'(t)\cdot U = 0,$$

because $\alpha'(t)$ is tangent to the surface. Hence, $f(t)$ is constant.

$$(q-p)\cdot U = (\alpha(1)-p)\cdot U = f(1) = f(0) = (\overset{\alpha(0)}{\overset{''}{p}}-p)\cdot U = 0$$

Hence, q is in the plane Γ containing p with normal u. Since q is an arbitrary point of M we see that $M \subseteq T$. ∎

Definition: A surface M is called all-umbilic if every point of M is umbilic.

Lemma: If M is a connected all-umbilic surface then M has constant Gaussian curvature $K \geq 0$.

Proof: Let E_1, E_2, E_3 be an adapted frame.

So at any point $p \in M$, $k_1(p) = k_2(p) = k(p)$. By the theorem we proved before §6.3, we have

$$E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2) \text{ and}$$

$$E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1).$$

Hence, $E_1[k_2] = E_2[k_1] = 0$ at all points.

So $dk(E_1) = dk(E_2) = 0$ $(k = k_1 = k_2)$, so

that $dk = 0$ on M. However, $K = k_1 k_2 = k^2$ and hence, $dK = 2k\,dk = 0$, so that K is constant. ∎

Note Title

Video 14

7.05.2020

Lemma: If M is a connected all-umbilic surface (i.e. every point of M is umbilic) then M has constant Gaussian curvature $K \geq 0$.

Theorem: If $M \subseteq \mathbb{R}^3$ is all umbilic and $K > 0$, then M is a part of a sphere of radius $1/\sqrt{K}$.

Proof: Let $p \in M$ and consider the point

$c = p + \frac{1}{k(p)} E_3(p)$, where E_1, E_2, E_3 is an adapted

frame on M and $k(p) = k_1(p) = k_2(p)$. Let $q \in M$ be any other point on M and choose a curve $\alpha: [0,1] \to M$ with $\alpha(0) = p$, $\alpha(1) = q$. Also let $\gamma(t)$ be the curve given by

$$\gamma(t) = \alpha(t) + \frac{1}{k(\alpha(t))} E_3(\alpha(t)).$$

By the previous lemma $K(p) = k_1(p) k_2(p) = k^2(p)$ is a constant function. Thus

$\gamma'(t) = \alpha'(t) + \frac{1}{k} E_3'$. However, $E_3' = U' = -S(\alpha'(t))$
$= -k\alpha'(t)$

so that S is a scalar function, because k is constant. Thus

$\gamma' = \alpha' + \frac{1}{k}(-k\alpha') = 0$, so that γ is constant.

So, $c = \gamma(0) = \gamma(1) = q + \frac{1}{k} E_3(q)$. Hence,

$d(c, q) = \| \frac{1}{k} E_3(q) \| = \frac{1}{k}$, for every $q \in M$.

Hence, M is contained in the sphere of radius $1/k$ and center c. Note that $1/k = 1/\sqrt{K}$, and the proof finishes. ∎

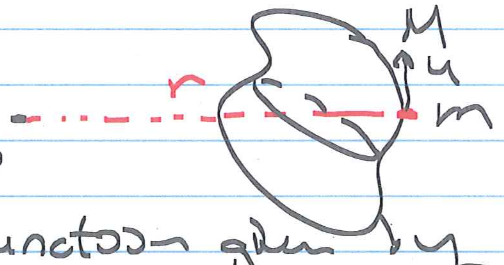
Hence we get

Corollary A surface M is all-umbilic if and only if M is a part of a plane or a sphere. In particular, if M is compact then M must be a sphere.

Theorem: On every compact surface M in $\mathbb{R}^3$ there is a point at which the Gaussian curvature is positive.

Corollary There is no compact surface in $\mathbb{R}^3$ with $K \leq 0$.

Proof of the Theorem:



Let $f: \mathbb{R}^3 \to \mathbb{R}$ be the function given by $f(p) = \|p\|^2$. Since f is continuous and M is compact $f|_M$ has a maximal value at a point $m \in M$.

Claim: Let $r = \|m\| > 0$. Then $K(m) \geq 1/r^2 > 0$.

Note that the claim proves the theorem.

Proof of the claim: Let $u \in T_m M$ be a unit vector and choose a unit speed curve α on M with $\alpha(0) = m$ and $\alpha'(0) = u$.

Thus the function $f(\alpha(t)) = \|\alpha(t)\|^2$ has its maximum at $t=0$. Thus,

$\frac{d}{dt}(f(\alpha(t))(0) = 0$ and $\frac{d^2}{dt^2}(f(\alpha(t))(0) \leq 0$.

$$0 = (f(\alpha(t)))'(0) = (\alpha(t)\cdot\alpha(t))'(0) = 2\alpha(0)\cdot\alpha'(0)$$
$$= 2m\cdot u$$

Since u is an arbitrary vector in T_pM we see that $u \perp T_pM$.

On the other hand, $\frac{d^2}{dt^2}(f(\alpha(t)) = 2\alpha'\cdot\alpha' + 2\alpha\cdot\alpha''$

and hence, $2\alpha'(0)\cdot\alpha'(0) + 2\alpha(0)\cdot\alpha''(0) \leq 0$. So,

$$u\cdot u + m\cdot\alpha''(0) \leq 0 \Rightarrow 1 + m\cdot\alpha''(0) \leq 0.$$

$\Rightarrow m\cdot\alpha''(0) \leq -1$. Since $\|m\| = r$, $\frac{m}{r}$ is a unit normal vector to M at m, we get

$k(u) = \frac{m}{r}\cdot\alpha''(0) \leq -\frac{1}{r}$ (since $\alpha'(0) = u$),

for all unit vectors $u \in T_pM$. In particular,

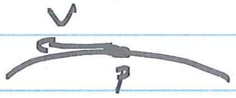
$K(m) = k_1(u)\,k_2(u_2) \geq \frac{1}{r^2} > 0$. ■

<u>Lemma</u>: (Hilbert) Let m be a point of $M \subseteq \mathbb{R}^3$ such that

1) k_1 has a local maximum at m,
2) k_2 has a local minimum at m,
3) $k_1(m) > k_2(m)$. Then $K(m) \leq 0$.

Proof: Since $k_1(m) > k_2(m)$, m is not an umbilic point and thus by a previous lemma there is a principal frame E_1, E_2, E_3 defined on a neighborhood of m.

Fact: let f be a function on M so that f has a maximum (minimum) at a point p. If V is a vector at p, then



1) $V[f] = 0$

2) $VV[f] \leq 0$ ($VV[f] \geq 0$, respectively.

Hence by this fact

$E_1[k_2] = E_2[k_1] = 0$ at m and
$E_1E_1[k_2] \geq 0$ and $E_2E_2[k_1] \leq 0$ at m.

Now, from the Codazzi equations, which says that

$$E_1[k_2] = (k_1 - k_2)\,\omega_{12}(E_2) \text{ and}$$

$$E_2[k_1] = (k_1 - k_2)\,\omega_{12}(E_1),$$ we obtain

$\omega_{12}(E_2) = \omega_{12}(E_1) = 0$, because $k_1 > k_2$

(*) Exercise 2 of §6.2: $K = \overbrace{E_2[\omega_{12}(E_1)]}^{\geq 0} - E_1[\overset{\leq 0}{\omega_{12}}(E_2)]$ at m.

Apply the field E_1 to the 1st Codazzi equation:

$$E_1E_1[k_2] = (E_1[k_1] - E_1[k_2])\,\overbrace{\omega_{12}(E_2)}^{=0} + \underbrace{(k_1 - k_2)}_{0<}\,E_1[\omega_{12}(E_1)].$$

However, at the point m, $\omega_{12} = 0$ and $k_1 - k_2 > 0$.

Thus $E_1[\omega_{12}(E_1)] \geq 0$. Similarly, taking E_2 derivative of the second Codazzi equation we obtain $E_2[\omega_{12}(E_2)] \leq 0$ at m. Hence, the proof finishes by $(*)$.

Video 15

Note Title 9.05.2020

Lemma (Hilbert) Let m be a point of $M \subseteq \mathbb{R}^3$ such that

1) k_1 has a local maximum at m,
2) k_2 has a local minimum at m,
3) $k_1(m) > k_2(m)$.

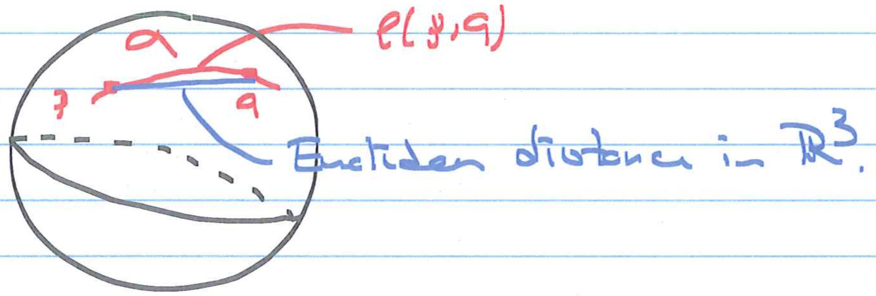
Then $K(m) \leq 0$.

Theorem (Liebmann) If $M \subseteq \mathbb{R}^3$ is a compact surface with constant Gaussian curvature K, then M is a sphere of radius $1/\sqrt{K}$ ($K>0$ since $K \subseteq \mathbb{R}^3$ is compact).

Proof: Since $M \subseteq \mathbb{R}^3$ is a compact surface M is orientable, which can't be proved by the tools of this course. So we have smooth unit normal vector field defined on all of M. Therefore principal curvature functions are globally defined on M and $k_1 \geq k_2 \geq 0$ at all points. In particular, k_1 has a maximum value at some point say p. However, $K = k_1 k_2$ is constant and thus k_2 has its minimum value at p. If $k_1(p) > k_2(p)$ then by the Hilbert's lemma $K(p) \leq 0$, a contradiction since $K>0$ is a constant function. So $k_1(p) = k_2(p)$. So $k_1(q) = k_2(q)$ at all $q \in M$. Hence, M is an all-umbilic surface. It follows that M is a part of a sphere of radius $1/\sqrt{K}$. By compactness of M and connectedness of S^2, M is equal to the sphere. ▬

§6.4. Isometries and Local Isometries:

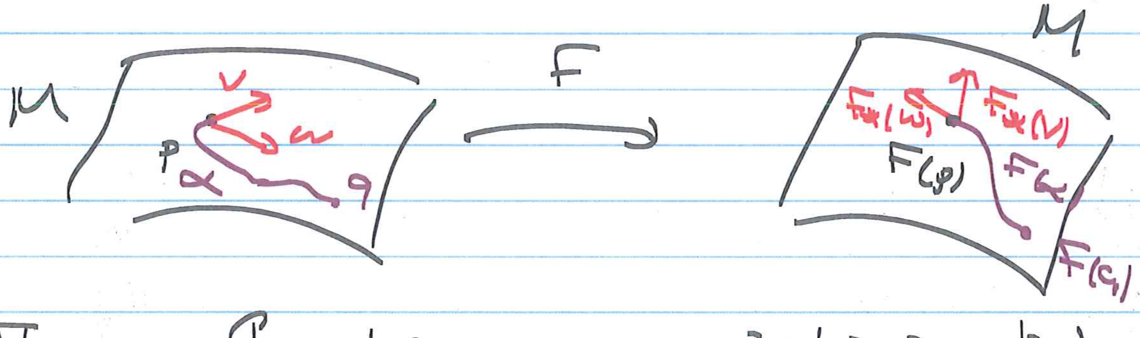
Definition: If p and q are points of $M \subseteq \mathbb{R}^3$, the intrinsic distance from p to q is defined to be the infimum of lengths $L(\alpha)$ of all curves α from p to q. It is denoted as $\rho(p,q)$.



Definition: An isometry $F: M \to \overline{M}$ of surfaces in $\mathbb{R}^3$ is a one to one and onto smooth mapping so that

$$F_*(v) \cdot F_*(w) = v \cdot w$$

for all $v, w \in T_pM$, $p \in M$.



Theorem: Isometries preserve intrinsic distance: If $F: M \to \overline{M}$ is an isometry, then

$$\rho(p,q) = \overline{\rho}(F(p), F(q))$$

Proof. If α is a smooth curve in M with

$\alpha(a) = p$ and $\alpha(b) = q$, then

$$L(\alpha) = \int_a^b \|\alpha'(t)\| dt.$$

Then $F(\alpha)$ is a curve in $\overline{M}$ with
$F(\alpha)(a) = F(\alpha(a)) = F(p)$ and $F(\alpha)(b) = F(\alpha(b)) = F(q)$
and

$$\begin{aligned} L(F(\alpha)) &= \int_a^b \|(F(\alpha(t)))'\| dt \\ &= \int_a^b \|F_*(\alpha'(t))\| dt \\ &= \int_a^b \|\alpha'(t)\| dt \\ &= L(\alpha), \text{ for all } \alpha \text{ in } M \text{ joining } p \text{ to } q. \end{aligned}$$

Then $L(\alpha) = L(F(\alpha)) \geq \overline{\rho}(F(p), F(q))$, for all α in M joining p to q.

Hence, $\rho(p,q) = \inf\{L(\alpha)\} \geq \overline{\rho}(F(p), F(q))$.

However, $F : M \to \overline{M}$ is a smooth bijection with F_* is an isomorphism at all points. Hence, by the Inverse Function Theorem F is a diffeomorphism, so that we have a smooth bijection

$$F^{-1} : \overline{M} \to M.$$

Since F is an isometry so is F^{-1}. Hence, by the above paragraph

$$\overline{\rho}(\overline{p}, \overline{q}) \geq \rho(F^{-1}(\overline{p}), F^{-1}(\overline{q})), \text{ for all } \overline{p}, \overline{q} \in \overline{M}.$$

Now take $\overline{p} = F(p)$ and $\overline{q} = F(q)$, then we get

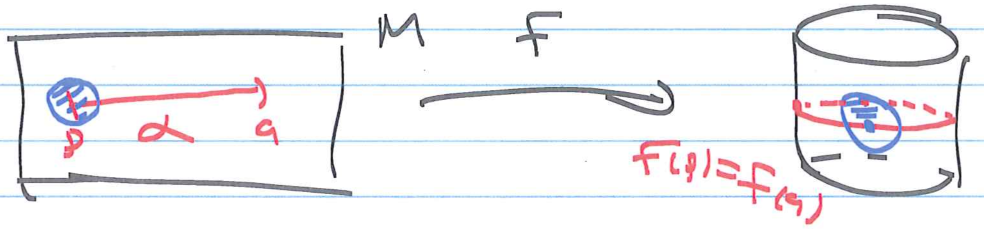
$$\bar\rho(F(p), F(q)) \geq \rho(p,q).$$

So $\rho(p,q) = \bar\rho(F(p), F(q))$. ∎

Definition: A local isometry $F: M \to N$ of surfaces is a mapping that preserves dot products of tangent vectors:

$$F_*(v)\cdot F_*(w) = v\cdot w.$$

Remark: If $F: M \to N$ is a local isometry then F_* is an isomorphism at all points of M and thus F is a local diffeomorphism at all points.



Lemma: Let $F: M \to N$ be any smooth mapping. For each surface patch $x: D \to M$ consider the composite mapping $\bar x = F(x): D \to N$. Then F is a local isometry if and only if for each patch x we have

$$E = \bar E,\ F = \bar F \text{ and } G = \bar G.$$

(Here, $\bar x$ need *not* to be a surface patch)

Proof: $\bar x(u,v) = F(x(u,v))$ and thus $\bar x_u = F_*(x_u)$ and $\bar x_v = F_*(x_v)$. So if F is a local isometry

then $\overline{E} = \overline{x}_u \cdot \overline{x}_u = F_*(x_u)\cdot F_*(x_u) = x_u \cdot x_u = E$,
$\overline{F} = \overline{x}_u \cdot \overline{x}_v = F_*(x_u)\cdot F_*(x_v) = x_u \cdot x_v = F$, and
$\overline{G} = \overline{x}_v \cdot \overline{x}_v = F_*(x_v)\cdot F_*(x_v) = x_v \cdot x_v = G$.

For the other direction, assume $E=\overline{E}$, $F=\overline{F}$ and $G=\overline{G}$ at all points. Now if $w_1, w_2 \in T_pM$ then

$$w_1 = a_1 x_u + b_1 x_v \quad \text{and} \quad w_2 = a_2 x_u + b_2 x_v$$

for some $a_i, b_i \in \mathbb{R}$, because T_pM is spanned by x_u and x_v.

Now $F_*(w_1)\cdot F_*(w_2) = F_*(a_1x_u + b_1x_v)\cdot F_*(a_2x_u+b_2x_v)$.

$$= \left(a_1 F_*(x_u) + b_1 F_*(x_v)\right)\cdot\left(a_2 F_*(x_u) + b_2 F_*(x_v)\right)$$

$$= a_1a_2 \overbrace{F_*(x_u)\cdot F_*(x_u)}^{\overline{E}=E} + (a_1b_2 + a_2b_1)\overbrace{F_*(x_u)\cdot F_*(x_v)}^{\overline{F}=F}$$
$$+ b_1b_2 \underbrace{F_*(x_v)\cdot F_*(x_v)}_{\overline{G}=G}$$

$$= a_1a_2\, x_u\cdot x_u + (a_1b_2 + a_2b_1)\, x_u\cdot x_v + b_1b_2\, x_v\cdot x_v$$

$$= (a_1x_u + b_1x_v)\cdot(a_2x_u + b_2x_v)$$

$$= w_1 \cdot w_2.$$

∴ F is a local isometry. ∎

Video 16

Note Title 10.05.2020

Definition: $F: M \to N$ local isometry if it is smooth and satisfies

$$F_*(v) \cdot F_*(w) = v \cdot w \quad \text{for all}$$

$v, w \in T_pM$, $\forall\, p \in M$.

Lemma: Let $F: M \to N$ be a smooth mapping. For each surface patch $x: D \to M$, consider the composite map $\bar{x} \doteq F(x): D \to N$. Then F is a local isometry if and only if for each surface patch x we have $E = \bar{E}$, $F = \bar{F}$ and $G = \bar{G}$

Example: 1) $M: x^2+y^2=r^2$, $x: \mathbb{R}^2 \to M$ a parametrization given by

$$x(u,v) = \left(r\cos\frac{u}{r},\, r\sin\frac{u}{r},\, v\right).$$

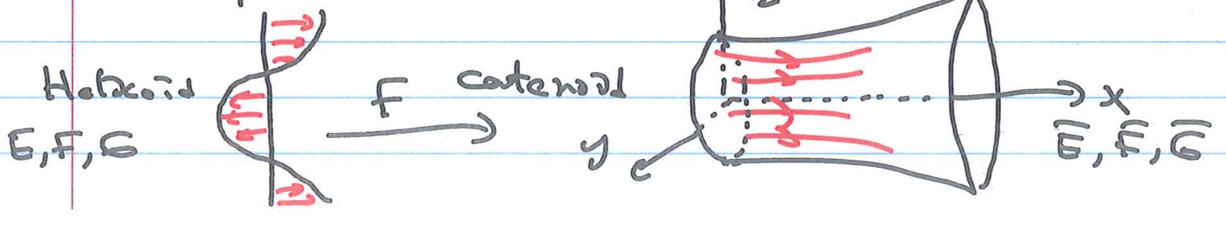
We've computed before that $E=1$, $F=0$, $G=1$. Hence, x is a local isometry since for the xy-plane $E=1$, $F=0$ and $G=1$, also.

2) Local isometry from a helicoid onto a catenoid.

Let $x(u,v) = (u\cos v, u\sin v, v)$ be a patch for the helicoid, and

$$y(u,v) = (g(u), h(u)\cos v, h(u)\sin v), \text{ where}$$

$g(u) = \sinh^{-1} u$ and $h(u) = \sqrt{1+u^2}$, be a parametrization for the catenoid.



$F(x(u,v)) = y(u,v)$ is an isometry, with

$$E = \bar{E} = 1, \quad F = \bar{F} = 0 \text{ and } G = \bar{G} = 1 + u^2 = h^2(u,v).$$

Definition: A mapping $F: M \to N$ of surfaces is conformal if there is a smooth real valued function $\lambda: M \to (0,\infty)$ so that

$$\|F_*(v_p)\| = \lambda(p)\,\|v_p\|, \text{ for all } v_p \in T_pM, \forall p \in M.$$

§6.5. Intrinsic Geometry of Surfaces in $\mathbb{R}^3$:

Lemma: The connection form $\omega_{12} = -\omega_{21}$ is the only 1-form on the surface that satisfies the first structural equations:

$$d\theta_1 = \omega_{12} \wedge \theta_2 \text{ and } d\theta_2 = \omega_{21} \wedge \theta_1.$$

Proof: Let E_1, E_2 be a frame on the surface.

Then $$d\theta_1(E_1,E_2) = \omega_{12}(E_1)\,\overbrace{\theta_2(E_2)}^{=1} - \omega_{12}(E_2)\,\overbrace{\theta_2(E_1)}^{=0}$$
$$= \omega_{12}(E_1)$$

and similarly,

$$d\theta_2(E_1,E_2) = -\omega_{21}(E_2) = \omega_{12}(E_2).$$

So, $\omega_{12}(E_1) = d\theta_1(E_1,E_2)$ and
$\omega_{12}(E_2) = d\theta_2(E_1,E_2)$.

This proves the lemma.

Lemma: Let $F: M \to \bar{M}$ be an isometry, and let E_1, E_2 be a tangent frame on M. If $\bar{E}_1, \bar{E}_2$ is the transferred frame field on $\bar{M}$, then

1) $\theta_1 = F^*(\bar{\theta}_1)$, $\theta_2 = F^*(\bar{\theta}_2)$;

2) $\omega_{12} = F^*(\bar{\omega}_{12})$.

Here, $\bar{E}_1 = F_*(E_1)$ and $\bar{E}_2 = F_*(E_2)$.

Proof. 1) $F^*(\bar{\theta}_i)(E_j) = \bar{\theta}_i(F_*(E_j)) = \bar{\theta}_i(\bar{E}_j) = \delta_{ij}$
$= \theta_i(E_j)$
for all i, j. Hence, $\theta_i = F^*(\bar{\theta}_i)$, $\forall i$.

2) By the previous it is enough to show that ω_{12} and $F^*(\bar{\omega}_{12})$ satisfy the same structural equations:

Since $d\bar{\theta}_1 = \bar{\omega}_{12} \wedge \bar{\theta}_2$ and $d\bar{\theta}_2 = \bar{\omega}_{21} \wedge \bar{\theta}_1$ we get

$$F^*(d\bar{\theta}_1) = F^*(\bar{\omega}_{12} \wedge \bar{\theta}_2)$$

$$dF^*(\bar{\theta}_1) = F^*(\bar{\omega}_{12}) \wedge F^*(\bar{\theta}_2)$$

$$d\theta_1 = F^*(\bar{\omega}_{12}) \wedge \theta_2.$$

Similarly, we get $d\theta_2 = F^*(\bar{\omega}_{21}) \wedge \theta_1$.

Hence, by the previous lemma $\omega_{12} = F^*(\bar{\omega}_{12})$. ∎

Theorem (Gauss's Theorema Egregium)

Gaussian curvature is an isometric invariant. Explicitly, if $F: M \to \bar{M}$ is an isometry then

$$K(p) = \bar{K}(F(p)), \text{ for all } p \in M.$$

Proof: Let $p \in M$ and E_1, E_2 a tangent frame field at p. Also let $\bar{E}_i = F_*(E_i)$, $i=1,2$. By the previous lemma

$$F_*(\bar{\omega}_{12}) = \omega_{12}$$

On the other hand, by §6.2

$$d\bar{\omega}_{12} = -\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2.$$

Thus, $d(F^*\bar{\omega}_{12}) = F^*(d\bar{\omega}_{12})$

$$d\omega_{12} = F^*(-\bar{K}\,\bar{\theta}_1 \wedge \bar{\theta}_2)$$

$$d\omega_{12} = -\bar{K}(F)\; F^*(\bar{\theta}_1) \wedge F^*(\bar{\theta}_2).$$

$$d\omega_{12} = -\bar{K}(F)\; \theta_1 \wedge \theta_2$$

$$-K\, \theta_1 \wedge \theta_2 = -\bar{K}(F)\, \theta_1 \wedge \theta_2$$

$$\Rightarrow K = \bar{K}(F) \quad \blacksquare$$

Corollary Any part of a sphere is not isometric to any part of a plane.

Video 17

Gauss-Bonnet Theorem

Note Title 11.05.2020

If Σ_g is a compact genus g surface in $\mathbb{R}^3$, then

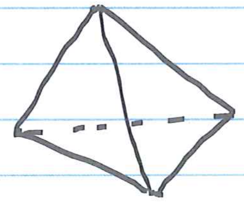
$$\int_{\Sigma_g} K(p)\, dS = 4\pi(1-g),$$ where $K(p)$ is the Gaussian curvature of Σ_g and dS is the area element on Σ_g.

Proof: We'll use a topological fact as follows:

Let Σ_g be triangulated so that the triangulation has v vertices, e edges and f faces. Then $v-e+f = 2-2g$ and this number is called the Euler characteristic of the surface Σ_g.

Notation: $\chi(\Sigma_g) = 2-2g$.

Example:



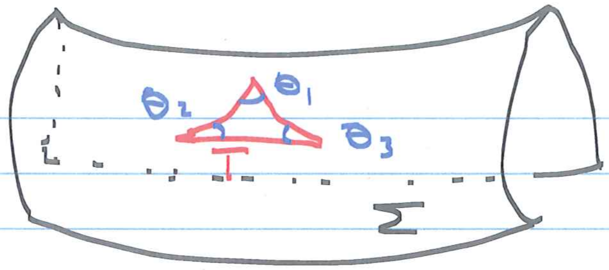
$\simeq S^2 = \Sigma_0$

$v=4$
$e=6$ $\qquad \chi(\Sigma_0) = 2-2\cdot 0 = 2 = v-e+f$
$f=4$

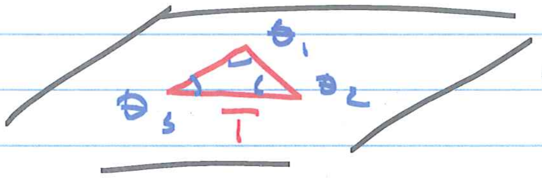
Also we need a geometric result: Let T be a triangle on a surface $\Sigma \subseteq \mathbb{R}^3$ so that its edges are geodesic curves on Σ. Then

$$\int_T K(p)\, dS = \theta_1+\theta_2+\theta_3-\pi,$$ where $\theta_1, \theta_2, \theta_3$ are the interior angles of T.

Ex:

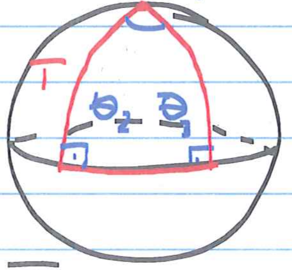


Ex



$$\theta_1+\theta_2+\theta_3-\pi = 0 = \int_T \overset{0}{\overset{\shortparallel}{K}}\, dS$$

Ex



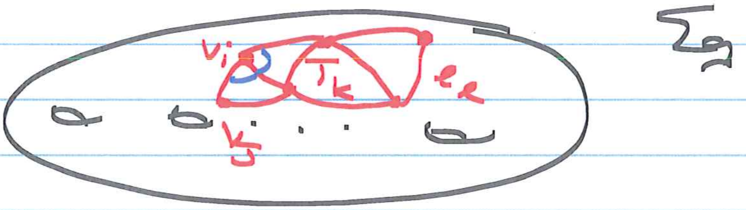
Σ : sphere of radius $r>0$.

$$K(p) = \frac{1}{r^2}$$

$$\theta_1+\overbrace{\theta_2+\theta_3}^{\pi}-\pi = \int_T K(p)\, dS = \frac{1}{r^2}\int_T dS = \frac{1}{r^2}\text{Area}(T)$$

$$\theta_1 r^2 = \text{Area}(T)$$

Back to the proof:



Take a geodesic triangulation of Σ_g with v vertices, e edges and f faces.
Then we know that

$$v-e+f = \chi(\Sigma_g) = 2-2g.$$

Let $T_1, T_2, \ldots, T_f$ be the list of all triangles in the triangulation. Since Σ_g is the union of T_i's we have $\Sigma_g = T_1 \cup T_2 \cup \cdots \cup T_f$.

Let $\theta_1^i, \theta_2^i, \theta_3^i$ be the interior angles of the triangle T_i.

Now, $$\int_{\Sigma_g} K(p)\,dS = \int_{\bigcup_{i=1}^{f} T_i} K(p)\,dS$$
$$= \sum_{i=1}^{f} \int_{T_i} K(p)\,dS$$
$$= \sum_{i=1}^{f} \left(\theta_1^i + \theta_2^i + \theta_3^i - \pi\right)$$
$$= -f\pi + \boxed{\sum_{i=1,\, j=1}^{f,\,3} \theta_i^j} = \text{sum of the all interior angles of all triangles.}$$
$$= -f\pi + V \cdot 2\pi$$

Moreover, each triangle has sides and every edge is the edge of exactly two triangles. Then $3f = 2e$.

So, $$\int_{\Sigma_g} K(p)\,dS = -f\pi + 2\pi v$$
$$= 2\pi\left(v - e + e - \frac{f}{2}\right)$$
$$= 2\pi\left(v - e + \frac{3f}{2} - \frac{f}{2}\right)$$
$$= 2\pi(v - e + f)$$
$$= 2\pi\, \chi(\Sigma_g) = 2\pi \cdot 2(1-g) = 4\pi(1-g).$$

Video 18

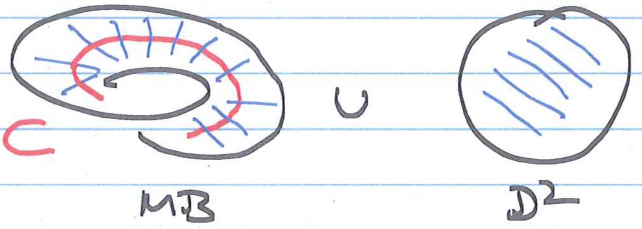
Some Applications:

Note Title 10.05.2020

1) $\mathbb{RP}^2$ does <u>not</u> embed into $\mathbb{R}^3$.

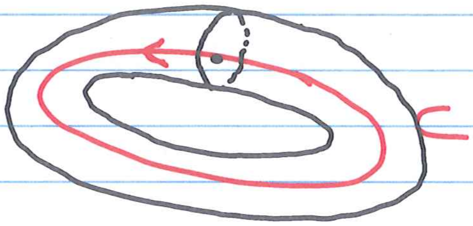
Proof: First assume that $\mathbb{RP}^2$ embeds into $\mathbb{R}^3$.

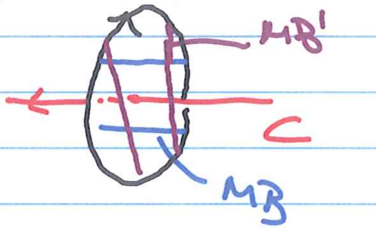
$\mathbb{RP}^2 = MB \cup_\partial D^2$



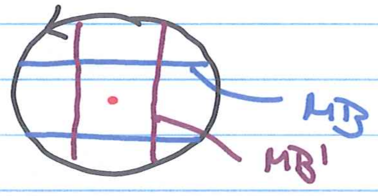
C: center of the Möbius Band.

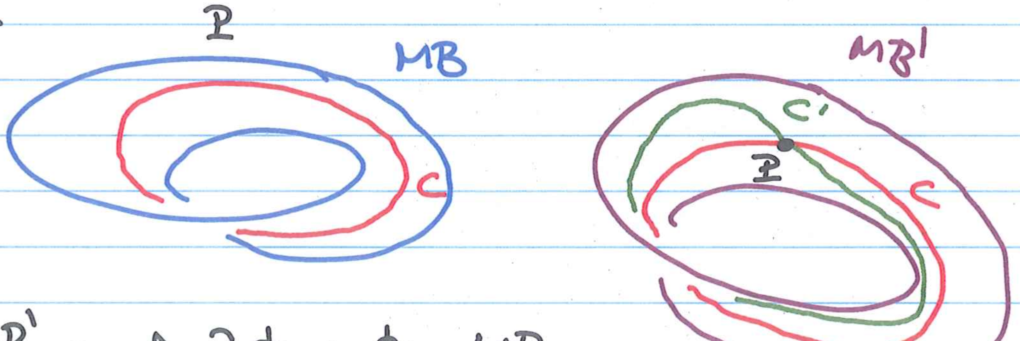
$C \subseteq MB \subseteq \mathbb{RP}^2 \subseteq \mathbb{R}^3$, consider the tubular neighborhood ν of C in $\mathbb{R}^3$.





Since C and $\mathbb{R}^3$ are both oriented the disk bundle ν is oriented. Now rotate each disk 90° degrees using the orientation to get another copy of the Möbius band. Note the center circle do not move under rotation.





$C' \subseteq MB'$ and intersects MB at one point p, where C' is a copy of C inside the rotated copy MB' of MB. Choosing the tubular neighborhood we see that C', which is a circle intersects MB and $\mathbb{RP}^2$ only at one point transversally. Hence the unoriented intersection of the closed manifolds $C' \simeq S^1$ and $\mathbb{RP}^2$ in $\mathbb{R}^3$ is

$$\text{Int}(C', \mathbb{RP}^2) = 1 \pmod 2$$

However, C' and $\mathbb{RP}^2$ are closed submanifolds of $\mathbb{R}^3$. Since $\mathbb{R}^3$ is unbounded by translating C' with vector we can make sure that C' and $\mathbb{RP}^2$ do not intersect at all. That is still a transversal intersection and thus

$$\text{Int}(C', \mathbb{RP}^2) = 0 \pmod 2.$$

This is clearly a contradiction!

Hence, $\mathbb{RP}^2$ cannot be embedded inside $\mathbb{R}^3$.

Remark: 1) $\mathbb{RP}^2 \subseteq \mathbb{RP}^3$ where $\mathbb{RP}^3$ is also oriented, however and $\text{Int}(C', \mathbb{RP}^2) = 1 \pmod 2$. Since $\mathbb{RP}^3$ is compact it is not possible to translate C' far enough so that C' and $\mathbb{RP}^2$ do not intersect any more.

2) $N = \mathbb{RP}^2 \times \mathbb{R}$, $\mathbb{RP}^2 \times \{p\} \longrightarrow \mathbb{RP}^2 \times \{p'\}$

$p \neq p' \Longrightarrow$ these two copies do not intersect.

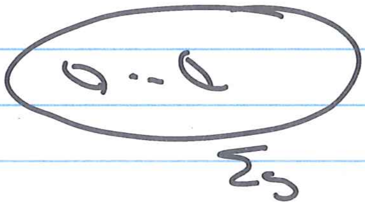
Q: Which parts of the above proof does not work in this case?

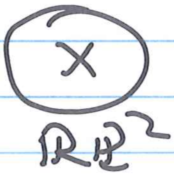
Answer: The tubular neighborhood of the center circle C in $N = \mathbb{RP}^2 \times \mathbb{R}$ is not orientable. Therefore, rotating each disc $\pi/2$-radian counterclockwise is not possible.

3) $\mathbb{R}^3 \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$ and therefore $\mathbb{RP}^2$ does not embed into S^3.

4) Now let N be any closed non-orientable surface. Then the above argument proves that N cannot embedded into $\mathbb{R}^3$ or S^3.

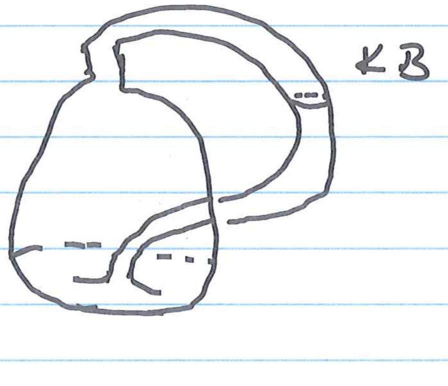
$N = \Sigma_g \# \mathbb{RP}^2$ or $\Sigma_g \#_2 \mathbb{RP}^2$





KB: $\Sigma_0 \#_2 \mathbb{RP}^2$

$\Rightarrow$ MB $\subseteq$ N and therefore we can repeat the above proof for N.



2) Theorem (Gauss-Bonnet Theorem)

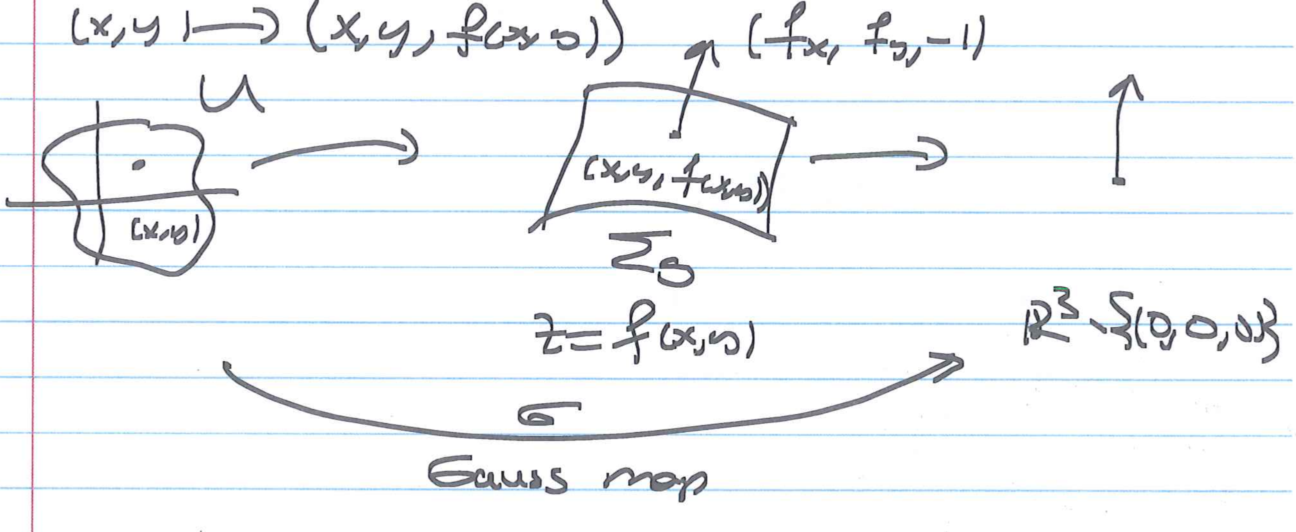
If Σ_g is a genus g orientable surface in $\mathbb{R}^3$ then

$$\int_{\Sigma_g} K(p)\,dS = 2\pi\,\chi(\Sigma_g) = 4\pi(1-g),$$

where $K : \Sigma_g \longrightarrow \mathbb{R}$ is the Gaussian curvature function on Σ_g.

Proof: Step 1: $\sigma : U \longrightarrow \mathbb{R}^3 \setminus \{(0,0,0)\}$

$U \subseteq \mathbb{R}^2$, $\sigma(x,y) = (f_x, f_y, -1)$ the Gauss map of the surface $\Sigma_g \subseteq \mathbb{R}^3$ parametrized by a local coordinate system



$H^2_{DR}(\mathbb{R}^3 \setminus \{(0,0,0)\}) \simeq \mathbb{R} = \langle [\omega] \rangle$, where

$$\omega = \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2(\mathbb{R}^3\setminus\{0\}).$$

$$\int_{S^2} \omega = 4\pi$$ (Exercise!)

$dS = \sqrt{1+f_x^2+f_y^2}\, dx\, dy$

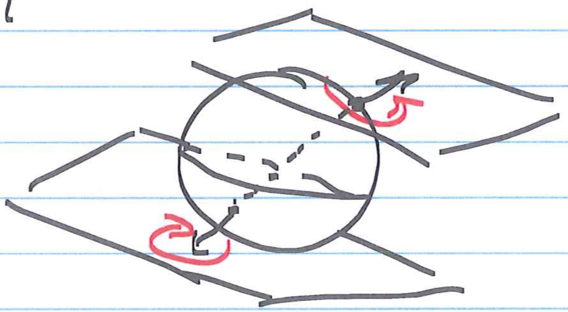
<u>Claim:</u> $\sigma^*(\omega) = K\, dS$, when

$K(x,y) = \dfrac{f_{xx} f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2}$ the Gaussian curvature.

Proof is left as an exercise.

Step 2) $\Sigma_g \subseteq \mathbb{R}^3$, $\sigma : \Sigma_g \longrightarrow \mathbb{R}^3 \setminus \{(0,0,0)\}$ Gauss map

$\frac{\sigma}{\|\sigma\|} : \Sigma_g \longrightarrow S^2$: the space of oriented 2-planes in $\mathbb{R}^3$.



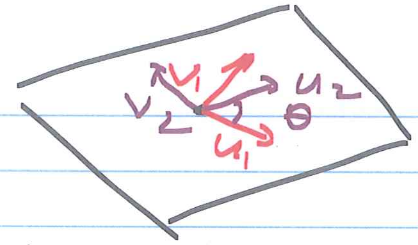
σ is homotopic to $\sigma/\|\sigma\|$, we can replace σ by $\sigma/\|\sigma\|$. So we assume that $\sigma : \Sigma_g \to S^2$.

$\sigma : \Sigma_g \longrightarrow S^2 = Gr^+_{\mathbb{R}}(3,2) \subseteq Gr^+_{\mathbb{R}}(n,2)$

$$\mathbb{R}^3 \subseteq \mathbb{R}^n$$
$$(x,y,z) \longmapsto (x,y,z,0,\dots,0)$$

$$Gr^+_{\mathbb{R}}(n,2) = \{(u,v) \in S^{n-1} \times S^{n-1} \mid u \perp v\} \big/ \sim$$

$(u_1,v_1) \sim (u_2,v_2) \iff \begin{cases} u_2 = \cos\theta\, u_1 - \sin\theta\, v_1 \\ v_2 = \sin\theta\, u_1 + \cos\theta\, v_1 \end{cases}$, $\theta \in \mathbb{R}$



$Gr^{+}_{\mathbb{R}}(n,2)$ is a smooth manifold of dimension $2(n-2)$. (Exercise!)

We have a map $\Phi : Gr_{\mathbb{R}}(n,2) \longrightarrow \mathbb{CP}^{n-1}$, by

$\Phi([u,v]) \doteq [u+iv] \qquad u+iv \in \mathbb{C}^n \setminus \{0\}$

$u, v \in \mathbb{R}^n$

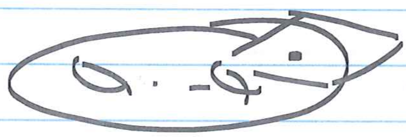
<u>Claim:</u> $\Phi(Gr_{\mathbb{R}}(n,2))$ is the quadratic hypersurface in $\mathbb{CP}^{n-1}$ given by $z_1^2 + z_2^2 + \dots + z_n^2 = 0$.

<u>Ex</u> At $n=3$, $Gr_{\mathbb{R}}(3,2) \simeq S^2$, $\Phi(Gr_{\mathbb{R}}(3,2))$ is the quadratic curve in $\mathbb{CP}^2$ given by $z_1^2 + z_2^2 + z_3^2 = 0$.

Let $F : \Sigma_g \times [0,1] \longrightarrow \mathbb{R}^n$ be a differentiable map.

$f_t \doteq F(-,t) \quad f_t : \Sigma_g \longrightarrow \mathbb{R}^n$ homotopy of maps.

Assume that each f_t is an immersion into $\mathbb{R}^n$.



$\longrightarrow \mathbb{R}^n$

$G_t : \Sigma_g \longrightarrow Gr_{\mathbb{R}}(n,2), \; p \longmapsto Df_{t*}(T_p \Sigma_g)$

Consider the composition $\Phi \circ \sigma_t : \Sigma_g \to \mathbb{CP}^{n-1}$

$$\Sigma_g \xrightarrow{\sigma_t} Gr_{\mathbb{R}}(n,2) \xrightarrow{\Phi} \mathbb{CP}^{n-1}$$

Let $a \in H^2_{DR}(\mathbb{CP}^{n-1})$ so that $\int_{\mathbb{CP}^1} a = 1/2$.

$\Rightarrow \int_{\Phi(Gr_{\mathbb{R}}(3,2))} a = 1$ because $\Phi(Gr_{\mathbb{R}}(3,2)) \to \mathbb{CP}^1$ is a double cover.

$\Rightarrow K\,dS = \sigma^*(\omega) = 4\pi(\Phi \circ \sigma_t)^*(a)$ as cohomology classes.

Conclusion: For any two immersions of Σ_g into $\mathbb{R}^n$ the integral

$$\int_{\Sigma_g} K dS \quad \text{gives the same result.}$$

Step 3) Proposition: Any two immersions of Σ_g into $\mathbb{R}^n$ $(n \geq 7)$ are homotopic through immersions.

Proof: The vector space of all polynomials in $\mathbb{R}[x,y,z]$ of degree at most d has dimension $s = \binom{3+d}{d}$. Take any point $P = (x_0, y_0, z_0) \in \mathbb{R}^3$.

By the linear change of coordinates $(x,y,z) \longrightarrow (x-x_0, y-y_0, z-z_0)$ we can assume that $P=(0,0,0)$.

Let $f_1, \dots, f_{k+3} \in \mathbb{R}^S$, the vector space of polynomials in x,y,z of degree $\leq d$.

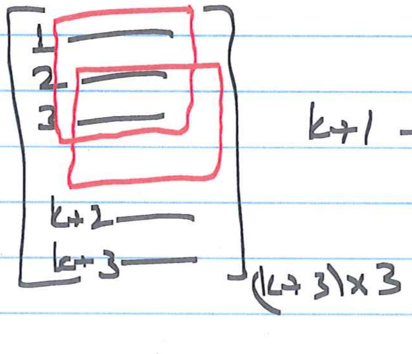
$$\phi = (f_1, \dots, f_{k+3}) : \mathbb{R}^3 \longrightarrow \mathbb{R}^{k+3}$$

The condition that $(0,0,\dots,0)$ is a critical point for ϕ is a linear condition on the first degree terms of f_i's.

$$D\phi(0,\dots,0) = \begin{bmatrix} \nabla f_1(0,0,\dots,0) \\ \vdots \\ \nabla f_{k+3}(0,0,\dots,0) \end{bmatrix}, \quad \nabla f_i(0) = \left(\frac{\partial f_i}{\partial x}(0), \frac{\partial f_i}{\partial y}(0), \frac{\partial f_i}{\partial z}(0)\right)$$

$$f_i = a_0 + a_1 x + a_2 y + a_3 z + O(2), \quad \nabla f_i(0) = (a_1, a_2, a_3)$$

$(0,0,\dots,0)$ is a critical point for Φ if and only if the matrix $D\Phi(0)$ has rank ≤ 2.



k+1 – independent conditions.

Hence, the subspace of all $(f_1, \dots, f_{k+3})$ in $\mathbb{R}^{S(k+3)}$ having $(0,0,\dots,0)$ as a critical point have codimension $k+1$.

Let $E = \{((x,y,z), f_2, f_1, \dots, f_{k+3}) \in \mathbb{R}^3 \times \mathbb{R}^{S(k+3)} \mid$

$$rk(D(f_1, \dots, f_{k+3})_{(x,y,z)} \leq 2\}.$$

$\pi : E \longrightarrow \mathbb{R}^3$, $((x,y,z), f_1, \dots, f_{k+3}) \longmapsto (x,y,z)$.

All the fiber of π have the same structure and they are unions of $\binom{k+3}{2}$ linear subspaces of codim $k+1$.

Thus the set of all polynomial maps

$$\phi = (f_1, \dots, f_{k+3}) : \mathbb{R}^3 \longrightarrow \mathbb{R}^{k+3},$$

which are not an immersion at some point, form a set in $\mathbb{R}^{S(k+3)}$ of codimension

$$(k+1) - 3 = k-2.$$

Hence, if $k \geq 4$ the set of all polynomial immersions $\phi = (f_1, \dots, k+3) : \mathbb{R}^3 \longrightarrow \mathbb{R}^{k+3}$ is path connected because $k \geq 4 \Rightarrow k-2 \geq 4-2 = 2$.

In particular, all immersions (polynomial)

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^7$$ is path connected.

So for our surface $\Sigma_g \subseteq \mathbb{R}^3$ restriction of any immersion $\mathbb{R}^3 \to \mathbb{R}^7$ to Σ_g is also an immersion. Hence, the space of all polynomial immersions of Σ_g into $\mathbb{R}^7$ is path connected.

$\Sigma_g \xhookrightarrow{\phi_0} \mathbb{R}^7$, $\Sigma_g \xhookrightarrow{\phi_1} \mathbb{R}^7$ two immersions $\Rightarrow \exists\ \phi_t$ homotopy so that each $\phi_t : \Sigma_g \hookrightarrow \mathbb{R}^7$ is an immersion.

In particular, any two embeddings

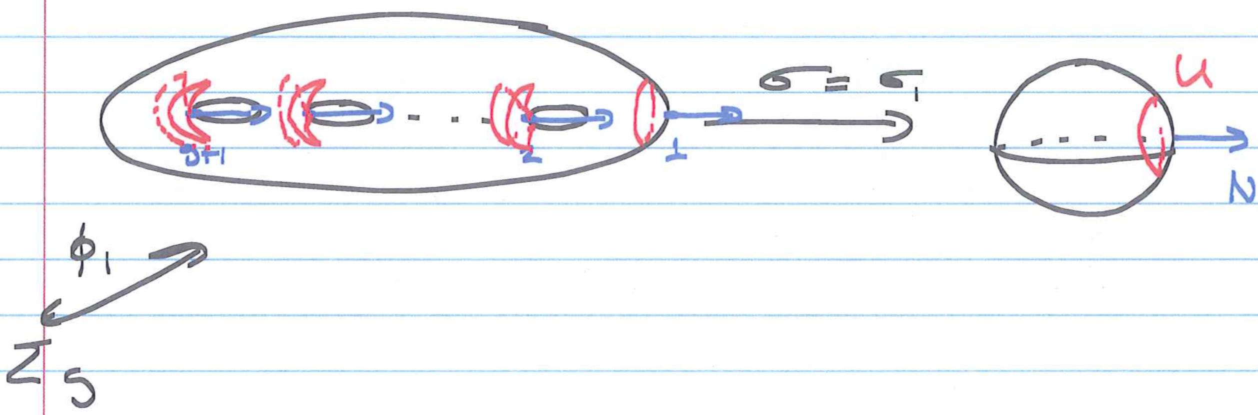
$\phi_0 : \Sigma_g \hookrightarrow \mathbb{R}^3$, $\phi_1 : \Sigma_g \hookrightarrow \mathbb{R}^3$ are homotopic through immersions into $\mathbb{R}^7$.

$\int_{\Sigma_g} \sigma_0^*(\omega) = \int_{\Sigma_g} \sigma_1^*(\omega)$, where σ_i is the Gauss map corresponding to the embedding ϕ_i.

Video 19

Note Title 13.05.2020

Step 4 $\phi_0: \Sigma_g \hookrightarrow \mathbb{R}^3$, $\phi_1: \Sigma_g \hookrightarrow \mathbb{R}^3$ two embeddings. ϕ_0 given embedding of Σ_g. ϕ_1 is the embedding looks like



$$\int_{\Sigma_g} \kappa\, dS = \int_{\Sigma_g} \sigma^*(\omega)$$

Replace ω be a form so that it is supported in U with the same integral.

$\sigma^{-1}(U) = V_1 \cup V_2 \cup \dots \cup V_{g+1}$ disjoint open sets.

$\sigma_i : V_i \to U$ is a diffeomorphism.

Then $\int_{\Sigma_g} \sigma^*(\omega) = \int_{\sigma^{-1}(U)} \sigma^*(\omega) = \sum_{i=1}^{g+1} \int_{V_i} \sigma^*(\omega)$,

where each $\int_{V_i} \sigma^*(\omega) = \pm \int_{U} \omega = \pm \int_{S^2} \omega = \pm 4\pi$

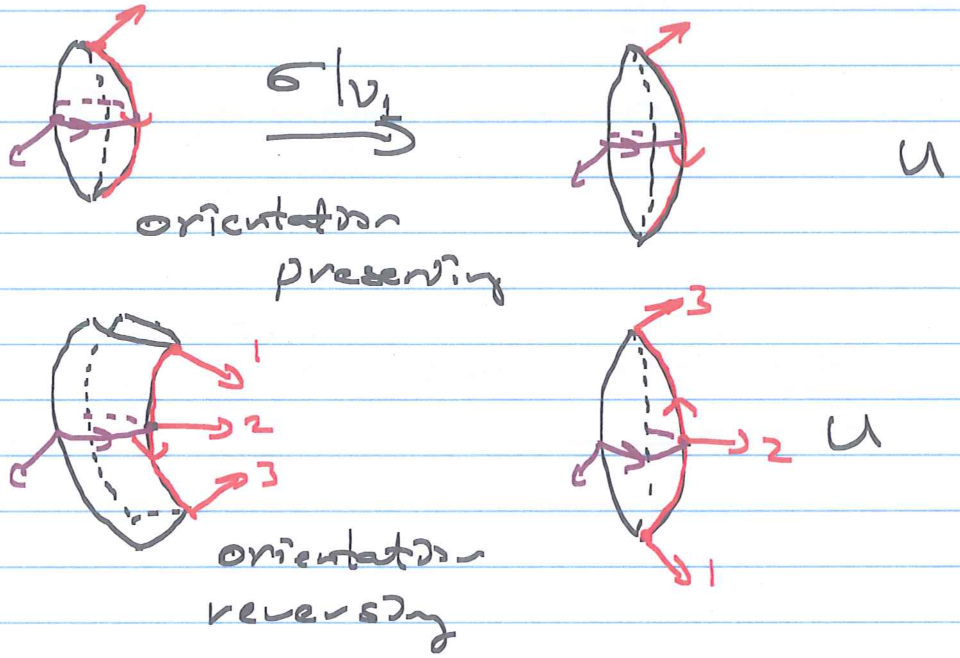
and the sign is ± 1 depending on whether

$\sigma : V_i \to U$ is orientation preserving or not.

V_1 :

V_j :

$i \geq 2$



Hence, $\int_{\Sigma_g} \kappa(p)\, dS = \int_{\Sigma_g} \sigma^*(\omega) = 4\pi - g\, 4\pi = 2\pi \chi(\Sigma_g).$

∎

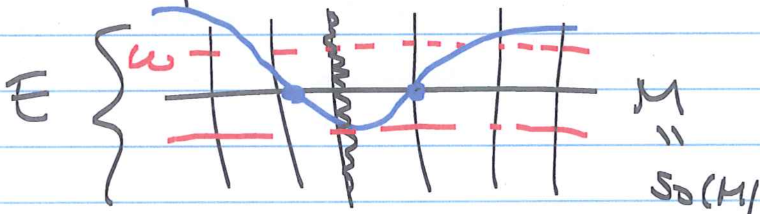
CHARACTERISTIC CLASSES

Euler class $\mathbb{R}^k \to E$ oriented vector bundle

$$\begin{tikzcd} \mathbb{R}^k \arrow[r] & E \arrow[d] \\ & M \end{tikzcd}$$

Let $s_0 : M \to E$ be the zero section.

$e(E)$: Poincaré dual of the zero section $s_0(M)$ in E.



$e(E) = [\omega]$, with supp(ω) lies in a tubular neighborhood whose integral along any fiber (oriented) is equal 1.

$\omega \in \Omega^k(M)$ $\quad e(E) \in H^k_{DR}(M)$.

Some Properties of the Euler Class:

1) $E_i \to M$ $i=1,2$, oriented vector bundles.

Then $E_1 \oplus E_2 \to M$ is also an oriented vector bundle.

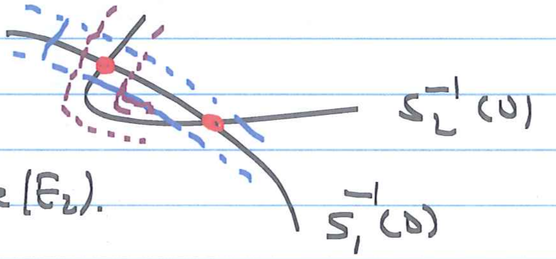
Proposition: $e(E_1 \oplus E_2) = e(E_1) \cdot e(E_2)$.

Proof: $s_i : M \to E_i$ section $i=1,2$.

$(s_1, s_2) : M \to E_1 \oplus E_2$ section.

$(s_1, s_2)^{-1}(0) = s_1^{-1}(0) \cap s_2^{-1}(0)$

Hence, $e(E_1 \oplus E_2)$ is the Poincaré dual of the intersection of the submanifolds $s_1^{-1}(0)$ and $s_2^{-1}(0)$.



Hence, $e(E_1 \oplus E_2) = e(E_1)\, e(E_2)$.

2) $f: M \longrightarrow N$ smooth map, $E \longrightarrow N$ oriented vector bundle. Then

Proposition $e(f^*(E)) = f^*(e(E))$

Proof

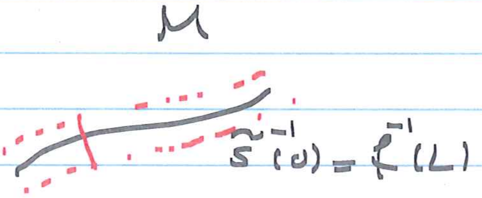
$$\begin{tikzcd}
f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\
M \arrow[r, "f"'] \arrow[u, bend left, "\tilde{s}"] & N \arrow[u, bend left, "s"] \arrow[u, bend right, "s"'] \\
p \arrow[r, mapsto] & f(p)
\end{tikzcd}$$

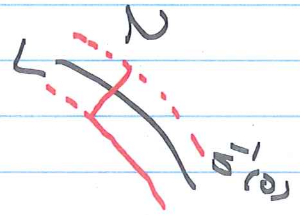
$f^*(E) = \{(p, v) \in M \times E \mid f(p) = \pi(v)\}$.

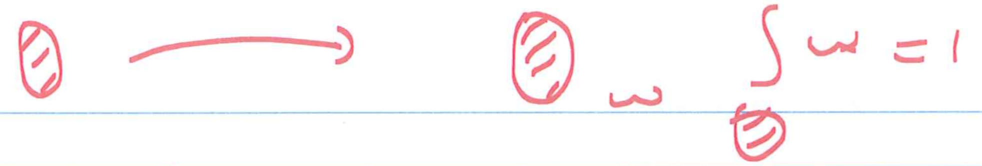
$\tilde{s}(p) \doteq (p, s(f(p))$

Hence, $\tilde{s}^{-1}(0) = f^{-1}(s^{-1}(0))$.

Hence, choosing f and s transverse to each other $\tilde{s}^{-1}(0)$ is a submanifold in M.







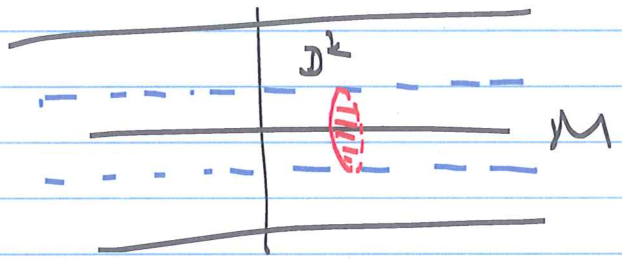
$\int f^*(\omega) = 1$

This proves the result.

3) For any oriented vector bundle E let $-E$ denote the bundle with opposite orientation. Then

$$e(-E) = -e(E).$$

Proof: E:



$$\int_{D^k} \omega = 1$$

$$\int_{-D^k} -\omega = +1 \qquad e(-E) = [-\omega] = -[\omega] = -e(E).$$

4) $E \to M$ oriented vector bundle and let $E^* \to M$ be the dual of $E \to M$.

$E^* = \hom(E, \mathbb{R})$.

Then $e(E^*) = (-1)^{\operatorname{rank}(E)/2} \, e(E)$.

Proof:
$$\begin{tikzcd} \mathbb{C} \arrow[r] & L \arrow[d] \\ & M \end{tikzcd}$$
complex line bundle

$\mathbb{C} = \mathbb{R}^2 \Rightarrow$ We may regard $L \to M$ as an oriented $\mathbb{R}^2$-bundle.

$L^* = \hom(L, \mathbb{C}) \longrightarrow M$.

$\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$ transition function for L.

$\varphi_{\alpha\beta}^{-1} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$, $\varphi_{\alpha\beta}^{-1}(x) = (\varphi_{\alpha\beta}(x))^{-1}$
transition function for L^*.

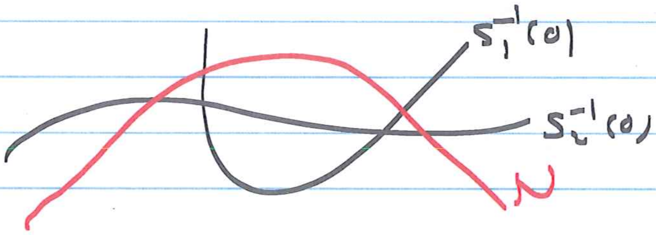
$L_1, L_2 \longrightarrow M$ cx. line bundle.

$L_1 \otimes L_2 \longrightarrow M$ another complex bundle whose transition function is the product of the transition functions of L_1 and L_2.

$s_i : M \longrightarrow L_i$ section of L_i.

$s_1 \cdot s_2 : M_1 \longrightarrow L_1 \otimes L_2$ section of $L_1 \otimes L_2$.

$(s_1 \cdot s_2)^{-1}(0) = s_1^{-1}(0) \cup s_2^{-1}(0)$.



$\mathrm{Int}(N, s_1^{-1}(0) \cup s_2^{-1}(0)) = \mathrm{Int}(N, s_1^{-1}(0)) + \mathrm{Int}(N, s_2^{-1}(0))$

$\Rightarrow PD(s_1^{-1}(0) \cup s_2^{-1}(0)) = PD(s_1^{-1}(0)) + PD(s_2^{-1}(0))$

$e(L_1 \otimes L_2) = e(L_1) + e(L_2)$.

$L \otimes L^* = \xi \longrightarrow M$ is the trivial bundle.
$0 = e(L \otimes L^*) = e(L) + e(L^*)$.

$e(L^*) = -e(L)$

$$e((L_1 \oplus L_2 \oplus \oplus L_k)^*) = e(L_1^* \oplus \dots \oplus L_k^*)$$
$$= e(L_1^*) \cdot \dots \cdot e(L_k^*)$$
$$= (-1)^k e_1(L_1) \dots e(L_k)$$
$$= (-1)^k e(L_1 \oplus \dots \oplus L_k).$$

Hence for an oriented vector bundle E of rank $2n$ we take orientation of E^* as follows:

$$e_1, \dots, e_{2n}, \longrightarrow (-1)^n e_1^*, e_2^*, \dots, e_{2n}^*.$$

$$e(E^*) = (-1)^n e(E), \quad \text{rank}(E) = 2n.$$

<u>Special Case:</u> $T_*M \longrightarrow M$ tangent bundle.

Then T_*M as a smooth manifold is oriented, with orientation:

$$\underbrace{x_1, \dots, x_n}_{\text{on } M}, \underbrace{a_1, \dots, a_n} \quad \text{coor. system on } T_*M.$$

$$a_j\left(\sum \xi_i \frac{d}{dx_i}(p)\right) = \xi_i.$$

<u>Orientation of T^*M.</u>

$$\left.\begin{array}{l} x_1, \dots, x_n, b_1, \dots, b_n \\ b_i\left(\sum \xi_i \, dx_i\right) = \xi_i \end{array}\right) \begin{array}{l}\text{This gives an} \\ \text{orientation on } T^*M.\end{array}$$

However, this is <u>not</u> compatible with the orientation we considered above.

Instead, we take as the canonical orientation on the cotangent bundle as

$x_1, b_1, x_2, b_2, \dots, x_n, b_n$.

<u>Remarks:</u> 1) The difference of orientations given by $x_1, \dots, x_n, b_1, \dots, b_n$ and

$x_1, b_1, x_2, b_2, \dots, x_n, b_n$ is $(-1)^{n(n-1)/2}$

2) T^*M has a canonical symplectic structure given by

$dx_1 \wedge db_1 + dx_2 \wedge db_2 + \dots + dx_n \wedge db_n$.