METU MATH. DEPT. MATH 341 **Video 1**

GRAPH THEORY FALL of 2021

Note Title 11.10.2021

Lecture notes by Ali Doğanaksoy, 3rd nov. 2019

§1. Preliminaries:

In this course we'll consider only "finite graphs".

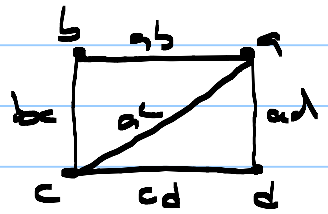
A (finite) graph G is a pair $(V(G), E(G))$ consisting of a nonempty set $V(G)$ and a (finite) collection $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$.

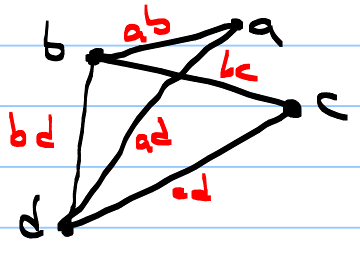
Elements of $V(G)$ are called vertices and elements of $E(G)$ are called edges of G. An edge $\{u,v\}$ is usually denoted uv, when u and v are distinct vertices. We'll often write V and E for $V(G)$ and $E(G)$, respectively. The number of elements in $V(G)$ is called the order of the graph and the number of elements in $E(G)$ is called the size of the graph.

A representation of a graph G in plane is a figure consisting of points and line segments, corresponding to vertices and edges, respectively. A line segment is drawn between two points corresponding to vertices u and v if and only if uv is in the edge set of G.

A given figure Γ in the plane, consisting of some points and some line segments joining these points is called a plane graph if there is a graph for which the given figure is a plane representation.

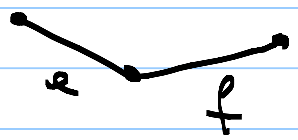
<u>Examples.</u> $(V(G), E(G)) = (\{a,b,c,d\}, \{\{a,b\},\{b,c\},\{c,d\}, \{d,a\},\{b,d\}\}$



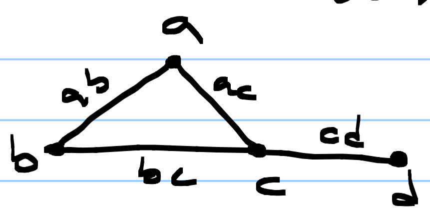


It may be the case that vertices and the line segments of Γ are not labelled. In such a case, Γ is called a plane graph if for some appropriate labelling of its objects it turns out to be a representation of some graph.

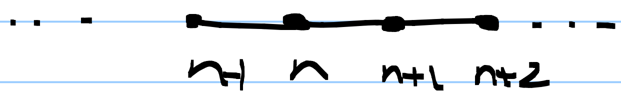
An edge $e = uv = \{u,v\}$ is said to join the vertices u and v. In that case, we also say that e is incident to u and v. Moreover, we say that u and v are adjacent or u is a neighbour of v.
Similarly, two edges incident to the same vertex are called adjacent.



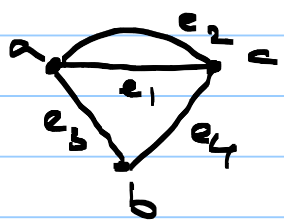
<u>Example:</u> G, $V(G) = \{a,b,c,d\}$,
$E(G) = \{ab, ac, bc, cd\}$



<u>Example:</u> 1) G, $V(G) = \mathbb{Z}$, $E(G) = \{\{n, n+1\} \mid n \in \mathbb{Z}\}$.



2) G, $V(G) = \{a,b,c\}$, $E(G) = \{e_1, e_2, e_3, e_4\}$



This is a multigraph.

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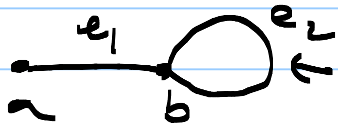
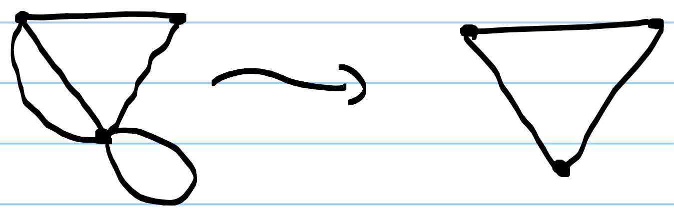
3) G, $V(G)=\{a,b\}$, $E(G)=\{e_1,e_2\}$



$\leftarrow$ e_2 is called a loop

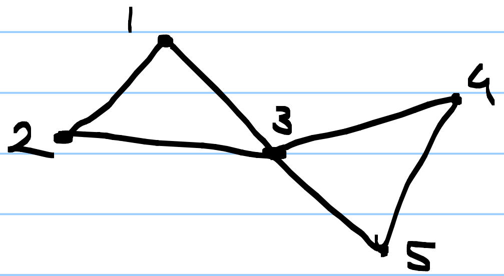
A graph, which has no multiedges or loops is called simple. If G is not simple, by removing loops and replacing each set of parallel edges with a single edge we obtain a simple graph, which is called the underlying graph of G.

<u>Example</u>: G:



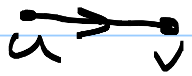
Underlying simple graph of G.

A labelled graph of order n is a graph to each vertex of which an integer $1, 2, \ldots, n$ is assigned such that no two vertices are assigned to the same number.



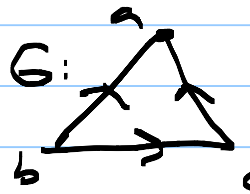
<u>Directed Graph</u>: Sometimes the edge set $E(G)$ of G is considered as a set of ordered pairs of vertices. Such graphs are called directed, meaning that each edge $(u,v) \in E(G)$ is directed

from the vertex u to the vertex v. In that case, we say that u dominates v and represent this by putting an arrow on the edge, pointing v.



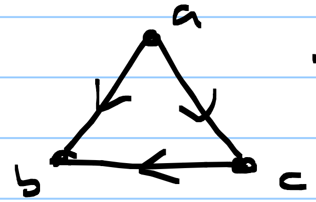
Given a directed graph G the graph G' obtained from G by reversing each arrow is called the symmetry of G.

Example: G:



$V(G)=\{a,b,c\}$
$E(G)=\{(b,a),(b,c),(c,a)\}$

G':

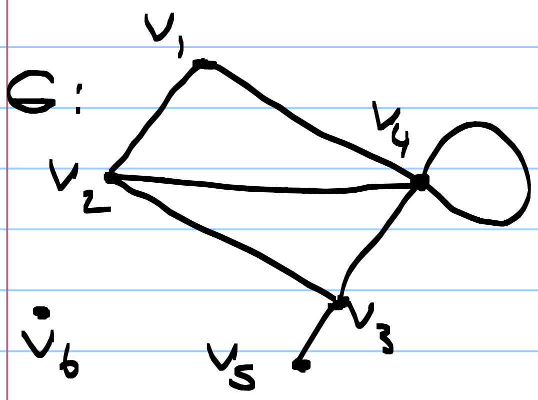


$V(G')=\{a,b,c\}$
$E(G')=\{(a,b),(c,b),(a,c)\}$.

If a graph contains no pairs of symmetric edges then it is called a tournament.

If $G=G'$ then G is called symmetric.
$\left((a,b)\in E(G) \Rightarrow (b,a)\in E(G)\right)$

Local Degree (Valency) For each vertex v in a graph G, the number of edges adjacent to v is called the valency of v, or the local degree of G at v. If a graph is a multigraph then a loop at v is counted twice. The valency of G at a vertex v is denoted as $d(v)$ or $d_G(v)$. By $\Delta = d_{max}(G)$ and $\delta = d_{min}(G)$ the maximum and minimum valencies in G are denoted, respectively.



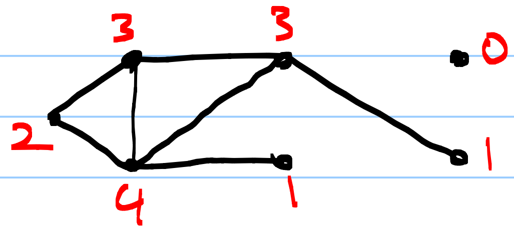
$d(v_1)=2$, $d(v_2)=3$
$d(v_3)=3$, $d(v_4)=5$
$d(v_5)=1$, $d(v_6)=0$

$\Delta=5$, $\delta=0$.

A vertex with valency zero is called isolated. Similarly, vertices with valency one are called end vertices.

A sequence of integers which is the valency sequence of some simple graph is said to be graphical.

Example:



Valency sequence 4, 3, 3, 2, 1, 1, 0.

Theorem: Let G be a graph with $|E(G)|=e$ and $V(G)=\{v_1,\ldots,v_n\}$. Then

$$\sum_{i=1}^{n} d(v_i) = 2e$$

Proof: Note that each edge contributes two to the sum of valencies, because each edge or loop has two "ends".

Corollary Any graph contains an even number of odd vertices.

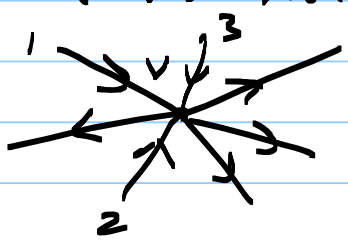
Proof $2e = \sum_{i=1}^{n} d(v_i) = \sum_{d(v_i)\text{ even}} d(v_i) + \sum_{d(v_i)\text{ odd}} d(v_i)$

Video 3

$\Rightarrow$ number of v_i's s.t. $d(v_i)$ is odd must be even. ∎

<u>Example:</u> The sequence 6, 6, 5, 5, 4, 4, 3, 2, 1, 1 is not graphical.

In a digraph (directed graph) D, indegree $d_{in}(v)$ is the number of edges in D incident to v pointing v, and $d_{out}(v)$ is the number of edges in D incident to v not pointing v.

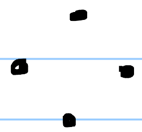


$d_{in}(v) = 3, \quad d_{out}(v) = 4$

$d(v) = d_{in}(v) + d_{out}(v)$

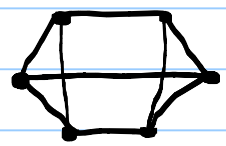
<u>Regular Graphs:</u> If all vertices of G have the same valency r, then G is said to be r-regular or a r-valent graph.

<u>Example:</u> 1)



0-valent graph

2)

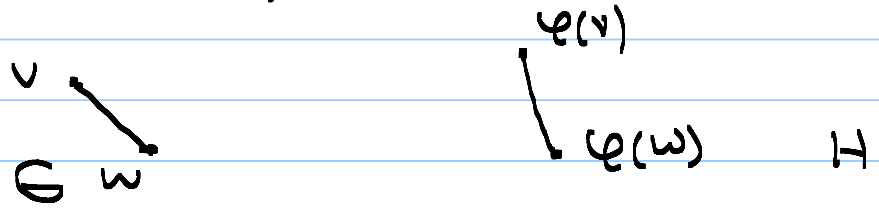


3-valent graph

Note that for a regular r-valent graph of order n we have $e = \frac{1}{2}nr$. Note that there is no r-valent regular graph of order n if nr is odd. For example, there is no 5-valent graph of order 9.

A regular 3-valent graph is also called a cubic graph.

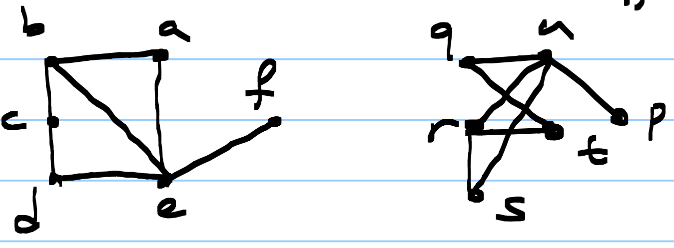
II - Graph Isomorphism: Two graphs G and H are said to be isomorphic (written $G \cong H$) if there exists a 1-1 correspondence $\varphi: V(G) \to V(H)$ such that for any $v, w \in V(G)$, $\varphi(v)\varphi(w) \in E(H)$ if and only if $vw \in E(G)$.



Example: G, $V(G) = \{a, b, c, d, e, f\}$
$E(G) = \{ab, bc, cd, de, be, ae, ef\}$

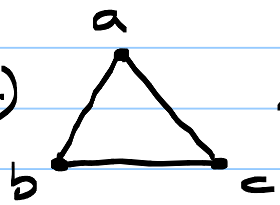
H, $V(H) = \{p, q, r, s, t, u\}$
$E(H) = \{pu, uq, ur, us, qt, rs, rt\}$

$\varphi: V(G) \to V(H)$ given by $a \mapsto s$, $b \mapsto r$, $c \mapsto t$, $d \mapsto q$, $e \mapsto u$, $f \mapsto p$.



An isomorphism of a graph G onto itself is called an automorphism of G. Note that the set of all automorphisms of a graph G is a group, called the automorphism group of the graph G.

Examples) 1-)

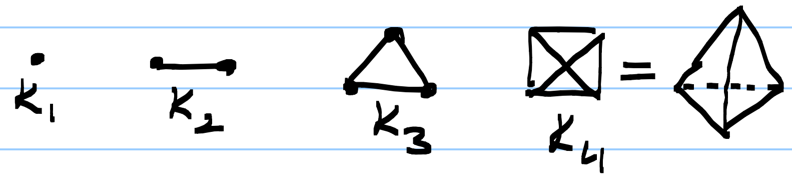


G

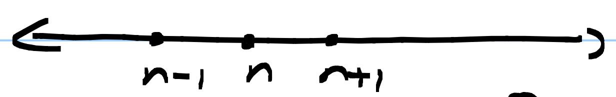
$Aut(G) = S_3 = \{(ab), (bc), (ac), (abc), (acb), id\}$

2) More generally, if $G = K_n$, $Aut(G) = S_n$

where $V(K_n)=\{v_1,\dots,v_n\}$, $E(K_n)=\{v_iv_j \mid 1\le i<j\le n\}$



2) G : $V(G)=\mathbb{Z}$, $E(G)=\{(n\,n+1) \mid n\in\mathbb{Z}\}$



$\mathrm{Aut}(G)=\langle\sigma,\tau\rangle$

$\sigma:\mathbb{Z}\to\mathbb{Z}$
$n\longmapsto -n$

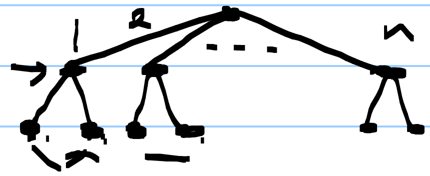
$\tau:\mathbb{Z}\to\mathbb{Z}$
$n\longmapsto n+1$

$$\begin{matrix} 0 & & 5\\ 1 & \xrightarrow{f} & 4\\ 2 & & 3\\ 3 & & 2\end{matrix}$$

$$\tau^5(n)=n+5$$

$f(n)=-n+5$, $f=\tau^5\sigma$

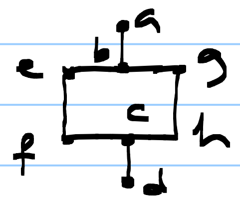
3)



G $\quad |V(G)|=1+n+2n=3n+1$
$|E(G)|=3n$

$\mathrm{Aut}(G)=S_n \ltimes (\mathbb{Z}_2)^n$ (semi-direct product)

<u>Exercise:</u> Determine the automorphism group of the graph



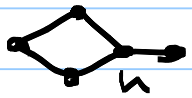
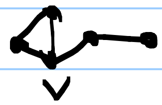
A graph is said to be vertex transitive (symmetric) if for any pair of vertices, there is an automorphism of the graph which maps one of these vertices to the other.

K_n is a symmetric graph.

Video 4

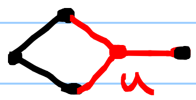
Remark: Some necessary but not sufficient conditions for two graphs G and H to be isomorphic are given as follows:

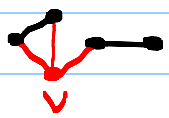
1) $|V(G)| = |V(H)|$
2) $|E(G)| = |E(H)|$
3) Non decreasing valency sequences are the same.

Example: Let G:  and H: 

Note that $|V(G)| = |V(H)| = 5$, $|E(G)| = |E(V)| = 5$ and both graphs have the same valency sequence $3,2,2,2,1$. However, $G \not\cong H$.

Proof that $G \not\cong H$: Let $\varphi: G \to H$ be an isomorphism. Then φ should map u to v since they are the only vertices with valency 3. Let's delete u, v and the edges connected to them, say

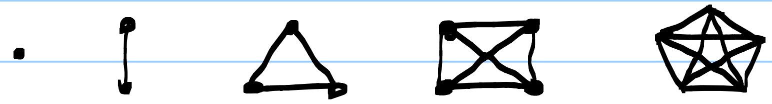
$G \setminus \{u\}$: 

 : $H \setminus \{v\}$

The restriction $\tilde{\varphi}: G \setminus \{u\} \to H \setminus \{v\}$ of φ is still an isomorphism. However, the valency sequences of $G \setminus \{u\}$ is $2,1,1,0$ and that of $G \setminus \{v\}$ is $1,1,1,1$, which are not the same. This contradiction implies that G and H are not isomorphic.

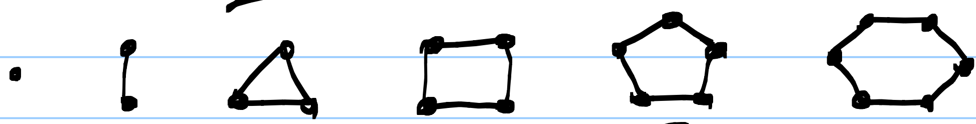
III – Special Graphs:

1) A simple graph in which every pair of vertices are joined by an edge is called a complete graph, denoted K_n. It is clearly an $(n-1)$-valent graph with $n(n-1)/2$ edges.



K_1 K_2 K_3 K_4 K_5

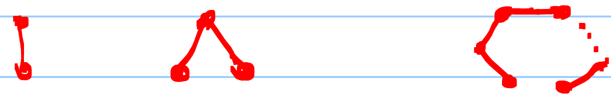
<u>Circuit Graphs</u>: C_n is the unique connected graph of order $n \geq 3$, which is 2-valent.
Thus, $e = \frac{1}{2} \cdot n \cdot 2 = n$.



C_1 C_2 C_3 C_4 C_5 C_6

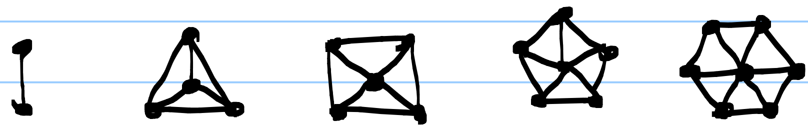
<u>Path graphs</u>:

P_2 P_3 P_n



$P_n = C_n \setminus \{uv\}$, where uv is an edge of P_n.

<u>Wheel Graphs</u>:



W_2 W_3 W_4 W_5 W_6

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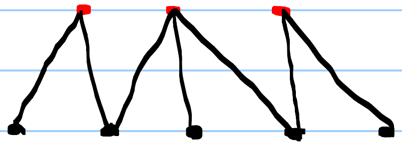
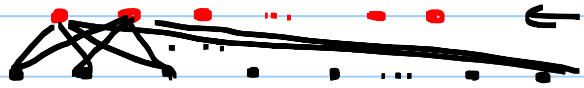
Bipartite Graph: A bipartite graph is a graph whose vertex set is partitioned into two disjoint parts such that no two vertices in the same part are adjacent.

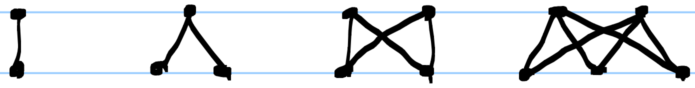


$K_{p,q}$: complete bipartite graph



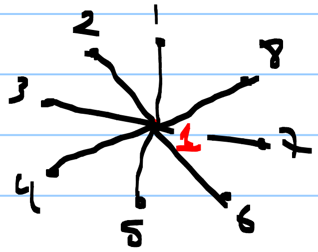
← p vertices

← q vertices



$K_{1,1}$ $K_{1,2}$ $K_{2,2}$ $K_{2,3}$

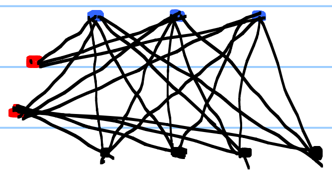
The graph $K_{1,q}$ is called a star graph



$K_{1,8}$

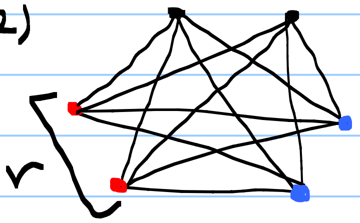
A complete r-partite graph is a graph whose vertex set is partitioned into r-sets and two vertices are joined by an edge if and only if they lie in different parts.

Ex:



The graph $K_{r(k)}$ has rk vertices and it is an $k(r-1)$-valent graph.

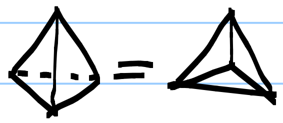
$K_{3(2)}$



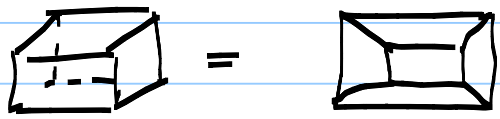
It has $\frac{1}{2}k^2 r(r-1)$ edges.

Each vertex is connected to $k(r-1)$ vertices. Since there are kr vertices and each edge has two ends the number of edges is $\frac{1}{2}k^2 r(r-1)$.

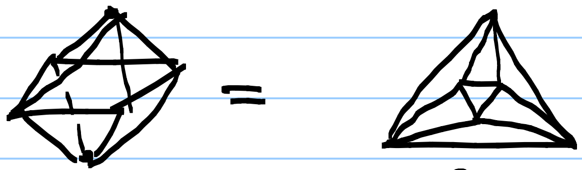
Polyhedral Graphs:



Tetrahedron



Hexahedron



Octahedron

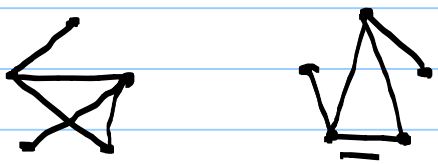
? (No!)

IV Obtaining New Graphs from a given graph:

Complement of a Graph: Given a simple graph G, the complement $\overline{G}$ of G is the graph with $V(\overline{G}) = V(G)$ and two vertices of which are adjacent if and only if they are not adjacent in G.

$$E(G) \cap E(\overline{G}) = \emptyset \text{ and } E(G) \cup E(\overline{G}) = E(K_n),\ n = |V(G)|.$$

Ex



G $\qquad$ $\overline{G}$

A graph which is isomorphic to the complement is called a self-complementary graph.

Proposition: Order of a self-complementary graph is of the form $4k$ or $4k+1$ for some positive integer k.

Proof: Let G is a self-complementary graph of order n then $E(G) \cup E(\overline{G}) = E(K_n)$ and $|E(G)| = |E(\overline{G})|$.
Hence,

$$|E(G)| + |E(\overline{G})| = |E(K_n)| \Rightarrow 2|E(G)| = \frac{n(n-1)}{2}$$

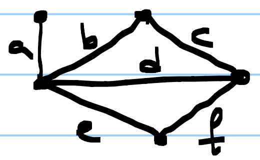
$\Rightarrow |E(G)| = \frac{n(n-1)}{4}$. Since $|E(G)|$ is an integer

$n(n-1)$ is divisible by 4. Since one of n or $n-1$ is odd the other one must be divisible by 4.
Hence, either n or $n-1$ is divisible by 4.
$\Rightarrow n = 4k$ or $n-1 = 4k \Rightarrow n = 4k+1$, for some $k \in \mathbb{N}$. ■

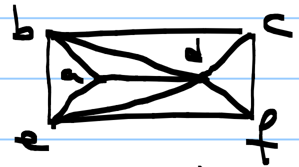
Corollary The number of edges of a self-complementary graph of order n is $n(n-1)/4 = \frac{|E(K_n)|}{2} = \frac{n(n-1)}{2 \cdot 2}$.

Line Graph: The line graph of a given graph G is a graph $L(G)$ with $V(L(G)) = E(G)$ and two vertices of which are adjacent if and only if the corresponding edges are adjacent in G.

Example



G



$L(G)$

Theorem: Let G be a graph of order n, then

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$|E(L(G))| = \frac{1}{2}\sum_{i=1}^{n} d^2(v_i) - e$, where e is the number of edges of G and $V(G)=\{v_1,\dots,v_n\}$.

Proof: Let v_i be a vertex of G. Since each pair of edges in G incident to v_i gives rise to an edge in $L(G)$, for each vertex v_i, there are $\binom{d(v_i)}{2}$ edges in $L(G)$. So,

$$|E(L(G))| = \sum_{i=1}^{n} \binom{d(v_i)}{2} = \sum_{i=1}^{n} \frac{d(v_i)(d(v_i)-1)}{2!}$$

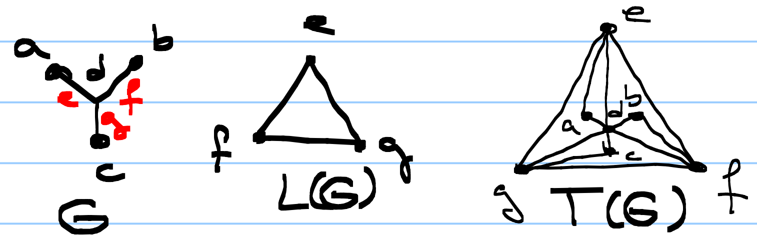
$$= \frac{1}{2}\sum_{i=1}^{n} d^2(v_i) - d(v_i)$$

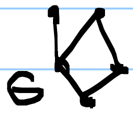
$$= \frac{1}{2}\sum_{i=1}^{n} d^2(v_i) - \underbrace{\frac{1}{2}\sum_{i=1}^{n} d(v_i)}_{=2e}$$

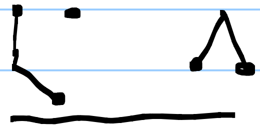
$$= \frac{1}{2}\sum_{i=1}^{n} d^2(v_i) - e. \blacksquare$$

Total Graph: The total graph of a graph G is the graph $T(G)$, whose vertices correspond to the edges and vertices of G and two vertices of which are joined if and only if the corresponding vertices or edges of G are adjacent or incident.

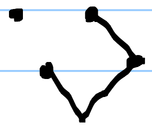
Ex



Subgraph: If G and H are graphs with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ we say that H is a subgraph of G, denoted $H \subseteq G$. A subgraph H of G has $V(H) = V(G)$ then H is called spanning subgraph of G. <u>Examples:</u> 



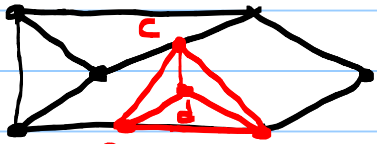
subgraphs



Spanning subgraph

A subgraph H of G such that two vertices of H are adjacent whenever they are adjacent in G is called an induced subgraph. If H is an induced subgraph of G with vertex set W, we say that H is a subgraph of G induced by W.

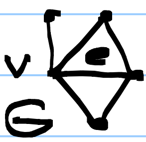
<u>Example:</u> G:

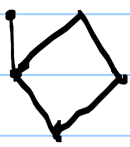
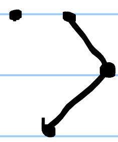


$H \cong K_4$ $\quad W = \{a, b, c, d\}$

A clique in a graph G is a complete subgraph of G. A clique K_3 is also called a triangle.

<u>Edge or Vertex Deletion:</u> Given a graph G. By removing an edge e of G we obtain the edge deleted graph $G-e$. For a vertex v of G, the vertex deleted subgraph is obtained from G by deleting the vertex with all the edges incident to v.

<u>Example:</u> 

$G \setminus e$:  $G \setminus v$: 

<u>Contraction:</u> Removing an edge $e = uv$ of a graph G and identifying the vertices u and v, is called contracting the edge e. The resulting graph

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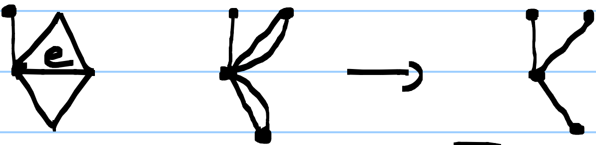
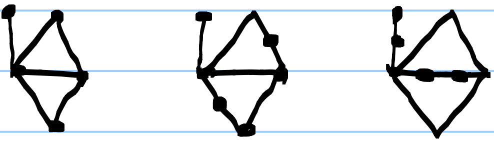
is denoted as $G \setminus e$. If a graph H is obtained from a graph G by contracting some edges, then we say that G can be contracted to H.

Example: 

G $\qquad$ $G \setminus e$ $\qquad$ The underlying simple graph of the resulting contracted graph

Homeomorphic Graphs: Replacing an edge $e = uv$ by two new edges uw and vw, where w is a new vertex is called inserting a vertex into an edge. If two graphs can be obtained from the same graph by inserting vertices into its edges, they are homeomorphic.

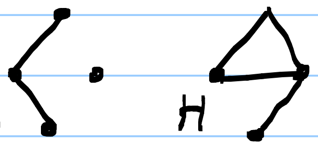
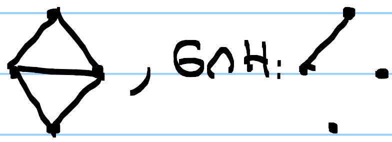
Example: 

Homeomorphic Graphs

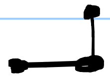
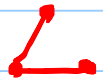
V. Binary Graph Operations:

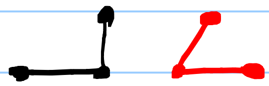
If G and H are graphs with $V(G) = V(H)$, the intersection and union of G and H, are $G \cap H$ and $G \cup H$, respectively, which are defined by

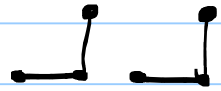
$$V(G \cap H) = V(G \cup H) = V(G) = V(H)$$
$$E(G \cap H) = E(G) \cap E(H) \text{ and}$$
$$E(G \cup H) = E(G) \cup E(H).$$

Example:  G $\quad$ H , $G \cup H$:  , $G \cap H$:

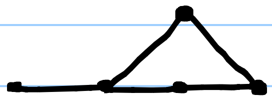
If G and H are graphs with $V(G) \cap V(H) = \emptyset$, then their disjoint union G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The disjoint union of k-copies of G is denoted by kG.

Ex: G:  , H: 

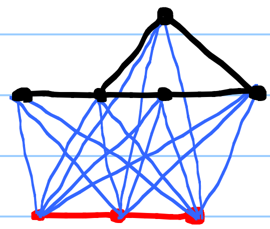
$G \cup H$: 

$2G$: 

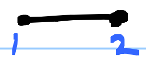
The join (or sum) $G+H$ of G and H is obtained from disjoint union of G and H by joining each vertex of G with each vertex of H.

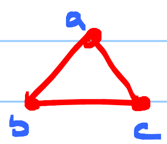
Example: G: 

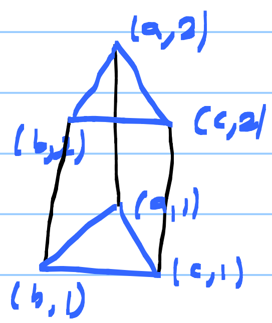
H: 

$G+H$: 

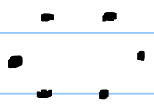
The Cartesian product $G \times H$ of G and H is defined by setting $V(G \times H) = V(G) \times V(H)$ and $(u,v) \sim (u',v')$ if and only if $[u = u'$ and $v \sim v']$ or $[v = v'$ and $u \sim u']$.

Example: G: 

H: 

$H \times G$ 

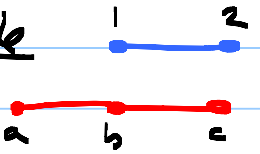
Example: $O_k \times G = kG$



The composition of G and H is the graph $G[H]$ defined as $V(G[H]) = V(G) \times V(H)$ and $(u,v) \sim (u',v')$ if and only if $[u \sim u']$ or $[u = u'$ and $v \sim v']$.

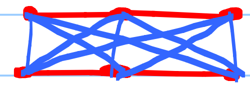
Note that in general $G[H] \neq H[G]$.

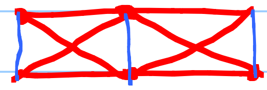
<u>Example</u>



P_2

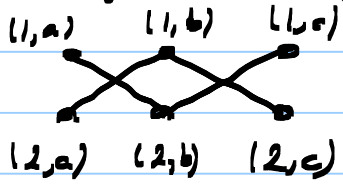
P_3

$P_2[P_3]$ 

$P_3[P_2]$ 

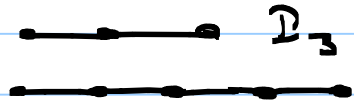
The conjunction of G and H is the graph $G \wedge H$ defined as $V(G \wedge H) = V(G) \times V(H)$ and $(u,v) \sim (u',v')$ if and only if $[u \sim u']$ and $[v \sim v']$.

$P_2 \wedge P_3$



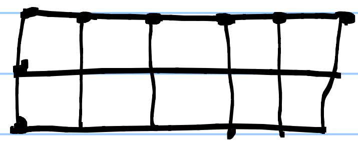
<u>Grid Graphs:</u> $P_n \times P_m = L_{m,n}$

$L_{3,5} = P_3 \times P_5$

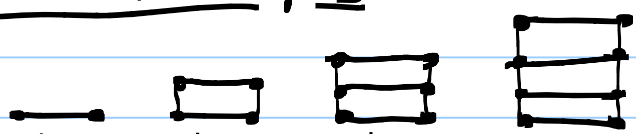


P_3

P_5



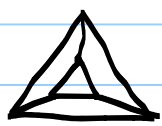
<u>Ladder Graphs</u>

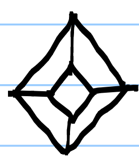


L_1 L_2 L_3 L_4

$L_{2,n} = P_2 \times P_n$

<u>Prism Graph:</u> $Y_n = C_n \times P_2$

$Y_3 = C_3 \times P_2$ 

$Y_4 = C_4 \times P_2$ 

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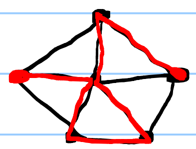
$Y_{m,n} \doteq C_m \times P_n$ Stacked prism graph.

VI. Walks, Trails, Paths and Cycles

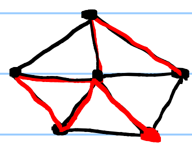
In a graph G, a sequence of vertices $v_{i_1}, \dots, v_{i_k}$ such that v_{i_j} and $v_{i_{j+1}}$ are adjacent for $j \geq 1$, $k \geq 1$, is called a walk of length k, from v_{i_1} to v_{i_k}.

A walk consisting of a single vertex is called a trivial walk. A walk is called a trail if no edge is repeated. A walk is called a path if no vertex is repeated. If the length of the walk is even (resp. odd), it is called an even (resp. odd) walk.

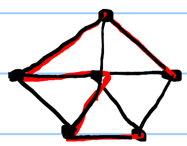
If $v_{i_1} = v_{i_k}$ then the walk $v_{i_1}, \dots, v_{i_k}$ is called closed, otherwise it is called open. A closed walk is also called a tour. A closed trail is called a circuit. Although in a path, repetition of vertices is not allowed, by letting $v_{i_1} = v_{i_k}$ exceptionally we obtain a closed path, which is also called a cycle.



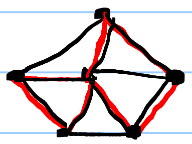
Walk



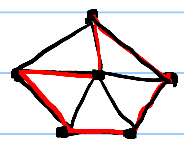
Trail



Path



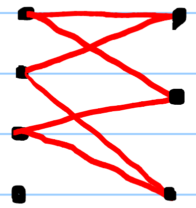
Circuit
=
Closed trail



Cycle
=
Closed Path

Theorem: A graph is bipartite if and only if it contains no odd cycles.

Example:



Proof: First assume that G is bipartite with (V_1, V_2) being the partition of the vertex set.

$$V(G) = V_1 \cup V_2, \quad V_1 \cap V_2 = \phi.$$

Let $\sigma = v_1 v_2 \cdots v_k v_1$ be a cycle. If $v_1 \in V_1$, then $v_2 \in V_2$. Similarly, $v_3 \in V_1$ and $v_4 \in V_2$ and so on. Since the last vertex $v_1 \in V_1$, $v_k \in V_2$. Hence, we see that k is an even integer. In particular, the number of edges of σ is k, an even integer.

Conversely, assume that G has no odd cycles.
<u>must show:</u> G is bipartite.

First assume that G is connected (any two vertices of G is connected by a path). Take any vertex v in G. Let

$V_1 = \{u \in G \mid$ there is an odd path $u = v_1 v_2 v_3 \cdots v_{2k-1} = v$, for some vertices$\}$,

and $V_2 = \{u \in G \mid$ there is an even path $u = v_1 v_2 \cdots v_{2k} = v$, for some vertices$\}$.

Since G is connected $V(G) = V_1 \cup V_2$, because any vertex u of G is connected to v via a path. Also note that $v \in V_1$.

<u>Claim:</u> No two vertices of V_1 (resp. V_2) are adjacent.

<u>Proof.</u> Assume on the contrary that two vertices u_1 and u_2 of V_1 are adjacent. Since $u_1, u_2 \in V_1$

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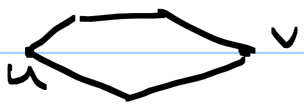
there are paths in G, $u_1 = v_1 v_2 \cdots v_{2k-1} = v$ and $u_2 = w_1 w_2 \cdots w_{2l-1} = v$. Then
$v = w_{2l-1} \cdots w_2 w_1 = u_2 u = v_1 v_2 \cdots v_{2k-1} = v$ path joining v to itself, so a cycle of length $2l + 2k - 2 - 1 = 2(l+k) - 3$, so that is an odd cycle, a contradiction.

Similar proof works also for V_2.

<u>Claim</u> $V_1 \cap V_2 = \phi$



<u>Proof</u> If $u \in V_1 \cap V_2$ then there are paths $u = v_1 v_2 \cdots v_{2k-1} = v$ and $u = w_1 w_2 \cdots w_{2l} = v$. Then again $u = v_1 v_2 \cdots v_{2k-1} = v = w_{2l} w_{2l-1} \cdots w_2 w_1 = u$ is again an odd cycle, a contradiction.

These two claims finish the proof for connected graphs. If G has more than one "component" then the result implies that each component of G is bipartite: $G = G_1 \cup G_2 \cup \cdots \cup G_l$. It is easy to see that G is bipartite since each G_i is.

→ To finish the proof we must take care the following potential problem: In both claims we form odd cycles by adding two path. However, if the paths have a common vertex except the end points, the walk we form by adding these two paths would be a cycle. For example, say the two path in the first claim have a common vertex as follows:

$u_1 = v_1 v_2 \cdots v_{j_1} \cdots v_{2k-1} = v$ and $u_2 = w_1 \cdots w_{j_2} \cdots w_{2l-1} = v$, where $v_{j_1} = w_{j_2}$. Then we have two closed walks:

$v_{j_1} v_{j_1+1} \cdots v_{2k-1} = v = w_{2l-1} \cdots w_{j_2} = v_{j_1}$ and
$v_{j_1} \cdots v_2 v_1 = u_1 u_2 = w_1 w_2 \cdots w_{j_2} = v_{j_1}$.

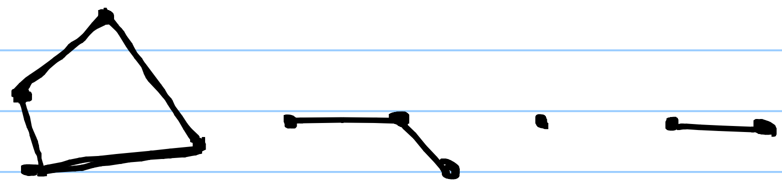
$v_1\ v_2 \quad v_5$

The first one has
$((2k-1)-j_1+1)+((2\ell-1)-j_2+1)-2 = 2(k_1+k_2)-(j_1+j_2)$
and the second one has j_1+j_2-1 vertices.
Their sum is $2(k_1+k_2)-1$, an odd integer. Hence, one of them is an odd closed walk. Continuing this we'll end up with an odd cycle, a contradiction.
This finishes the proof. ∎

VII. Connectedness

Let G be a graph. If for any pair of vertices of G there is a path joining these vertices then we say that G is connected. In a graph G a maximal connected subgraph is called a (connected) component of G. The number of connected components of G will be denoted as $c(G)$.

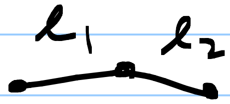
Example: G:



G has 4 connected components; $c(G)=4$

Theorem: If $e < n-1$ for a graph G, then G is disconnected, where $e=|E(G)|$, $n=|V(G)|$.

Proof: Let $\ell_1, \dots, \ell_e$ be the list of all edges of G. Each ℓ_i has two ends. Assume on the contrary that G connected. Then ℓ_1 has a common edge with one of the remaining edges, say ℓ_2.



Then l_1 and l_2 has three vertices. Now w.l.o.g. l_3 has a common vertex with either l_1 or l_2. So l_1, l_2, l_3 has at most 4 vertices. By induction we see that $l_1, \ldots, l_e$ has all together at most $e+1$ vertices. So $e+1 \geq n$, which contradicts with our assumption that $e+1 < n$. Hence, G cannot be connected.

<u>Theorem</u> If $e > \frac{(n-1)(n-2)}{2}$, then G is connected.

<u>Proof</u>: A connected graph with k vertices can have at most $\binom{k}{2} = \frac{k(k-1)}{2}$ edges, because between any two vertices there can be at most one edge. So if G is disconnected, where each connected component having $k<n$ vertices then G has at most

$$\binom{k}{2} + \binom{n-k}{2} \text{ edges.}$$

So

$$e \leq \binom{k}{2} + \binom{n-k}{2} = \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2}$$

$$= \frac{k^2 - \cancel{k} + n^2 - 2nk + k^2 - n + \cancel{k}}{2}$$

$$= \frac{n(n-1)}{2} - k(n-k).$$

By assumption $\frac{(n-1)(n-2)}{2} < e \leq \frac{n(n-1)}{2} - k(n-k)$

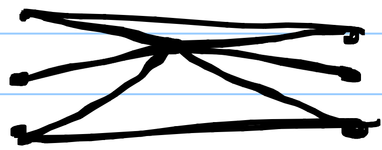
$\Rightarrow 0 < (n-1) - k(n-k) \Rightarrow n-1 > k(n-k)$ for some $k = 1, \ldots, n-1$. This is a contradiction since

has the smallest area among all rectangles with circumference $2n$.

Definition: A graph with $n = |V(G)| > k+1$ is said to be k-connected if any pair of vertices are connected by at least k pairwise vertex disjoint paths. The connectivity of G, denoted $\kappa(G)$, is defined to be the largest value of k for which G is k-connected.

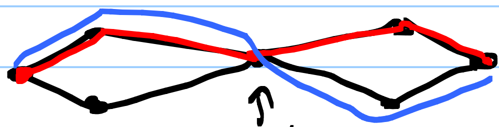
A graph G is k-edge connected if every pair of vertices are connected by at least k-edge disjoint paths. The largest value of k for which G is k-edge connected is denoted by $\lambda(G)$ and is called the edge-connectivity of G. From the definitions it follows that $\lambda(G) \geq \kappa(G)$.

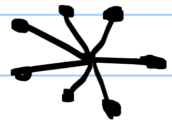
Example:

G 

$\kappa(G) = 1$

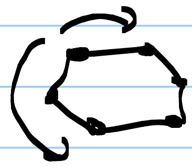
$\lambda(G)$



Example: $K_{1,q}$, $\kappa = \lambda = 1$ 

$K_{p,q}$, $\kappa = \lambda = \min\{p,q\}$ $(p,q \geq 2)$

K_n, $\kappa = \lambda = n-1$

C_n : $\kappa = \lambda = 2$ $(n \geq 2)$ 

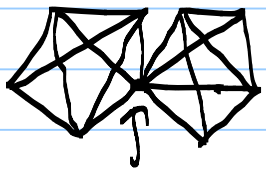
W_n : $\kappa = \lambda = 3$

Theorem (Menger's Theorem)
Let G be a connected graph with $n \geq k+1$. Then
a) G is k-connected if and only if G cannot be made disconnected by the removal of $k-1$ or fewer vertices.
b) G is k-edge connected if and only if G cannot be disconnected by the removal of $k-1$ or fewer edges.

In a connected graph if the removal of a vertex makes G disconnected, v is called a cut point (or cut vertex). In general, a vertex v of a graph G is called a cut point of G if the number of components of $G \setminus v$ is larger than the number of components of G. If G is connected and $G \setminus \{v_1, \ldots, v_k\}$ is disconnected, then the set $\{v_1, \ldots, v_k\}$ is called a separating set of vertices.

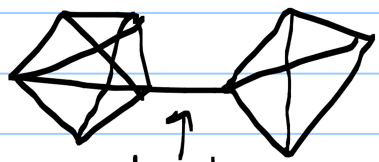
A 2-connected non-trivial graph is called non-separable. In other words, a graph is non-separable if it has no cut points. A block of a graph is a maximal non-separable subgraph. Then, it follows from Menger's Theorem that, connectivity $\kappa(G)$ of G is the minimum number of vertices whose removal disconnects G (or reduces G to a single vertex).

Similarly, if for an edge e, the number of components of $G \setminus e$ larger than the number of components of G, e is called a bridge of G. Again it follows from the Menger's Theorem that, edge connectivity $\lambda(G)$ of G is the minimum number of edges whose removal disconnects G.



cut point

$\kappa(G) = 1$



bridge

$\lambda(G) = 1 = \kappa(G)$

A directed graph is called strongly regular if from each vertex there is some directed path to any other vertex.

Theorem If H is spanning subgraph of G, then
$$\kappa(H) \le \kappa(G).$$

$(\ V(H) = V(G)$ and $E(H) \subseteq E(G))$

Theorem (Whitney's Inequality) For any graph G,
$$\kappa(G) \le \lambda(G) \le \delta(G)$$

Theorem If $\delta(G) \ge [n/2]$, then $\lambda(G) = \delta(G)$, where n is the number of vertices of G.

Theorem: If $\delta(G) \ge n-2$, then $\kappa(G) = \delta(G)$.

Theorem: For any graph G with order $n \ge 2$,

1) $1 \le \lambda(G) + \lambda(\overline{G}) \le n-1$ $\quad \left(\begin{array}{l} E(G) \cup E(\overline{G}) = E(K_n) \\ E(G) \cap E(\overline{G}) = \emptyset \end{array} \right)$

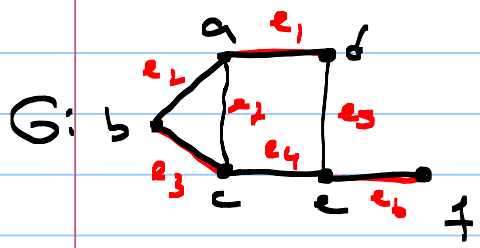
2) $0 \le \lambda(G)\lambda(\overline{G}) \le M(n)$, where
$$M(n) = \begin{cases} \left[\frac{n-1}{2}\right]\left[\frac{n-1}{2}\right] & \text{if } n \equiv 0,1,2 \pmod 4 \\ \frac{n-3}{2} \cdot \frac{n+1}{2} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Video 11

VIII. Matchings

A set of vertices no two adjacent in a graph G is called independent set of vertices.
The cardinality of a largest independent set of vertices is called the independence number of the graph, denoted by $\alpha_0(G)$ or just α_0.

Example: Let the vertices of a graph G represent 6 students and let an edge of G stand for the friendship relation. An independent set of vertices of this graph is a subset of the students that no one knows the others.



$\{a, c\}$ is not independent, but $\{a, e\}$ and $\{c, d, f\}$ are independent sets.

$\alpha_0(G) = 3$.

Example: Now suppose that the six students of previous example are to be grouped in six pairs.
Naturally, none of them would like to be paired with another who is not known by her. Say, if we have the pairing (a,c) and (e,f), the students b and d are to stay at home. If we wish a complete pairing, we may consider (a,d), (b,c) and (e,f).

Note that having a pairing of adjacent vertices is equivalent to having a set of edges no two adjacent.
For example, in the above example the pairing $\{(a,c), (e,f)\}$ corresponds to the edge set $\{e_6, e_7\}$ and the pairing $\{(a,d), (b,c), (e,f)\}$ corresponds to

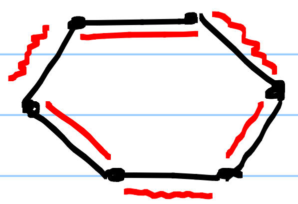
$\{e_1, e_3, e_7\}$.

For a graph G, a set M of edges no two adjacent is called a matching or an independent set of edges. The two ends of an edge $e \in M$ are said to be matched under M.

A matching M is called a maximal matching if $|M| \geq |M'|$ for any matching M'.

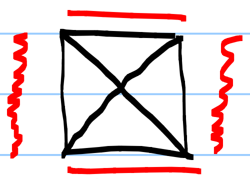
A matching M is said to saturate a vertex $v \in G$ if v is an end vertex of some edge in M. Also, M saturates $A \subseteq V(G)$ if it saturates all vertices in A. A matching M that saturates $V(G)$ then M is called a perfect matching. Clearly, every perfect matching is a maximum matching. The size of a maximum matching is called the edge-independence number of G.

For any even n, the complete graph K_n, the edge graph C_n, and the path graph P_n, all have perfect matchings.



C_n

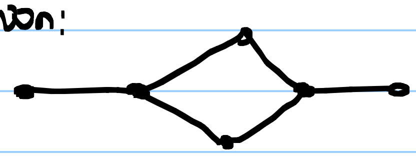
P_n



K_4

<u>Remark</u>: Clearly, a graph that has a perfect matching must have even number of vertices.

The graph below shows that this is **not** a sufficient condition:



This graph has 6 vertices and has no perfect matching.

It is clear that if a graph has a component with an odd number of vertices then it cannot have a perfect matching. A component with odd number of vertices is called odd, and the number of odd components of a graph G is denoted as $c_{odd}(G)$.

Theorem: (Tutte–Berge Formula)
The number of edges in any maximal matching of a graph G is

$$m(G) = \frac{1}{2}|V(G)| + \frac{1}{2}\min_{S \subseteq V(G)} (|S| - c_{odd}(G \setminus S))$$

Remark: If a graph G has perfect matching then $m(G) = \frac{1}{2}|V(G)|$. It follows from the above theorem that if G has perfect matching then

$$\min_{S \subseteq V(G)} (|S| - c_{odd}(G - S)) = 0.$$ Hence, we have

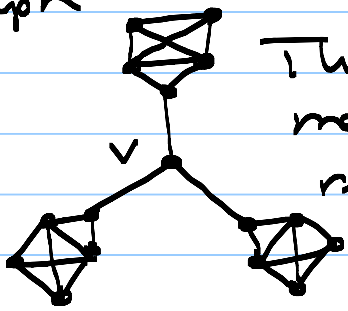
Theorem (Tutte – 1947)

A non trivial graph G has a perfect matching if and only if it has even number of vertices and there is no set S of vertices such that the number of odd components of $G-S$ exceeds $|S|$.

Example: The smallest regular graph with odd degree

Video 12

that has no perfect matching is the Sylvester graph



This graph has no perfect matching because, when we remove the central vertex the remaining graph has three connected components ($3 \geq 1$).

$S = \{v\}$

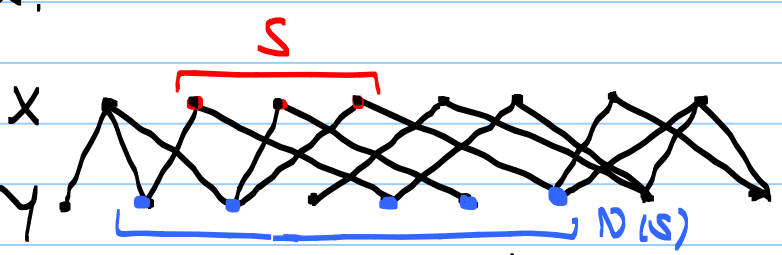
The matching of bipartite graph are particularly important. For a subset $S \subseteq V(G)$ of a graph G, define

$$N_G(S) = \{v \mid uv \in E(G), \text{ for some } u \in S\}.$$

Note that if G is bipartite, and (X, Y) is the corresponding partition of vertices ($V(G) = X \cup Y$) then $N_G(S) \subseteq Y$ for any subset $S \subseteq X$.

<u>Theorem</u> (Hall, 1935)
Let G be a bipartite graph with the corresponding partition (X, Y). Then G contains a matching M saturating X if and only if $|S| \leq |N_G(S)|$, for all $S \subseteq X$.

<u>Example:</u>

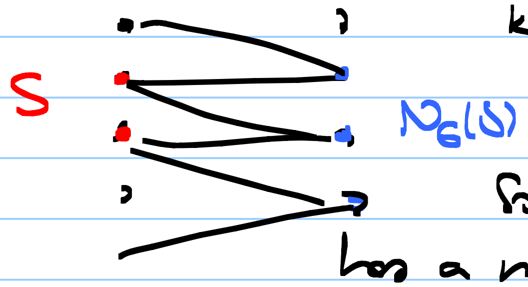


Note that for any $S \subseteq X$ we have $|S| \leq |N_G(S)|$.
Thus G has a matching M saturating X.

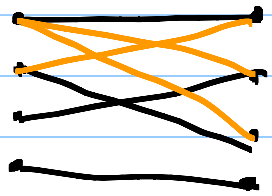
Corollary (Frobenius, 1917)
If G is a k-regular bipartite graph with $k>0$ then G has a perfect matching.

Proof: Let (X,Y) be the partition for the k-regular bipartite graph G. Since G is k-regular $k|X| = e = k|Y|$, where $e = |E(G)|$. Hence $|X| = |Y|$ since $k>0$. Let $S \subseteq X$ and denote the set of edges with an end in S by E_1 and denote by E_2 the set of edges with an end in $N_G(S)$. Clearly, $E_1 \subseteq E_2$ and hence

$$k|N_G(S)| = |E_2| \geq |E_1| = k|S|$$
$$\Rightarrow |N_G(S)| \geq |S|.$$



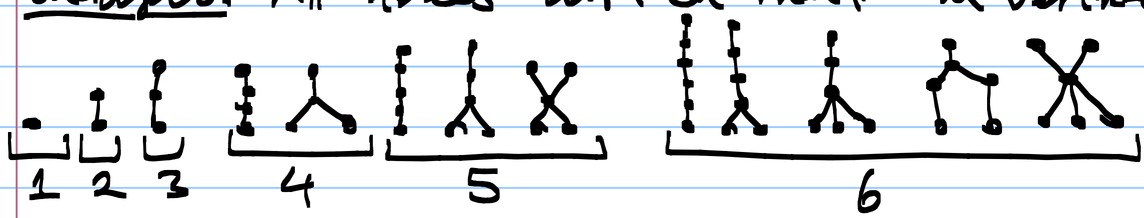
So by Hall's Theorem G has a matching M saturating X. Since $|X| = |Y|$ this matching is perfect.



∎

IX. TREES: A graph which has no cycles is called acyclic. A tree is a connected acyclic graph. A disjoint union of trees is called a forest.

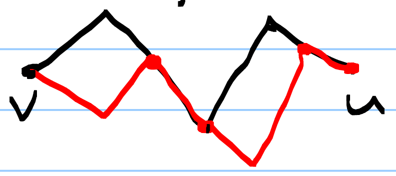
Examples: All trees with at most six vertices.



Video 13

<u>Theorem:</u> A graph is a tree if and only if there is exactly one path between every pair of vertices.

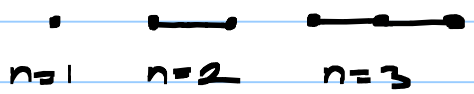
<u>Proof:</u> ($\Rightarrow$) Assume that T is tree. By definition any tree is connected and thus between any two vertices of T there is a path joining these vertices. Suppose on the contrary that two vertices, say v and u are connected by two distinct paths. The composition of these two distinct paths will contain a cycle, which is a contradiction.



Conversely, if T is a graph so that any two vertices of T are connected by a unique path then clearly T is connected. If T had a cycle, say $v_1 v_2 \cdots v_k v_{k+1} \cdots v_1$. Then the vertices v_1 and v_2 are joined by two distinct path $v_1 v_2$ and $v_2 \cdots v_k \cdots v_1$, which is a contradiction. Hence, T has no cycles so that T is a tree. ∎

<u>Theorem:</u> The number of edges of a tree with n vertices is $n-1$.

<u>Proof:</u> Induction on n. The result clearly holds for $n=1, 2$ or 3.



Now let's assume that the results holds for some $n \geq 3$. So any tree with n vertices has $n-1$ edges. Let T be a tree with $n+1$ vertices.

<u>must show:</u> T has n edges.

Take any edge e of T joining the vertices

v_1 and v_2. By the previous result the path $v_1v_2 = e$ is the only path joining v_1 and v_2. In particular, $T \setminus e$ is not connected.

<u>Claim</u>: $T\setminus e$ has two components.

<u>Proof</u>: We know that $T\setminus e$ is not connected, and v_1 and v_2 are contained in different components of $T\setminus e$. Say T_i is the component containing v_i, $i=1,2$.

Let $v \in T$ be any vertex with $v \neq v_i$, $i=1,2$. Then there is a unique path joining v to v_1, say $vu_1u_2 \cdots u_kv_1$ in T. We have two cases:

<u>Case 1</u> $u_j \neq v_2$ for all $j=1,\dots,k$. Then e is not contained in this path and thus this path lies in $T\setminus e$. Since v_1 is an end point of this path this path lies inside the connected component T_1. In particular, $v \in T_1$.

<u>Case 2</u> $u_j = v_2$ for some $j=1,\dots,k$. So we have

$$vu_1 \cdots u_{j-1} \overset{v_2}{\overset{\shortparallel}{u_j}} \underline{u_{j+1} \cdots u_k v_1}.$$ This would indeed imply that $j=k$:

$$vu_1 \cdots u_k = v_2 v_1.$$

In particular, we have $vu_1 \cdots u_{j-1}u_j = v_2$ and path does not contain the edge e. Since this path lies in $T\setminus e$ v lies in the connected component T_2.

So, $T\setminus e \subseteq T_1 \cup T_2 \subseteq T\setminus e \Rightarrow T\setminus e = T_1 \cup T_2$. Hence, $T\setminus e$ has only two components. ∎

Note that $|V(T_1)|+|V(T_2)|=|V(T)|$

So, $|E(T)|=|E(T\setminus e)|+1$
$$=|E(T_1)\cup E(T_2)|+1$$
$$=|E(T_1)|+|E(T_2)|+1$$
$$=(|V(T_1)|-1)+(|V(T_2)|-1)+1$$
$$=|V(T_1)|+|V(T_2)|-1$$
$$=|V(T)|-1.$$

This finishes the proof. $\blacksquare$

Theorem. A connected graph with n vertices and $n-1$ edges is a tree.

Proof: Let T be a connected graph with n vertices and $n-1$ edges. Suppose that T contains a cycle $v_1v_2\cdots v_k=v_1$, where $v_i\neq v_j$ if $1\leq i<j\leq k-1$. Note that $k\geq 3$. Let $e=v_2v_3$. Then $T\setminus e$ is still connected. Note that
$|V(T\setminus e)|=|V(T)|$ but $|E(T\setminus e)|=|E(T)|-1=n-2$.
Keep doing this until $T\setminus\{e_1,\ldots,e_l\}$ does not contain any cycle. So $T\setminus\{e_1,\ldots,e_l\}$ is a tree with n vertices and $n-l-1$ edges. By the previous Theorem $n-l-1=n-1$, so that $l=0$. Thus T were a tree at the beginning. $\blacksquare$

Theorem Each edge of a tree is a bridge.

Theorem: A graph G with n vertices and $n-1$ edges having no cycles is connected.

Video 14

Proof: Let $G = T_1 \cup T_2 \cup \cdots \cup T_k$, where each T_i is a connected component, with $T_i \neq T_j$ for $i \neq j$. Must show: $k=1$. Since T_i's are disjoint

$$|V(G)| = \sum_{i=1}^{k} |V(T_i)| \text{ and } |E(G)| = \sum_{i=1}^{k} |E(T_i)|.$$

Since each T_i is connected and has no cycle (because T has no cycles) each T_i is tree. Then $|V(T_i)| = |E(T_i)| + 1$, for each $i = 1, \ldots, k$.

$$\begin{aligned} \text{So, } n = |V(G)| &= \sum_{i=1}^{k} |V(T_i)| \\ &= \sum_{i=1}^{k} \underbrace{|E(T_i)| + 1} \\ &= |E(G)| + k \\ &= n - 1 + k. \end{aligned}$$

Thus $k=1$ so that $G = T_1$. This finishes the proof. ∎

Theorem: Any tree with at least two vertices has at least two leaves (end vertices).

Proof: Suppose that T has $n > 1$ vertices, say $v_1, v_2, \ldots, v_n$. Then T has $n-1$ edges.

We know that the sum of valencies is $\sum_{i=1}^{n} d(v_i) = 2e = 2(n-1)$. Since T is connected $d(v_i) \geq 1$. For any vertex that is <u>not</u> an end $d(v_i) \geq 2$. Hence, $d(v_i) = 1$ for at least two values of i. Thus T has at least two ends. ∎

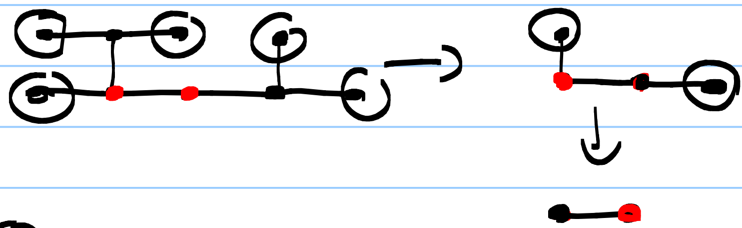
Recall that in a graph G, a vertex with minimum eccentricity is called a center.

<u>Theorem:</u> Every tree has either one or two adjacent centers.

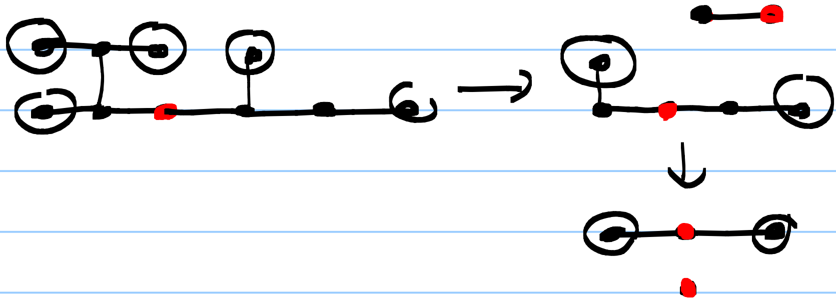
<u>Proof.</u> Let u be a fixed vertex of G. Considered as a function of v, $d(u,v)$ has a maximum at some vertex v only if v is leaf; i.e., $d(v)=1$.

Hence, deleting all the leaf vertices of a tree T we obtain another tree T_1 so that the eccentricities of the remaining vertices are all decreased by one, provided that T_1 is non empty. In particular, centers of T are still centers of T_1. Repeating this process we end up finally either with one vertex or an edge, whose end points are the two centers of T. ∎

<u>Examples</u> 1)



2)



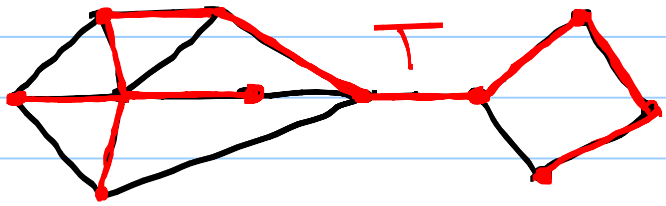
Trees with center K_1 are called unicentral (or central) and trees with center K_2 are called bicentral trees.

A tree T is said to be a spanning tree of a connected graph G, if T is a subgraph of G and T contains all vertices of T.

Video 15

Example

G



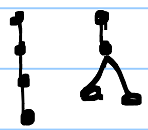
Theorem: Every connected graph has at least one spanning tree.

Proof: Let G be a connected graph. If G has no cycle then G is a tree, spanning tree for itself.

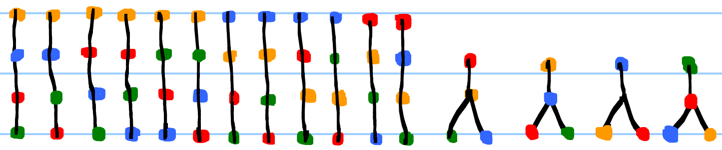
If G has a cycle then delete an edge of the cycle to obtain a connected subgraph G' with $V(G) = V(G')$. We repeat this process till we obtain a subgraph $\tilde{G}$ so that $V(G) = V(\tilde{G})$ and $\tilde{G}$ has no cycles. In particular, $\tilde{G}$ is a tree and hence a spanning subtree. ∎

Labelled and Unlabelled Trees:

Below you see all labelled and unlabelled trees with four vertices.



2 unlabelled trees with 4 vertices



16 labelled trees with 4 vertices

Here two labelled trees are regarded as different if there is no isomorphism mapping one labelling to the other.

In general the number of (non isomorphic) different unlabelled graphs of any order is not known.

Below is a list of known ones:

# of vertices	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
# of unlabelled trees	1	2	3	2	3	6	11	23	47	104	235	531	1301	7741	19320

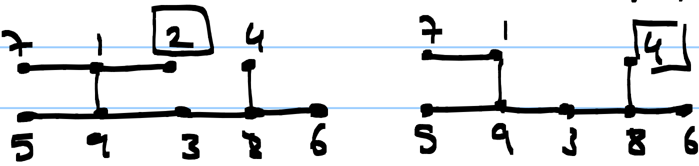
On the other hand the number labelled tree is known:

Theorem: There are n^{n-2} labelled trees with n vertices.

Proof: Let T be a tree with n vertices and the vertices be labelled as $1, 2, \dots, n$. Note that n^{n-2} is the number of sequences of length $n-2$ that can be formed from $N = \{1, 2, \dots, n\}$. Thus it is enough to establish a 1-1 correspondence between the set of labelled trees of length n and the set such sequences.

Let T be a labelled tree. Let $s_1 \in N$ be the smallest label of the end vertices of T and let t_1 be the label of the other vertex of the edge s_1 is incident. Delete from T the vertex with label s_1 and let s_2 be the label of the end vertex of $T \setminus s_1$ with smallest label. Again let t_2 be the vertex of $T \setminus s_1$ s_2 is adjacent to. We continue this process until we obtain t_{n-2} has been defined so that a tree with two vertices is left. So we have obtained the sequence $(t_1, t_2, \dots, t_{n-2})$.

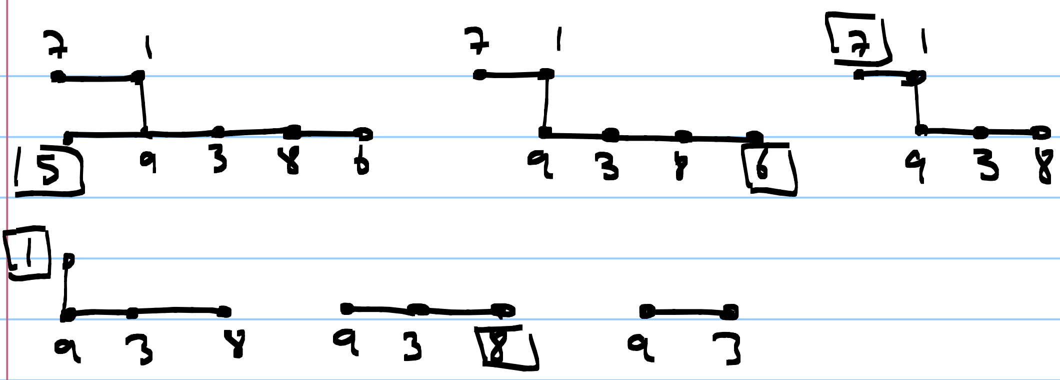
Example:



$s_1 = 2$ $s_2 = 4$ $s_3 = 5$ $s_4 = 6$ $s_5 = 7$ $s_6 = 1$ $s_7 = 8$

$t_1 = 1$ $t_2 = 8$ $t_3 = \underline{9}$ $t_4 = 8$ $t_5 = 1$ $t_6 = \underline{9}$ $t_7 = 3$

Video 16



So our sequence is $\mathcal{V}=(1,8,9,8,1,9,3)$.

Next we describe the reverse operation. Note that any vertex v of T occurs $d(v)-1$ times in the sequence $(t_1,\dots,t_{n-2})$. Let $(t_1,t_2,\dots,t_{n-2})$ be given. Let s_1 be the smallest labelled vertex in $N\setminus\{t_1,\dots,t_{n-2}\}$, where $N=\{1,2,\dots,n\}$. Then join s_1 to t_1. Let s_2 be the smallest vertex of $N\setminus\{s_1,t_2,\dots,t_{n-2}\}$ and join s_2 to t_2. Repeating this $n-2$ times we obtain $n-2$ edges $s_1t_1, s_2t_2, \dots, s_{n-2}t_{n-2}$. T is now obtained by adding the edge joining the remaining vertices of $N\setminus\{s_1,s_2,\dots,s_{n-2}\}$.

One needs to check that these two operations are inverses of each other. This finishes the proof. ∎

Back to the above example. Suppose we are given the sequence

$$\mathcal{V}=(1,8,9,8,1,9,3)=(t_1,t_2,t_3,t_4,t_5,t_6,t_7).$$

Then $N-\mathcal{V}=\{2,4,5,6,7\}$.

1) $s_1=2$, $t_1=1$ $\quad (s_1,t_1)=(2,1)$

2) Next, $s_2=\min N\setminus\{s_1,t_2,\dots,t_9\}$.
$=4$
$t_2=8$ so we have $(s_2,t_2)=(4,8)$

3) $s_3=\min N\setminus\{s_1,s_2,t_3,\dots,t_9\}$
$=\min N\setminus\{2,4,9,8,1,9,3\}=5$, $t_3=9$

$(s_3, t_3) = (5,9)$

4) $s_4 = \min N \setminus \{s_1, s_2, s_3, t_4, \dots, t_9\}$
$= \min N \setminus \{2, 4, 5, 8, 1, 9, 3\}$
$= 6$ and $t_4 = 8$ so that $(s_4, t_4) = (6,8)$

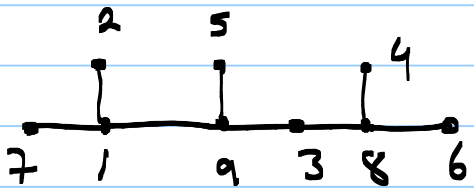
5) $s_5 = \min N \setminus \{s_1, s_2, s_3, s_4, t_5, \dots, t_9\}$
$= \min N \setminus \{2, 4, 5, 6, 1, 9, 3\}$
$= 7$ and $t_5 = 1$ so that $(s_5, t_6) = (7, 1)$.

6) $s_6 = \min N \setminus \{s_1, s_2, s_3, s_4, s_5, t_6, \dots, t_9\}$
$= \min N \setminus \{2, 4, 5, 6, 7, 9, 3\}$
$= 1$, $t_6 = 9$ so that $(s_6, t_6) = (1, 9)$.

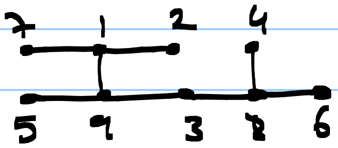
7) $s_7 = \min N \setminus \{2, 4, 5, 6, 7, 1, 3\}$
$= 8$ and $t_7 = 3$ so that $(s_7, t_7) = (8, 3)$.

8) Finally, $N \setminus \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\} = N \setminus \{2, 4, 5, 6, 7, 1, 8\}$
$= \{3, 9\}$

So we have the edges
$(2,1), (4,8), (5,9), (6,8), (7,1), (1,9), (8,3), (3,9)$



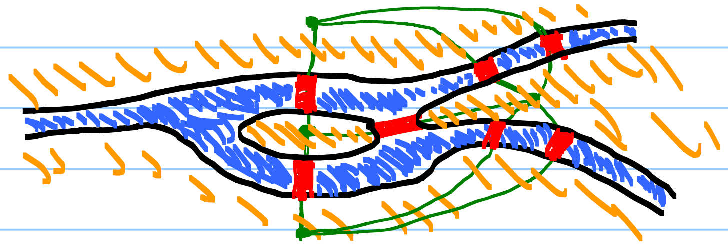
$\cong$



Video 17

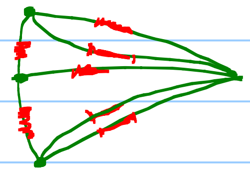
X. Eulerian Tours and Hamiltonian Cycles:

Seven bridges of Königsberg of Prussia (Kaliningrad, Russia)



Question: People asked if it is possible to walk around the city by crossing each bridge exactly once? The walk need not to start and ended the same point.

Euler solved this problem as follows: He first converted the problem to a problem about the "green graph" below. The problem is now to draw the graph without retracing any edge and without picking the pencil up off the paper. In terms of the terminology of graphs, to find a trail that contains all the edges of the graphs.



Note that all the vertices of the graph are odd (i.e., all valencies are odd). Pick any vertex, say one with degree 3. The first time we reach this vertex we need to leave the vertex by another edge. But the next time we arrive there is no edge by which we can leave. So this vertex cannot be an intermediate vertex; it can only be the first or the last vertex of the walk. Hence it is impossible to draw the above graph in one pencil stroke without retracing.

Eulerian Graph: A trail which contains all the edges of a graph is called a Eulerian trail and a closed such trail is called a Eulerian tour. A graph that admits an Eulerian tour is said to be traceable. Above discussions shows that for a graph to be traceable, two of the vertices must odd and all the other vertices must be even. For the graph to be Eulerian, all the vertices must be even. This necessary condition is also sufficient:

Theorem A connected graph is Eulerian if and only if every vertex is even.

Proof: Since the tour enters a vertex through some edge and leaves by another edge, necessity of the condition is clear. To show the sufficiency start with a vertex v and begin making a tour. Keep going never using the same edge twice until it is not possible to go further. Since every vertex is even the end point of the tour must be v. If all the edges are used, then the proof is completed. Otherwise, each connected component of the subgraph consisting of the unused edges is a graph whose all vertices are even. Apply the same procedure to each component to obtain a second tour. If this tour starts in a vertex of the first tour, the two tours can be combined to produce a new tour. Since the graph has finitely many edges the procedure must stop after finitely many steps. This finishes the proof.

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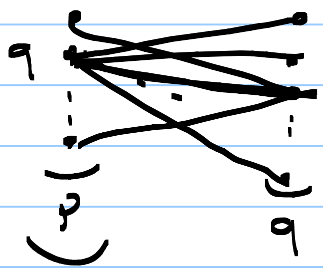
Corollary A connected graph is traceable if and only if it has exactly two odd vertices. In this case, the end points of the trail are the odd vertices.

Theorem: A connected digraph is Eulerian if and only if the invalency of each vertex is equal to the out valency.

Examples: 1) K_n is Eulerian if and only if n is odd ($n \geq 3$).

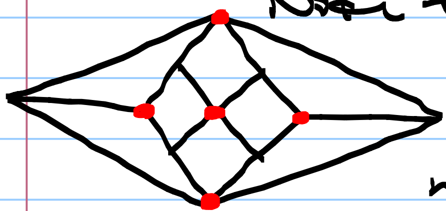
2) $K_{p,q}$ is Eulerian if and only if p and q are both even ($p, q \geq 2$)



Hamiltonian Graphs: A path is called Hamiltonian if it contains all the vertices. A cycle is called Hamiltonian if it contains all the vertices.

Remark: To check a graph is Hamiltonian or not is quite difficult compared to checking a graph is Eulerian. Only some necessary and sufficient conditions are known.

Example: The graph below is not Hamiltonian.



Note that this graph is bipartite having 5 red and 6 black vertices, so that it has odd number vertices.

We know that a bipartite graph does not admit a odd cycle and hence the graph above does not admit a cycle containing all the vertices.

In general, if $p+q$ is odd then $K_{p,q}$ is not Hamiltonian.

<u>Theorem:</u> If G is Hamiltonian then, for every nonempty proper subset $S \subseteq V(G)$
$$c(G-S) \leq |S|,$$ where $c(G-S)$ is the number of connected components of $G-S$.

<u>Proof:</u> Let $\mathcal{C}$ be a Hamiltonian cycle in G. Then for every nonempty proper subset S of $V(G)$ we have
$$c(\mathcal{C} \setminus S) \leq |S|.$$
However, $\mathcal{C}$ is a spanning subgraph of G, so $c(G-S) \leq c(\mathcal{C}-S)$. To see this take any component G_i of $G-S$. If $v \in G_i$ is a vertex then $v \in \mathcal{C} \setminus S$ also because $\mathcal{C}$ is spanning. Then there is a component C_i of $\mathcal{C} \setminus S$ containing v and thus contained in G_i. If G_i and G_j are different components of $G-S$ then C_i and C_j will also disjoint and thus distinct. In particular, there is an injection $G \setminus S \to \mathcal{C} \setminus S$ sending G_i to C_i so that $|G \setminus S| \leq |\mathcal{C} \setminus S|$.

Hence, $|G \setminus S| \leq |\mathcal{C} \setminus S| \leq |S|$ and this finishes the proof. $\square$

<u>Example:</u> The graph below is not Hamiltonian. Let S be the set of blue vertices of G, so that $|S| = 2$ and $c(G-S) = 3$. Note that this graph is bipartite and has even number of vertices.

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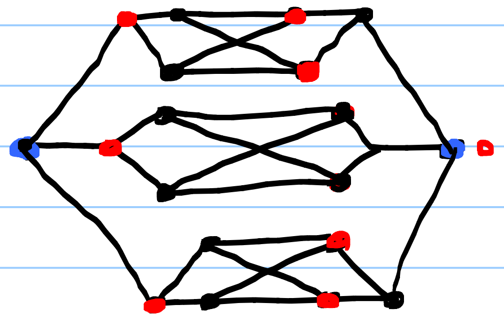
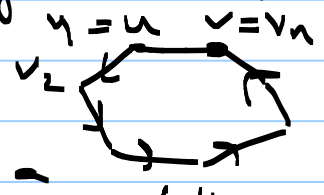


Theorem (Ore, 1962)
Let G be a simple connected graph with $n \geq 3$. For any pair of non adjacent vertices u and v if $d(u)+d(v) \geq n$, then G is Hamiltonian.

Proof Pick any two non adjacent vertices of G, and add a new edge between them. Keep on doing this until reaching a graph G^* that has a Hamiltonian cycle. This process must stop because eventually we'll reach the complete graph on n vertices K_n, which is obviously Hamiltonian. Let G' be the graph obtained immediately before G^*, and suppose uv is the edge added to G' to obtain G^*. Note that if we can prove that G' is Hamiltonian the proof will finish.

So $G^* = G' + uv$ has a Hamiltonian cycle say $\mathcal{C}$. If $e = uv \notin \mathcal{C}$, then $\mathcal{C}$ lies in G' and thus we are done. If $e = uv \in \mathcal{C}$ we may assume that $\mathcal{C}$ has the form



$\mathcal{C}: u \to \cdots\cdots \to v \to u$

Then $u = v_1 \to v_2 \to \cdots \to v_n = v$ is a Hamiltonian path in G'. Define the following sets:

$S = \{v_i \mid uv_{i+1} \in E(G')\}$ and $T = \{v_i \mid v_i v \in E(G')\}$.

Note that $v = v_n \notin S \cup T$, and thus $|S \cup T| \leq n-1$.

On the other hand

$$|S\cap T| = |S| + |T| - |S\cup T| = \underbrace{d(u)+d(v)}_{\geq n} - |S\cup T| \geq 1$$

so that there is a vertex v_i such that $uv_{i+1} \in E(G')$ and $v_i v \in E(G')$. But now

$$u \to v_{i+1} \to v_{i+2} \to \cdots \to v_{n-1} \to v_n = v \to v_i \to v_{i-1} \to \cdots \to v_1 = u$$

is a Hamiltonian cycle in G'.

This finishes the proof. ∎

Corollary (Dirac, 1952)

If G is a simple graph with $n \geq 3$ and $\delta > n/2$ then G is Hamiltonian.

Proof $\delta = \min\{d(v) \mid v \in V(G)\}$. Then by assumption for any two vertices $u, v \in G$, $d(u)+d(v) > n$. So by the above theorem G is Hamiltonian.

Corollary (Bondy and Chvátal, 1976)

Let G be a graph with $n \geq 3$. For any pair of vertices u and v with $d(u)+d(v) \geq n$, G is Hamiltonian if and only if $G+uv$ is Hamiltonian.

Proof If G is Hamiltonian then clearly $G+uv$ is Hamiltonian. For the other direction, just apply the argument in the proof of previous theorem taking $G^* = G+uv$ to deduce that $G' = G$ is Hamiltonian ∎

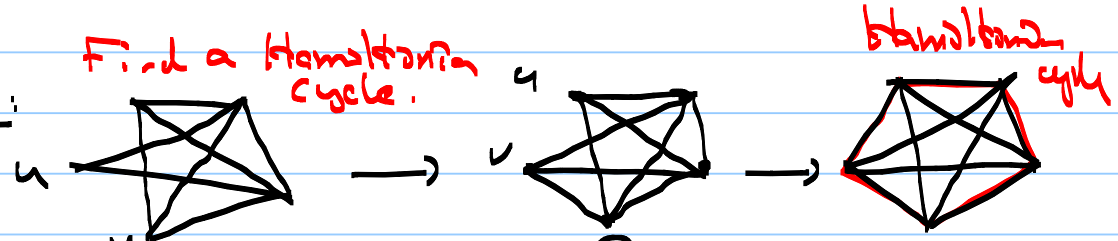
Definition (Closure)

For a given graph G with n vertices, define inductively a sequence $G_0, G_1, \ldots, G_k$ of graphs such that $G_0 = G$ and $G_{i+1} = G_i + uv$, for $i = 1, \ldots, k-1$, where u and v are nonadjacent vertices of G such that

$d(u)+d(v) \geq n$. This procedure stops when no new edges can be added to G_k for some k. The result of this procedure is called the closure of G, and it is denoted as $cl(G)$.

It is known that $cl(G)$ is uniquely determined and hence well-defined. A graph G with $G = cl(G)$ is called closed.

Ex: Find a Hamiltonian cycle.



$G = G_0$ $\qquad G_1 \qquad G_2 = K_5 = cl(G)$

$d(v) = 3$, $d(u)+d(v) = 6 \geq n = 5$

Theorem A simple graph G with $n \geq 3$ is Hamiltonian if and only if $cl(G)$ is Hamiltonian.

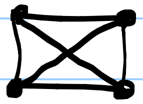
Proof Again the direction "$\Rightarrow$" is clear. Conversely, if $cl(G)$ is Hamiltonian then G is Hamiltonian since an edge uv is added only if $d(u)+d(v) \geq n$ (and u, v are nonadjacent). Thus removing the edge uv will not make the graph non-Hamiltonian. ■

Corollary Let G be a graph of order $n \geq 3$. If $cl(G)$ is the complete graph K_n, then G is Hamiltonian.

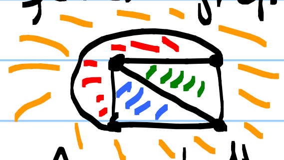
XI Planarity

A graph in the plane such that no two edges intersect (except at vertices) is said to be planar. Such a graph is also called a plane graph or an embedding of a graph into the plane.

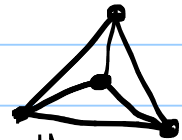
Example K_4 is a planar graph.



not an embedding of K_4



An embedding of K_4

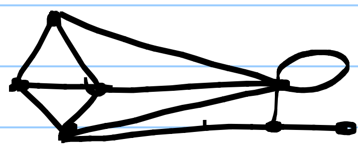


Another embedding of K_4.

Remark We'll see that K_5 and $K_{3,3}$ are not planar.

A planar graph partitions the plane into a number bounded regions together with an unbounded region. Each region is called a face of the plane graph. The unbounded region is called the exterior region and the others are called interior faces. The boundary of each face is a cycle. The set and the number of faces are denoted as $F(G)$ and f, respectively.

Example



A non simple planar graph with 7 vertices, 12 edges and 7 faces.

In the proof below we'll use so called Jordan Simple Closed Curve Theorem often.

Theorem (Jordan Simple Closed Curve Theorem)
Let $\alpha : [a,b] \longrightarrow \mathbb{R}^2$ be a simple closed curve (i.e. α is continuous, $\alpha(a) = \alpha(b)$ and $\alpha(t_1) = \alpha(t_2)$ with $t_1 \neq t_2$ then $\{t_1, t_2\} = \{a, b\}$)
Then $\mathbb{R}^2 \setminus \mathrm{Im}(\alpha)$ has two connected components. One of them is bounded and homeomorphic to D_2 (open disk) and the other one is unbounded

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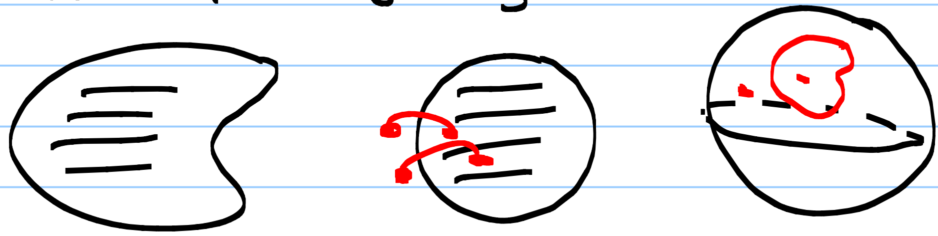
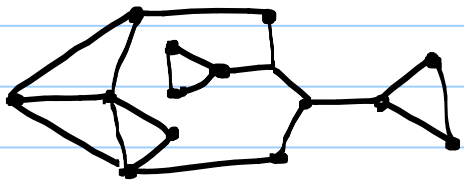
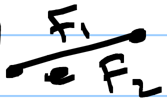
and homeomorphic to $\mathring{D}_2 \setminus \{0\}$, where
$\mathring{D}_2 = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2<1\}$.



Theorem: For a planar graph G,
a) Two distinct faces of G are disjoint, their boundaries can intersect only on edges or vertices.
b) G has a unique exterior face,
c) a bridge belongs to the boundary of one face,
d) any edge which is not a bridge belongs to the boundary of two faces,
e) each cycle of G surrounds at least one internal face.
f) G has no interior face if and only if it is acyclic.



Sketch of proof a) Faces are connected components of $\mathbb{R}^2\setminus G$ and thus they are disjoint.
b) Since G is compact it lies inside a ball and hence there is a unique unbounded face.
c) 

If an edge e belongs to the boundary of two faces F_1 and F_2 then one of them must be bounded, say F_1. Then the boundary of F_1 is a cycle. This implies that $G\setminus e$ is connected, which is a contradiction. Hence, the proof finishes.
d) Let e be an edge that is not a bridge.

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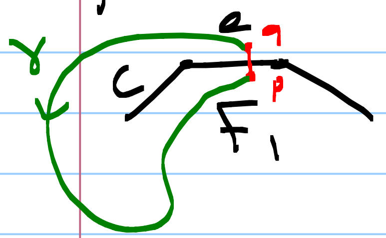
The e belongs to a cycle C, which bounds a face say F_1. By the Jordan Closed Curve Theorem $\mathbb{R}^2 \setminus C$ has two connected components, one bounded, the other unbounded, where the bounded one is a disc. Let's choose two points p and q as the picture below. If there is a continuous path γ joining p to q so that $\mathrm{Int}\,\gamma \cap C = \phi$



then the union of $\mathrm{Im}(\gamma)$ with the line segment $\bar{pq}$ is a closed curve intersecting C in exactly one point, which is not possible by "Homology Theory".
This implies that p and q are in different connected components of $\mathbb{R}^2 \setminus G$.

e) Each cycle is a simple closed curve and thus by the Jordan Closed Curve Theorem it bounds a bounded region.

f) This is left as an exercise!

<u>Corollary</u> If a planar graph has two disjoint faces with the same boundary, then the graph is a cycle.

<u>Theorem</u> (Euler's Polyhedral Formula)
Let G be a connected plane graph with v vertices, e edges and f faces. Then $v + f = e + 2$.

<u>Proof</u>: Induction on the number of faces f. If $f = 1$ then it is the unbounded face. In particular, G has no cycles. Since G is connected and has no cycles, G must be tree. Then $e = v - 1$ so that
$v + f = (e+1) + 1 = e + 2$, which gives the result for $f = 1$.
Now now that the theorem holds for all planar

graphs with less than f_0 $(f_0 \geq 2)$ faces. We must prove the result for $f = f_0$. So let G be a connected graph with f_0 faces. Pick one edge u of G, which is not a bridge, which exists since G has at least one cycle. Then $G' = G \setminus u$ is connected with $v' = v$ vertices and $e' = e - 1$ edges. Moreover since the edge u is not a bridge it belongs to the boundary of two distinct faces. Hence, $G \setminus u$ has $f_0 - 1$ faces. By induction hypothesis

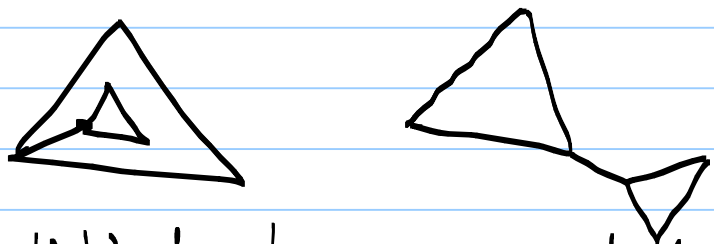
$$v' + f' = e' + 2 \text{ which implies}$$

$$v + f_0 - 1 = e - 1 + 2 \Rightarrow v + f_0 = e + 2,$$ which is what we wanted to prove. This finishes the proof. ∎

Let G be a connected planar graph on v vertices with e edges and let G_1 and G_2 be two plane representations of G with f_1 and f_2 faces, respectively. Then by the Euler's Polyhedral Theorem we have

$$v + f_1 = e + 2 = v + f_2$$ so that $f_1 = f_2$. This common number is called the number of faces of the planar graph.

Ex



Two distinct plane representations of the same planar graph. Both have 3 faces.

Corollary Let G be a graph with v vertices, e edges, f faces and c components. If G is planar then $v + f = e + c + 1$.

Proof: Let $G_1, \dots, G_c$ be the connected components of G. Let v_i, e_i and f_i denote the number of vertices, edges and faces of G_i. Then by Euler's formula

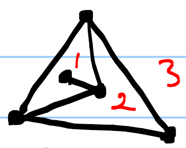
$v_i + f_i - 1 = e_i + 1$, $i = 1, \dots, n$, where $f_i - 1$ is the number interior faces of G_i. Then taking sum over i we get

$$\sum_{i=1}^{c} v_i + \sum_{i=1}^{c} (f_i - 1) = \sum_{i=1}^{c} e_i + \sum_{i=1}^{c} 1$$

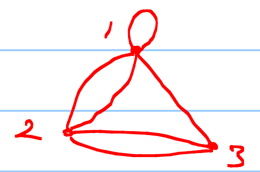
$$v + f - 1 = e + c$$ so that $v + f = e + c + 1$. ■

__Dual of a Plane Graph:__ Given a plane graph G, we define the dual graph G^* to be a new graph as follows. Each vertex of G^* corresponds to an edge of G, and each edge of G^* correspond to an edge of G. Two vertices of G^* are joined if and only if the corresponding faces are incident.

__Example:__

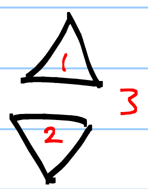


G

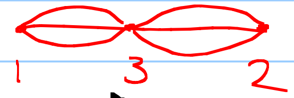


G^*

__Example__



G



G^*

__Remark:__ It is clear that the dual of a disconnected graph is connected.

Hence the dual of a dual of graph cannot be isomorphic to G provided that G is not connected.

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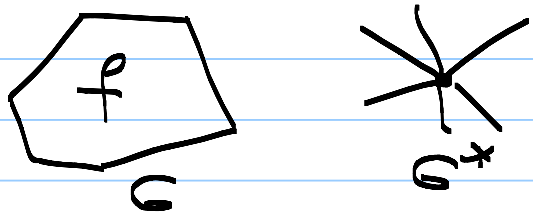
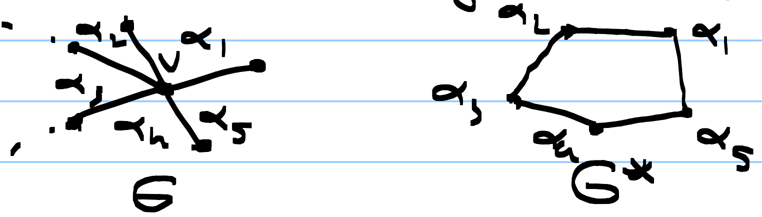
Proposition: A planar graph G is bipartite if and only if its dual G^* is Eulerian.

Proof: G is bipartite if and only if G has no odd cycles. Thus in any plane embedding of G, each face has an even degree which is equivalent to saying that each vertex of G^* has an even degree, which means that G^* is Eulerian. ∎

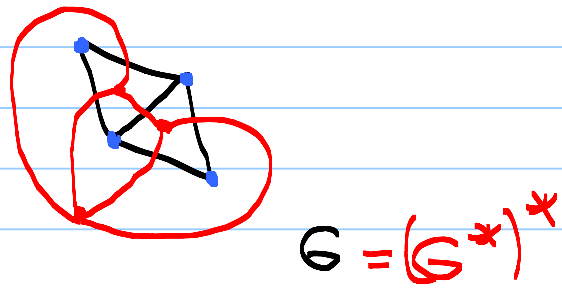


Theorem: If G is a connected planar graph, then
$$(G^*)^* \cong G.$$

Proof: Let G have parameters v, e and f and G^* have parameters v^*, e^* and f^*. By the definition $v^* = f$ and $e^* = e$. On the other hand, G and G^* are both planar graphs and by the Euler formula
$$v + f = e + 2 = e^* + 2 = v^* + f^* = f + f^*$$
so that $f^* = v$. Hence, $(G^*)^*$ is a connected graph with parameters $v^{**} = v$, $e^{**} = e$ and $f^{**} = f$. Moreover, two faces of G are adjacent if and only if the corresponding vertices of G^* are adjacent.

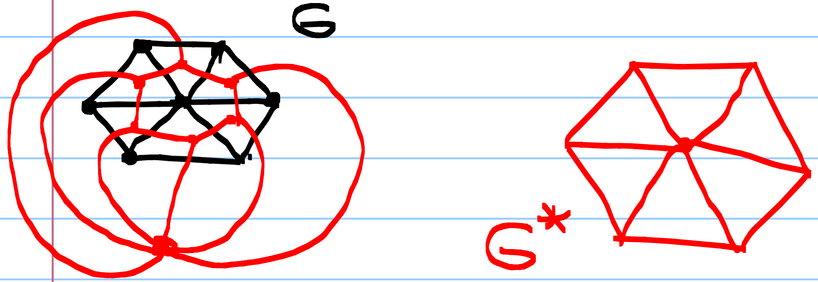


Example:

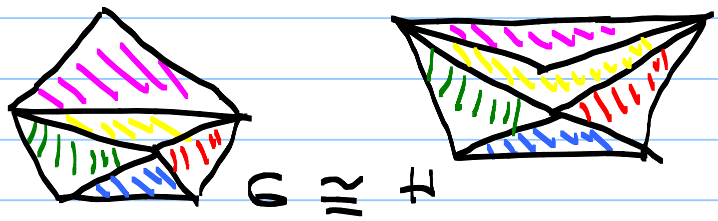


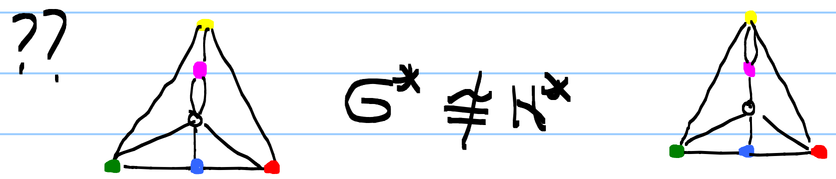
A graph G is said to be self dual if $G^* \cong G$.

Example: A wheel graph is self dual.



Remark: The graphs below G and H are isomorphic but their duals are not! So the dual depends on the embedding of the graph.





They are not isomorphic since G^* has a vertex of valency 5 whereas H^* has no such vertex.

Each face of a plane graph is an open connected subset of the plane and boundary of a face consists of edges and vertices of the graph.

For any face f_i, the number of edges in the boundary is called the degree of the face, denoted by $d(f_i)$. A bridge in a face contributes 2 to the degree of that face.

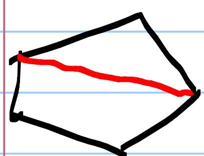
Theorem Let G be a planar graph, where V and F are the set of vertices and faces of G. Then we have

$$\sum_{\varphi \in F} d(\varphi) = 2e = \sum_{x \in V} d(x)$$

A simple planar graph G is said to be maximal planar if it contains largest possible number of edges. In other words, if whenever any pair of non-adjacent vertices are joined by a new edge, the resulting graph is non-planar, then we say that G is maximal planar.

In a similar manner, we say that a graph G is minimal non planar if it is non-planar and $G \setminus u$ is planar for any edge u of G.

If a face of a planar graph has degree larger than 3, we can draw a new edge in that face.



In other words, a maximal planar graph cannot have a face with degree 4 or more. Therefore, all faces of a maximal planar graph are triangles. Such a graph is called a triangulation. To summarize we have

Proposition A planar graph of order at least 3 is maximal planar if and only if it is a triangulation

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Theorem: Let G be a simple planar connected graph on $v \geq 3$ vertices with e edges. If all faces has degree 3 then $e = 3(v-2)$.

Proof: Since each face is a triangle $3f = 2e$. Plugging $f = \frac{2}{3}e$ in Euler's formula $v + f = e + 2$ we get

$$v + \frac{2}{3}e = e + 2 \Rightarrow \frac{1}{3}e = v - 2 \Rightarrow e = 3(v-2).$$

Corollary: Let G be simple planar connected graph on $v \geq 3$ vertices and e edges. Then

$$e \leq 3(v-2).$$

Proof: For any non triangle face of G draw extra edges to make it a maximal simple planar graph with $e+k$ edges, where k is the number of edges added to G. Since now every face is a triangle and thus by the above theorem

$$e + k = 3(v-2).$$

Hence, $e \leq e + k = 3(v-2)$. ∎

Corollary Let G be a simple planar connected graph on v vertices with e edges. If g is girth of G then $e \leq \frac{g}{g-2} \cdot (v-2)$, where g is length of a shortest cycle in G.

Proof $2e/f$ = the average order of a face of $G \geq g$.

Then $2e \geq fg$. Since $v + f = e + 2$ we get

$$e = (v-2) + f \leq (v-2) + \frac{2e}{g}.$$

$$\Rightarrow e - \frac{2e}{g} \leq v-2$$

$$\Rightarrow e\left(1-\frac{2}{g}\right) \leq v-2 \Rightarrow e\left(\frac{g-2}{g}\right) \leq v-2.$$

$$\Rightarrow e \leq \frac{g}{g-2}(v-2). \quad \blacksquare$$

Corollary: If G is a simple planar connected graph having no triangle face then $e \leq 2(v-2)$.

Proof: $2e/f \geq 4 \Rightarrow 2e \geq 4f \Rightarrow \frac{e}{2} \geq f$

Euler's formula: $v+f = e+2$

$$\Rightarrow e = (v-2)+f \leq v-2+\frac{e}{2}$$

$$\Rightarrow \frac{e}{2} \leq v-2. \Rightarrow e \leq 2(v-2). \quad \blacksquare$$

Theorem: For any simple planar graph G, $\delta \leq 5$.

Proof: Without loss of generality we may assume that G is connected and $v \geq 3$. Then $2e \geq \delta v$, because $2e = \sum_{v \in V(G)} d(v)$ and $\delta \leq d(v)$ for all $v \in V(G)$.

By one of the above Corollary $e \leq 3(v-2)$ we get

$\delta v \leq 2e \leq 6(v-2)$, which implies that

$\delta \leq 6\left(1-\frac{2}{v}\right) < 6$. Since δ is an integer

we see that $\delta \leq 5$. $\blacksquare$

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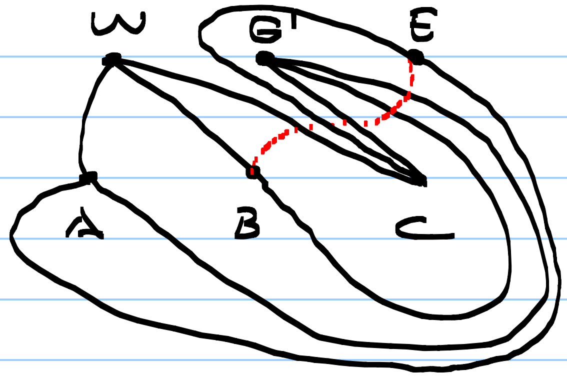
Two Problems

1) Möbius (1840): There was a king with 5 sons. In his will he stated that after his death the sons should divide the kingdom into five regions so that the boundary of each region should have a frontier line in common with each of the other four regions. Can the terms of the will be satisfied?

2) Henry Ernest Dudeney (1917)

Water–Gas–Electricity Problem



Is it possible to lay down pipes from W, G and E to each house so that the pipes do not intersect?

Solution: 1) Taking the dual note that the first problem is equivalent to asking if K_5 is planar. To see this simply replace each piece (face) with a vertex and join two vertices if the corresponding faces have a common edge.

2) The second problem is equivalent to asking if $K_{3,3}$ is planar.

Both questions are answered in the negative.

Lemma: K_5 and $K_{3,3}$ are both non-planar.

Proof: For K_5, $v=5$, $e=10$ and hence $e>3(v-2)$. However, by the first corollary above $e\le 3(v-2)$

if K_5 were planar. Hence, K_5 cannot be planar.

For $K_{3,3}$, $v=6$ and $e=9$ and hence $e>2(v-2)$. Since $K_{3,3}$ has no triangle face by a corollary above $e \le 2(v-2)$, which implies that $K_{3,3}$ cannot be planar. ∎

<u>Remark:</u> Note that any graph G containing K_5 or $K_{3,3}$ as a subgraph cannot be planar either.

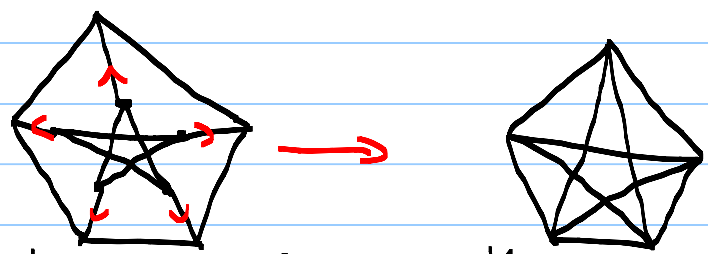
The converse of the above statement is known as Kuratowski's Theorem.

<u>Theorem</u> (Kuratowski, 1930)
A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

<u>Example:</u> Any complete graph K_n, $n \ge 5$ and $K_{p,q}$, $p,q \ge 3$, are non-planar.

<u>Theorem</u> (Wagner, 1937)
A graph is planar if and only if it contains no subgraph contractible to K_5 or $K_{3,3}$.

<u>Example:</u> The Petersen graph contains $K_{3,3}$ as a subgraph and hence it is non planar. It also contracts to K_5 and hence it is non planar by Wagner's Theorem also.



Petersen graph $\qquad$ K_5

Measures of Nonplanarity:
We'll describe a method to measure how a given nonplanar graph is far from being planar.
We'll mention 4 methods and examine the first 3 of them.
Assume that a nonplanar graph G is given. To understand how this graph is nonplanar we ask the following questions:

- What is the smallest number of planar graphs whose union is G?

- At least how many edges can we delete from G to obtain a planar graph?

- What is the smallest number of crossings when we represent G in the plane?

- At least how many bridges can we use to avoid crossing of edges?

Thickness of a graph: Given a graph G, the thickness (or the depth of G) denoted as $\theta(G)$ is the smallest value of t such that it is possible to find planar graphs $G_1, G_2, \ldots, G_t$, each having the same vertex set with G, such that $G = \bigcup_{i=1}^{t} G_i$. Thus the thickness of a planar graph is one. A graph with thickness 2 is called biplanar. Since we don't require connectedness of G_i, $\theta(G)$ is just the minimum number of planar subgraphs of G, whose union is G.

Remark: The concept of thickness originated from

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a 1962 conjecture of Frank Harary: For any graph on nine points, either itself or its complementary graph is non planar. Note that this problem is equivalent to determining whether K_9 is biplanar. It turns out that $\theta(K_9)=3$, that is K_9 is not biplanar. Hence, the Harary's conjecture is true.

Example (Lecture notes p. 32) $\theta(K_9)=\theta(K_{10})=3$.

Recall that for a simple connected planar graph G we must have $e \le 3(v-2) = 3v-6$. So we get the following lower bound for $\theta(G)$.

Corollary For a simple graph $\theta(G) \ge \lceil e/(3v-6) \rceil$.

Proof Assume that $\theta(G)=k$. So, $G = G_1 \cup \dots \cup G_k$ and each G_i is planar.

$V(G_i) = V(G)$, $\quad v_i = |V(G_i)| = n, \ \forall n.$

Let $|E(G_i)| = e_i$, then $e \le \sum_{i=1}^{k} e_i$.

Since each G_i is planar $e_i = |E(G_i)| \le 3v-6$.

So, $e \le \sum_{i=1}^{k} e_i \le \sum_{i=1}^{k} 3v-6 = k(3v-6)$.

$$\Rightarrow k \ge \frac{e}{3v-6} \text{ and hence } k \ge \left\lceil \frac{e}{3v-6} \right\rceil.$$

Remark: Since for any positive integers a, b we have $\lceil a/b \rceil = \left\lfloor \frac{a+b-1}{b} \right\rfloor$ we have

$$\theta(G) \ge \left\lfloor \frac{e+3v-7}{3v-6} \right\rfloor.$$

Proposition (Vasak 1976, Alekseev and Gonchakov, 1976)

$\theta(K_9) = \theta(K_{10}) = 3$ and for all other values $n \geq 1$, we have

$$\theta(K_n) = \left\lceil \frac{n+2}{6} \right\rceil = \left\lfloor \frac{n+7}{6} \right\rfloor$$

Example $\theta(K_5) = 2$, and K_5 is biplanar.

Proposition (Beineke et al., 1964; Harary 1994)

For positive integers p, q, $\theta(K_{p,q}) = \left\lceil \frac{pq}{2(p+q-2)} \right\rceil$.

In particular, $\theta(K_{n,n}) = \left\lceil \frac{n^2}{4(n-1)} \right\rceil$

$$= \left\lfloor \frac{n^2+4n-5}{4(n-1)} \right\rfloor = \left\lfloor \frac{n+5}{4} \right\rfloor.$$

Proposition (Harary, 1994) The thickness of the hypercube Q_n is

$$\theta(Q_n) = \left\lceil \frac{n+1}{4} \right\rceil.$$

Skewness of a graph: Given a non planar graph G, the minimum of edges we have to remove from G in order to obtain a planar graph is called the skewness of G, denoted as $sk(G)$. For a planar graph G, we set $sk(G)=0$.

It is clear that skewness of a disconnected graph is equal to the sum of skewness of the components.

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Proposition: (Cimikowski, 1992)
Let G be a graph of order $v \geq 3$ and e edges with girth g. Then $sk(G) \geq e - \frac{g(v-2)}{g-2}$.

Proof: Let G' be the planar graph obtained from G by removing $sk(G)$ edges. It is clear that the girth of G' is at least g. Since G' has $e - sk(G)$ edges we have

$$e - sk(G) \leq g'(v-2)/g'-2 \leq g(v-2)/g-2, \text{ where}$$

g' is the girth of G', the function $f(x) = \frac{ax}{x-2}$ is decreasing if $a > 0$. ∎

Corollary Skewness $sk(G)$ of any simple graph of order $v \geq$ with e edges satisfies

$$sk(G) \geq e - 3(v-2).$$

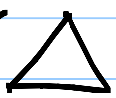
Proof: By the previous result $sk(G) \geq e - \frac{g}{g-2}(v-2)$.

On the other hand, since $g \geq 3$ and thus $2g \geq 6$.
This implies

$$2g - 6 \geq 0 \Rightarrow 3g - 6 \geq g.$$
$$\Rightarrow 3 \geq \frac{g}{g-2}$$

Hence, $sk(G) \geq e - \frac{g}{g-2}(v-2) \geq e - 3(v-2)$. ∎

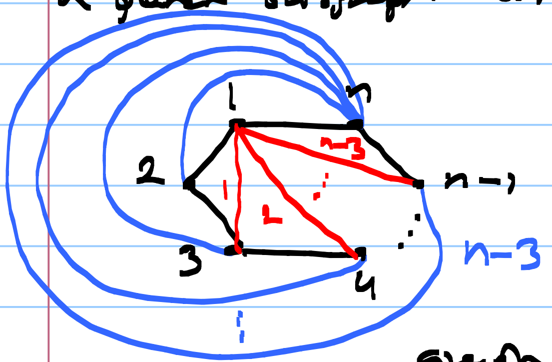
Corollary $sk(K_n) = \frac{1}{2}(n-3)(n-4)$, for $n \geq 3$.

Proof: (
K_3 $g(K_3) = 3$. It follows that $g(K_n) = 3$ for any $n \geq 3$, since $K_3 \subseteq K_n$.)

So, $sk(K_n) \geq e - 3(v-2) = \binom{n}{2} - 3(n-2)$.

$\Rightarrow sk(K_n) \geq \binom{n}{2} - 3n + 6$.

On the other hand, the complete graph K_n has a planar subgraph with $3n-6$ edges:



$n + (n-3) + (n-3) = 3n-6$.

Hence, removing $\binom{n}{2} - (3n-6)$ edges we obtain a planar graph so that $sk(K_n) \leq \binom{n}{2} - (3n-6)$.

Thus, $sk(K_n) = \binom{n}{2} - (3n-6) = \frac{n(n-1)}{2} - 3n + 6$

$$= \frac{n^2 - 7n + 12}{2} = \frac{(n-3)(n-4)}{2}.$$

<u>Example</u> $sk(K_7) = \frac{4 \cdot 3}{2} = 6$. $sk(K_6) = \frac{3 \cdot 2}{2} = 3$.

$sk(K_5) = \frac{2 \cdot 1}{2} = 1$.

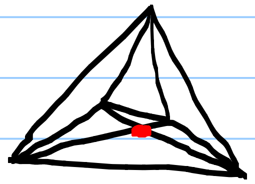
<u>Proposition</u> (Cimikowski, 1992)

a) $sk(K_{p,q}) = (p-2)(q-2)$, $p, q \geq 3$,
b) $sk(Q_n) = 2^n(n-2) - n2^{n-1} + 4$, $n \geq 4$.

<u>Crossing Number of a Graph</u>:

Among all representations of a graph G in plane, the minimum value of the crossing number is defined to be the crossing number $Cr(G)$ of the graph. For a planar graph G, $Cr(G) = 0$.

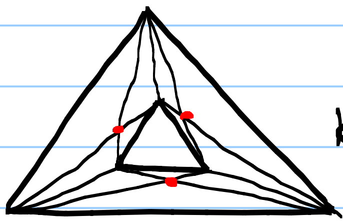
Example The nonplanar graphs K_5 and $K_{3,3}$ has crossing number 1.



Proposition: For any graph G, $Cr(G) \geq sk(G)$.

Proof. Consider an embedding of G in plane which has exactly $Cr(G)$ crossings. To obtain a planar graph, when we remove an edge of G which crosses some other edge, at least one crossing is removed. Hence, by removing at least $sk(G)$ edges, we remove all the crossings. ∎

Example: For $n \geq 3$, $Cr(K_n) \geq sk(K_n) = \binom{n}{2} - 3n + 6$ and hence $Cr(K_6) \geq \binom{6}{2} - 18 + 6 = 3$. The plane representation of K_6 below has 3 crossings. Hence, $Cr(K_6) = 3$.



K_6 with 3 crossings.

Similar arguments shows that $Cr(K_7) = 9$.

Example. $Cr(K_{p,q}) \geq (p-2)(q-2)$, $p, q \geq 3$.

So, $Cr(K_{3,3}) \geq (3-2)(3-2) = 1 \Rightarrow Cr(K_{3,3}) = 1$, since $K_{3,3}$ has a diagram with one crossing.

$Cr(K_{4,4}) \geq (4-2)(4-2) = 4$. Indeed, $Cr(K_{4,4}) = 4$.

The above formula shows that $Cr(K_{6,7}) \geq 4 \cdot 5 = 20$. However, $Cr(K_{6,7}) = 54$.

<u>Conjecture</u> (Guy, 1972)

$$Cr(K_n) = \frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor .$$

This proved for $n \leq 12$ by Pan and Richter in 2007. For large values of n, the conjectural number is known to be an upper bound for $Cr(K_n)$.

<u>Conjecture</u> (Zarankiewicz)

$$Cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor .$$

In 1971 Kleitman proved the above conjecture for $K_{5,m}$ and $K_{6,m}$. Moreover, the conjectured number is known to be an upper bound for $Cr(K_{m,n})$.

<u>Theorem</u> For any graph G with v vertices and e edges
a) $Cr(G) \geq e - 3v$ and b) $Cr(G) \geq e^3/64v^2$.

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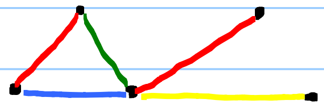
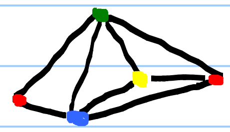
XII - COLORING

Although map coloring can be regarded as coloring of faces of a graph embedded in the plane, in graph theory, unless otherwise stated explicitly, a coloring is a vertex coloring, namely assigning colors to the vertices of the graph such that no two adjacent vertices sharing the same color.

Vertex coloring can be generalized to edge coloring and face coloring as well. An edge coloring of a graph is just a vertex coloring of its line graph.



G



Line graph of G

Similarly, face coloring of a plane graph is just a vertex coloring of its dual. Thus any type coloring can be reduced to some vertex coloring.

For a graph G, a function $\sigma: V(G) \to \{1, 2, \ldots, k\}$ such that $\sigma(u) \neq \sigma(v)$, for all $uv \in E(G)$, is called a (proper) k-coloring (of vertices). Note that the coloring function σ is not necessarily onto. A k-coloring of a graph $G(V, E)$ partitions the vertex set V into k disjoint sets $V_1, V_2, \ldots, V_k$, which are called the color classes.

For a given integer k, if a coloring function exists for G, G is said to be k-colorable. The smallest value of k for which G is k-colorable is called the chromatic number of G and denoted as $\chi(G)$.

Some immediate observations:

1) A graph G is 1-chromatic if and only if it is totally disconnected.

2) If $|V(G)|=n$, then $\chi(G) \leq n$.

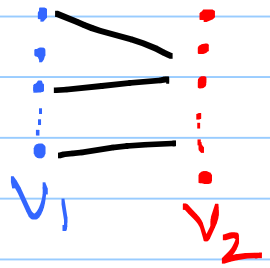
3) $\chi(K_n)=n$, because any two edges are connected by an edge.

4) $\chi(K_{p,q})=2$ if $K_{p,q}$ is not totally disconnected.

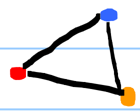
5) Any tree T with $n \geq 2$ vertices is a bipartite graph and hence $\chi(T)=2$.

<u>Theorem</u> A graph is 2-chromatic if and only if it is bipartite.

<u>Proof</u>: Assume G is 2-chromatic. Then $V=V_1 \cup V_2$. No two vertices in V_1 or in V_2 are connected by an edge. Thus G is a bipartite graph.



<u>Corollary</u> A graph G is 2-colorable if and only if it has no odd cycle.



<u>Lemma</u>: In any k-coloring of a k-chromatic graph G, there is at least one edge between any two color classes.

<u>Proof</u>: If there were no edges between two color classes, then the same color could be used for both classes. This finishes the proof. ■

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Theorem: For any graph G with e edges,

$$\chi(G) \le 1/2\,(1+\sqrt{8e+1}).$$

Proof: Let $\chi(G)=k$. Then by the above lemma to each pair of colors there correspond an edge of G. Thus, $e \ge \binom{k}{2}$, that is $e \ge \frac{1}{2}k(k-1)$.

$$\Rightarrow k^2-k \le 2e \Rightarrow (k-1/2)^2 - 1/4 \le 2e$$

$$\Rightarrow (k-1/2)^2 \le 2e+1/4 = \frac{8e+1}{4}$$

$$\Rightarrow k-1/2 \le \frac{\sqrt{8e+1}}{2}$$

$$\Rightarrow \chi(G) = k \le \frac{1}{2}(1+\sqrt{8e+1}). \quad \blacksquare$$

Note that for a subgraph H of a graph G we have $\chi(H) \le \chi(G)$. If $\chi(H) < \chi(G)$, for all subgraphs $H \subsetneq G$, then G is said to be k-critical.

Proposition: A k-critical graph is connected.

Proof: Suppose G is not connected. Then one of the components, say G_1 with $\chi(G_1)=k$, which is a contradiction. $\blacksquare$

Theorem: If G is k-critical, then $\delta(G) \ge k-1$.

Proof: Assume on the contrary that $\delta(G) < k-1$. Let $v \in V(G)$ with $d(v)=\delta$. Since G is k-critical $G \setminus v$ is $k-1$-colorable. Let $\{V_1, \ldots, V_{k-1}\}$ be a partition which corresponds to a $(k-1)$-coloring of $G \setminus v$. Then, v is adjacent in G to $\delta < k-1$ vertices and hence it must be non adjacent in G to every vertex of some color class V_j. Now $\{V_1, V_2, \dots, V_j \cup \{v\}, \dots, V_{k-1}\}$ is a $k-1$-coloring of G, which contradicts to the fact $\chi(G) = k$. This finishes the proof. ∎

<u>Corollary</u> Every k-chromatic graph has at least k vertices of degree at least $k-1$.

<u>Proof</u>: Let G be a k-chromatic graph, and let H be a k-critical subgraph of G. Here H can be obtained as follows: Just delete edges e from G so that the chromatic number of $G - e$ does not get smaller. Finally, the process will stop when we obtain the desired subgraph. Now by the above theorem $\delta(H) \geq k-1$. Since, H is k-chromatic H has at least k vertices (with valency $\geq k-1$). This finishes the proof. ∎

<u>Corollary</u> For any graph G, $\chi(G) \leq \Delta + 1$.

<u>Proof</u>: If $\chi(G) = k > \Delta + 1$ then G had at least one vertex with degree $k-1 > \Delta$, by the above corollary. However, this is a contradiction. Hence $\chi(G) \leq \Delta + 1$. ∎

<u>A Coloring Algorithm</u>: Although there is no efficient algorithm for coloring a graph with minimum number of colors, we have an algorithm known as greedy graph coloring algorithm that gives a coloring of a given graph with at most $\Delta + 1$ colors.

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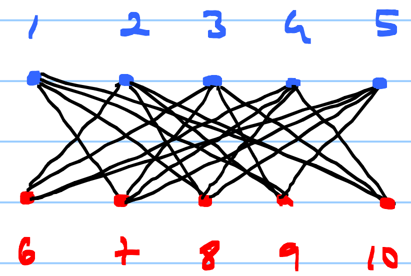
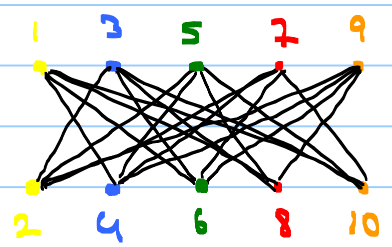
Step 1: Color the first vertex with the first color.

Step 2: Consider the remaining $n-1$ vertices one by one and do the following. Color the currently picked vertex with the lowest numbered color which has not been used to color any of its adjacent vertices.

Example: Greedy graph coloring algorithm does not guarantee the use of minimum number of colors necessary, as the examples below demonstrates.

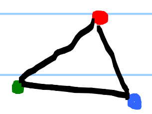




$\Delta(G) = 4$.
So G is $4+1 = 5$-colorable.
$\chi(G) = 2$.

Corollary For any n there is at least one graph of size n for which $\chi(G) = \Delta + 1$.

Proof $\chi(K_n) = n = \Delta(K_n) + 1$.
Also $\chi(C_{2k+1}) = 3 = \Delta(C_{2k+1}) + 1$



Theorem (Brooks, 1941)
If G is a connected simple graph which is neither an odd cycle nor a complete graph then $\chi(G) \leq \Delta(G)$.

Lemma: Let G be a graph with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$.
$$\chi(G) \leq \max_i \{\min(i, d_i+1)\}.$$

Proof: Let $\chi(G)=k$. By a previous corollary G has at least k vertices of degree at least $k-1$. So $d_k \geq k-1$ and thus $(d_k+1 \geq k)$
$$\max_i \{\min(i, d_i+1)\} \geq \min(k, d_k+1) \geq k = \chi(G).$$
∎

Recall that α_0 is the independence number, namely the size of the largest set of mutually non adjacent vertices. For a graph G, we denote the chromatic number of its complement $\overline{G}$ by $\overline{\chi}$.

Theorem: For any graph of order n, we have $2\sqrt{n} \leq \chi + \overline{\chi} \leq n+1$ and $n \leq \chi\overline{\chi} \leq \left(\frac{n+1}{2}\right)^2$.

Proof: A χ-coloring of a graph partitions the vertex set into $k=\chi$ nonempty and independent sets $V_1, V_2, \ldots, V_k$. Clearly, $|V_i| \leq \alpha_0$, for all $i=1,\ldots,k$.
Hence,
$$n = \sum_{i=1}^{k} |V_i| \leq \sum_{i=1}^{k} \alpha_0 = k\alpha_0 = \chi\alpha_0.$$

Thus, $\frac{n}{\chi} \leq \alpha_0$.

$\overline{G}$ has a complete subgraph of order α_0 and hence $\chi(\overline{G}) = \overline{\chi} \geq \alpha_0$. In particular,

$$\chi\bar{\chi} \geq \chi\alpha_0 \geq n.$$

From the arithmetic-geometric mean inequality

$$\frac{\chi+\bar{\chi}}{2} \geq \sqrt{\chi\bar{\chi}} \geq \sqrt{n} \Rightarrow 2\sqrt{n} \leq \chi+\bar{\chi}.$$

On the other hand, if $d_1 \geq d_2 \geq \dots \geq d_n$ is the degree sequence of G then the degree sequence of $\bar{G}$ is

$n-1-d_n \geq n-1-d_{n-1} \geq \dots \geq n-1-d_1$. Now by the lemma

$$\chi+\bar{\chi} \leq \max_i\{\min(i, d_i+1)\} + \max_i\{\min(i, n-d_{n+1-i})\}.$$

$$= \max_i\{\min(i, d_i+1)\} + \max_i\{\min(n+1-i, n-d_i)\}$$

$$= \max_i\{\min(i, d_i+1)\} + n+1 + \max_i\{\min(-i, -(d_i+1))\}.$$

$k = \max\{1, 2, \dots, k, d_{k+1}+1, \dots, d_n+1\}$ $-\{d_1+1, \dots, d_k+1, k+1, \dots, n\} \leq 0$

$$= \max_i\{\min(i, d_i+1)\} + n+1 - \min\{\max(i, d_i+1)\}$$

$n-1 \geq d_1 \geq d_2 \geq \dots \geq d_n \geq 0$

$$\leq n+1$$

$1 \leq d_1+1, \dots, k \leq d_k+1, k+1 > d_{k+1}+1, \dots$

$n > d_n+1.$

Thus, $\chi+\bar{\chi} \leq n+1$ and hence, $2\sqrt{n} \leq \chi+\bar{\chi} \leq n+1$.

Moreover, since $\chi+\bar{\chi} \geq 2\sqrt{\chi\bar{\chi}} \geq 2\sqrt{n}$ we have

$$\left(\frac{n+1}{2}\right)^2 \geq \left(\frac{\chi+\bar{\chi}}{2}\right)^2 \geq \chi\bar{\chi} \geq n.$$

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Planar Graph Coloring:
The so called Four Color Theorem for planar graphs is the most famous problem in the history of graph theory. The problem originally considers the proper face colorings of plane graphs. However, any such coloring is equivalent with a proper vertex coloring of the dual graph. So we can keep on examining the vertex coloring problems focusing on planar graphs.

Theorem (Six Color Problem):
Every planar graph is 6-colorable.

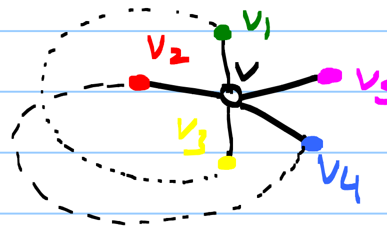
Proof: Proof is by induction on the number of vertices. Note that $\chi(K_7 - e) \leq 6$. So any graph which is isomorphic to a subgraph of $K_7 - e$ is 6-colorable. Assume that any planar graph with n vertices is 6-colorable. By the above sentence we may assume that $n \geq 7$. Now let G be a planar graph with n+1 vertices. We know that for any planar graph $\delta(G) \leq 5$. So G has a vertex v with degree at most 5. By the induction hypothesis G−v is 6-colorable. In any coloring of G−v with 6 colors, at most 5 colors are used for the neighbors of v. Then v can be colored with the sixth color. ∎

Theorem (Five Color Theorem, 1890)
Every planar graph is 5-colorable.

Proof: Proof is by induction on the number of vertices n. Clearly, the claim holds for n=5. Assume that the claim holds for any planar graph with n vertices ($n \geq 5$) and let G be a planar graph with $n+1$ vertices. Since G is planar $\delta(G) \leq 5$. Choose a vertex v with smallest $d(v)$. We have two cases to consider:

<u>Case 1</u>: $d(v) \leq 4$. By assumption $G-v$ is 5-colorable. In any coloring of $G-v$ with five colors, at most 4 colors are used for coloring the neighbors of v. Thus v can be colored with the fifth color. This finishes the proof in this case.

<u>Case 2</u>: $d(v)=5$. In some 5-coloring of $G-v$, if we can show that the neighbors of v can be colored with at most 4 colors then the proof will finish as above. Label the neighbors of v in a cyclic manner as v_1, v_2, v_3, v_4 and v_5 and assume that they are colored as green, red, yellow, blue and pink, respectively.



Let H be the subgraph of $G-v$ induced by all green and yellow vertices. If v_1 and v_3 are in different components of H, then colors of all green and yellow vertices in the component of v_1 can be switched. The new coloring is still proper because there are no edges between components of H containing v_1 and v_3. Now the neighbors of v are colored using 4 colors and thus v can be colored with the fifth color, green.

Thus we only have to consider the case where v_1 and v_3 are in the same component of H. In this case there is 2-colored (green and yellow) path from v_1 to v_3. A similar argument implies that there is a two colored path (red and blue) from v_2 to v_4. Since this a planar graph the two paths has to intersect at a vertex. However,

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this is a contradiction, because the first path has only green and yellow vertices which the second path has only red and blue vertices. This contradiction finishes the proof. ∎

Kempe Chain: A two colored chain used in the above proof is called a Kempe chain.

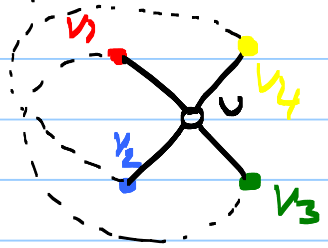
Theorem (Four Color Theorem, Appel-Haken, 1976) Every planar graph is 4-colorable.

Proof: It is clear that every planar graph with $n \leq 5$ vertices is four colorable (note that $\chi(K_5)=5$, $\chi(K_5-e)=4$ and any planar graph with 5 vertices is a subgraph of K_5-e).

Now assume that every graph with n vertices is four colorable and G be a graph with $n+1$ vertices. Again $\delta(G) \leq 5$ and thus we may pick a vertex v of degree at most 5. Let v be a vertex with smallest degree.

Case 1 $d(v) \leq 3$. By assumption $G-v$ is four colorable and in any coloring of $G-v$ with four colors, at most 3 colors are used to color the neighbors of v. Thus, v can be colored with the fourth color.

Case 2 $d(v)=4$. In some coloring of $G-v$ if we can show that the 4 neighbors of v can be colored with at most 3 colors, then the proof finishes immediately. So we may assume that the four neighbors v_1, v_2, v_3 and v_4 of v can be enumerated according to their cyclic positions around v and also let these colors be red, blue, green and yellow. Now we use Kempe chain again.



Let H be the subgraph of $G \setminus v$ induced by all red and green vertices. If v_1 and v_3 are in different components, then colors of all red and green in the component of v_1 can be switched. The resulting coloring is still proper and for coloring the neighbors of v only three colors are sufficient so that v can be colored with the fourth color, red.

So we may assume that v_1 and v_3 are in the same component of H. Similarly, we may assume that there is a blue-yellow Kempe chain from v_2 to v_4. Since these two chains must intersect as in the previous proof we obtain a contradiction. This finishes the proof in this case.

<u>Case 3:</u> $d(v) = 5$.

This part is proved by Appel and Haken by the help of computers using so called "Unavoidable Sets and Reducible Configurations" (120 hours of computer time, in 1976). ∎

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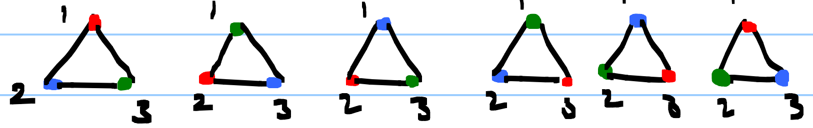
Chromatic Polynomial

Chromatic polynomial was introduced and developed by Birkhoff as a tool to count the proper colorings of a graph to attack the four color conjecture in 1912.

Two colorings of a graph with the same color set is considered different if some vertex has different colors in the two colorings. An example is given below:



We denote the number of distinct k-colorings of a simple graph G by $\chi_G(k)$. Then $\chi_G(k) > 0$ if and only if G is k-colorable. Note that the chromatic number of G is the smallest k so that $\chi_G(k) > 0$.

For a given number of k colors, $\chi_G(k)$ is defined to be the number of distinct k colorings of G. Thus

$$\chi_G : \mathbb{N} \to \mathbb{N}, \quad \mathbb{N} = \{0, 1, 2, \dots\}.$$

Theorem: For any simple graph G, the function χ_G is a polynomial in k.

Proof. A coloring of G gives a partition of the vertex set $V(G)$. If the partition has p parts, then number of colorings for this partition is clearly

$$k(k-1)\cdots(k-p+1).$$

Note that two colorings corresponding to different partitions of the vertex set $V(G)$ are clearly distinct. So the number of all distinct colorings of G will be the sum of the polynomials in k, corresponding to finitely many distinct partitions of $V(G)$. The sum is again a polynomial. This finishes the proof. ∎

From now on $\chi_G(k)$ will be called the chromatic polynomial of G.

<u>Corollary</u> (of the proof of the previous theorem)

The chromatic polynomial $\chi_G(k)$ has degree $n = |V(G)|$.

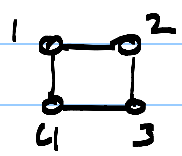
<u>Example</u> For the null graph O_n ($|V(O_n)| = n$, $E(O_n) = \emptyset$), $\chi_{O_n}(k) = k \cdot k \cdots k = k^n$.

On the other hand, $\chi_{K_n}(k) = k \cdot (k-1) \cdots (k-n+1)$

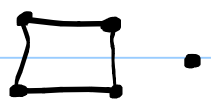
$\chi_{K_n}(k) = 0$, for $k = 0, \ldots, n-1$.

In particular, $\chi_{C_3}(k) = k(k-1)(k-2)$. $\chi_{\underset{K_3}{\overset{\shortparallel}{C_3}}}(3) = 6$.

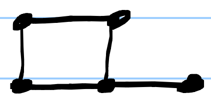
Let's compute $\chi_{C_4}(k)$.



$$\begin{aligned}\chi_{C_4}(k) &= k \cdot (k-1)(k-2)(k-2) \\ &\quad + k(k-1)(k-1) \\ &= k(k-1)(k-1+(k-2)^2) \\ &= k(k-1)(k^2-3k+3)\end{aligned}$$

<u>Example</u> 1) G:  $\chi_G(k) = \chi_{C_4}(k) \cdot k$

$$= k^2(k-1)(k^2-3k+3)$$

2) G:  $\chi_G(k) = \chi_{C_4}(k) \cdot (k-1)$

$$= k(k-1)^2(k^2-3k+3).$$

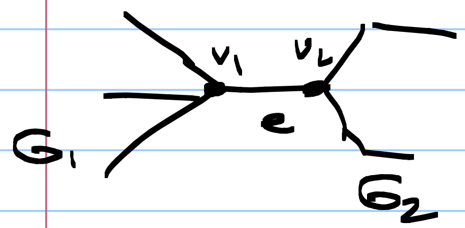
Remark: 1) If G is the disjoint union of the graphs $G_1, G_2, \dots, G_r$ then

$$\chi_G(k) = \chi_{G_1}(k) \cdot \chi_{G_2}(k) \cdot \dots \cdot \chi_{G_r}(k).$$

2) If $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v\}$

then $\chi_G(k) = \chi_{G_1}(k) \cdot \chi_{G_2}(k) \cdot \frac{1}{k}$

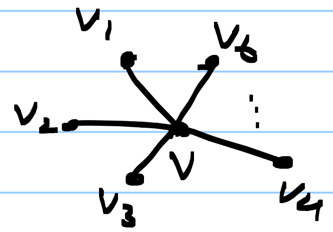
3) If G is a graph so that $G - e = G_1 \dot\cup G_2$ is a disjoint union for some edge e, then



$$\chi_G(k) = \chi_{G_1}(k) \cdot \chi_{G_2}(k) \cdot \frac{k-1}{k}$$

4) If G has a vertex v of degree t which is adjacent to all vertices of a clique K_t in G, then

$$\chi_G(k) = (k-t)\, \chi_{G-v}(k)$$



$K_{v_1, \dots, v_t}$

Theorem (Deletion–Contraction Argument)
Let e be an edge of a simple graph G. Chromatic polynomial of G, $G-e$ and $G \setminus e$ satisfy

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G \setminus e}(k).$$

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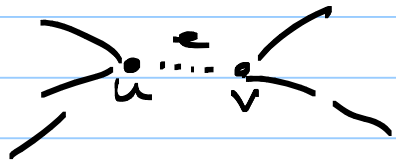
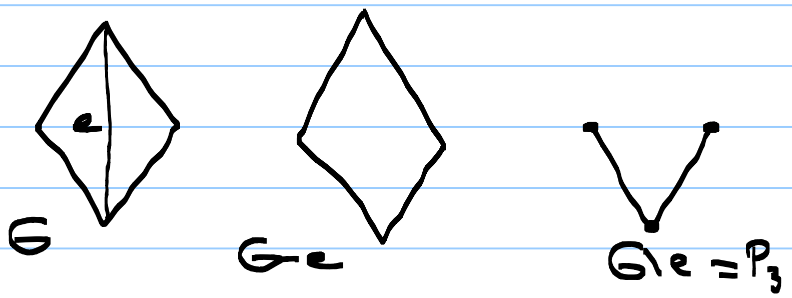
Proof: Let $e = uv$. A proper coloring σ of $G-e$ either assigns different colors to u and v or the same color to both. In the former case,



σ is a proper coloring of G, while in the latter case σ is a proper coloring of $G\backslash e$.

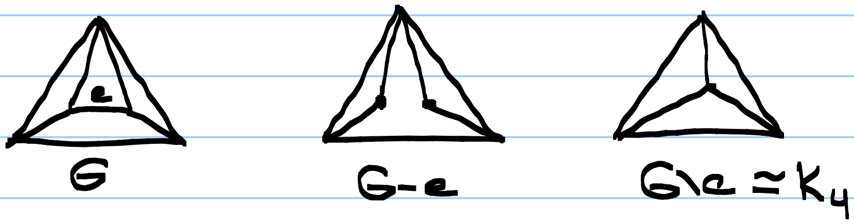
Hence we see that $\chi_{G-e}(k) = \chi_G(k) + \chi_{G\backslash e}(k)$ and this finishes the proof. ∎

Example



$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G\backslash e}(k)$$
$$= \chi_{C_4}(k) - \chi_{P_3}(k)$$
$$= k(k-1)(k^2-3k+3) - k\cdot(k-1)^2$$
$$= k(k-1)(k^2-4k+4).$$

Example:



So, $\chi_G(k) = \chi_{G-e}(k) - \chi_{G\backslash e}(k)$

$$= k(k-1)(k-2)^3 - k(k-1)(k-2)(k-3)$$
$$= k(k-1)(k-2)(k^2-5k+7).$$

__Theorem__ (Coefficients of the Chromatic Polynomial)

Let $\chi_G(k) = a_n k^n + \dots + a_1 k + a_0$ be the chromatic polynomial of a simple graph G with n vertices, m edges and c components. Then we have,

a) $a_n = 1$

b) $a_{n-1} = -m$

c) $a_0 = 0$

d) $a_1 = 0$ if and only if G is disconnected

e) $a_c \neq 0$ and $a_{c-1} = a_{c-2} = \dots = a_0 = 0$

f) coefficients alternate in signs.

__Proof__: a) A partition with l elements of the vertex set $\{v_1, \dots, v_n\}$ contributes to $\chi_G(k)$ with $k(k-1)\dots(k-l+1)$. So the highest degree contribution comes from the partition of $V(G)$ consisting of singletons, and that term is $k(k-1)\dots(k-n+1)$, whose leading coefficient is k^n. Hence, $a_n = 1$.

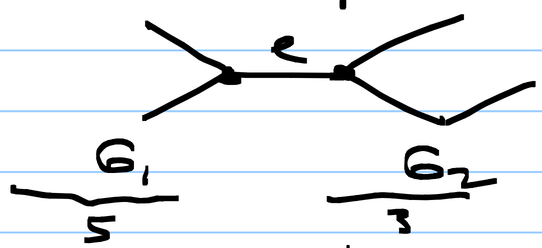
b) $a_{n-1} = -m$: By part (a), when we contract an edge we obtain a graph on $n-1$ vertices and leading term of the chromatic polynomial is k^{n-1}. Since, $\chi_G = \chi_{G-e} - \chi_{G/e}$ deleting an edge should increase the coefficient of k^{n-1} in χ_G by 1. Moreover, this process will end when we delete all m edges, in which case the coefficient would be zero since $\chi_{O(n)} = k^n$ is zero. Thus, the coefficient of k^{n-1} in χ_G has to be $-m$.

c) No graph can be colored with zero colors and thus $\chi_G(0) = a_0 = 0$.

d) If G is disconnected then by part (c), $\chi_G(k)$

is divisible by k^2, because, $G = G_1 \dot\cup G_2$, then
$\chi_{G_1}(0) = 0 = \chi_{G_2}(0) \Rightarrow \chi_G(k) = \chi_{G_1}(k)\chi_{G_2}(k)$
admit 0 as a zero of order two.
Hence, $a_1 = 0$.

Now assume that G is connected. We must show that $a_1 \neq 0$: Proof is by induction on the number of edges m. If $m=1$, then $G = P_2$: •—• so that $\chi_G(k) = k(k-1) = k^2 - k$. Now assume that the result holds for all connected graphs with $m \geq 1$ edges and let G be a connected graph with $m+1$ edges. Choose an edge e of G. If $G-e$ is disconnected then
$G - e = G_1 \cup G_2$ where each G_i is connected and e is a bridge.



Now, $\chi_G(k) = \frac{k-1}{k}\chi_{G_1}(k)\chi_{G_2}(k)$ and $a_1 \neq 0$ for both G_1 and G_2.
Hence, $a_1 \neq 0$ for G also.

Now assume $G-e$ is connected. Then we have by the deletion–contraction argument

$\chi_G(k) = \chi_{G-e}(k) - \chi_{G\setminus e}(k)$, where $G\setminus e$ is also connected.

By part 1-4) the coefficients of the chromatic polynomials are alternating and
$\deg \chi_{G-e}(k) = n$, while $\deg \chi_{G\setminus e}(k) = n-1$.

So, $\chi_{G-e}(k) = k^n + c_{n-1}k^{n-1} + \dots + c_1 k \quad (c_1 \neq 0)$ and
$\chi_{G\setminus e}(k) = k^{n-1} + b_{n-2}k^{n-2} + \dots + b_1 k \quad (b_1 \neq 0)$

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So, $c_1 b_1 < 0$. Hence, a_1 of $\chi_G(k) = \chi_{G-e} - \chi_{G\setminus e}$ is $c_1 - b_1 \neq 0$.

e) The chromatic polynomial $\chi_G(k) = \chi_{G_1}(k) \cdots \chi_{G_c}(k)$ when $G_1, \dots, G_c$ are the connected components of G. Since each G_i is connected $\chi_{G_i}(k)$ is divisible by k ($a_0 = 0$) but not divisible by k^2 ($a_1 \neq 0$). Hence, $\chi_G(k)$ is divisible by k^c but not k^{c+1}.

f) The claim is true for P_1: $\chi_{P_1}(k) = k$. Proof is by induction on the number $n+m$, where n is the number of vertices and m is the number edges. It holds, for $n+m=1$ ($\Leftrightarrow G = P_1$). Now assume that the result holds for $n+m \leq N$, and assume that G is a graph with $n+m = N+1$. Choose an edge e in G. Then the result holds for both $G-e$ and $G\setminus e$ so that $\chi_{G-e}(k)$ and $\chi_{G\setminus e}(k)$ are both alternating. Note that if G has degree ℓ then $\deg \chi_{G-e} = \ell$ but $\deg \chi_{G\setminus e} = \ell - 1$. Say

$$\chi_{G-e}(k) = k^\ell + \underset{\overset{\wedge}{0}}{b_{\ell-1}} k^{\ell-1} + \underset{\overset{\vee}{0}}{b_{\ell-2}} k^{\ell-2} + \dots + b_1 k + b_0, \text{ and}$$

$$\chi_{G\setminus e}(k) = k^{\ell-1} + \underset{\overset{\wedge}{0}}{c_{\ell-2}} k^{\ell-2} + \underset{\overset{\vee}{0}}{c_{\ell-3}} k^{\ell-3} + \dots + c_1 k + c_0$$

$$\chi_G = \chi_{G-e} - \chi_{G\setminus e} = k^\ell + (\underset{-}{b_{\ell-1}} - 1) k^{\ell-1} + (\underset{+}{b_{\ell-2} - c_{\ell-2}}) k^{\ell-2}$$
$$+ (\underset{-}{b_{\ell-3} - c_{\ell-3}}) k^{\ell-3} + \dots$$

This finishes the proof. $\blacksquare$

Theorem: For any tree T, $\chi_T(k) = k(k-1)^{n-1}$, where n is the degree of T.

Proof: Induction on n. For $n=1$, $T: \bullet$ and here, $\chi_T(T) = k = k^1(k-1)^0$. Now assume the result for n and let G be a tree with $n+1$ vertices. Let v be an end vertex for T. Then $T-e$ is again a tree with $n-1$ vertices. So, $\chi_{T-e} = k(k-1)^{n-2}$. Hence,

$$\chi_T = (k-1)\chi_{T-e} = (k-1)\,k\,(k-1)^{n-2} = k(k-1)^{n-1}. \quad \blacksquare$$

Corollary All trees are chromatically equivalent.

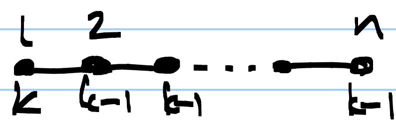
Theorem: For $n \geq 3$ the cycle graph C_n has chromatic polynomial

$$\chi_{C_n}(k) = (k-1)^n + (-1)^n(k-1).$$

Proof is by induction on n based on the deletion-contraction argument. For $n=3$, we have $\chi_{C_3}(k) = k(k-1)(k-2) = (k-1)^3 - (k-1)$.

If e is an edge of C_n, then $C_n - e = P_n$ and $C_n \setminus e = C_{n-1}$. Hence,

$$\chi_{C_n - e} = \chi_{P_n} = k(k-1)^{n-1}$$

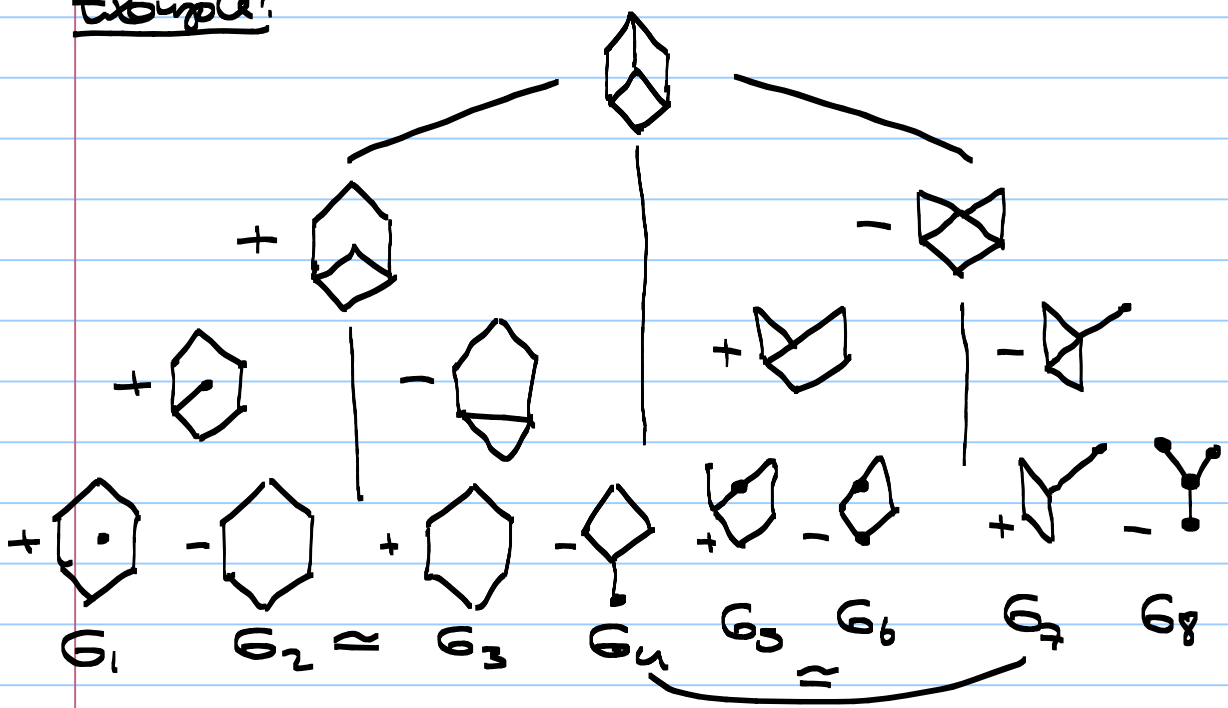


Hence
$$\chi_{C_n} = \chi_{C_n - e} - \chi_{C_n \setminus e}$$
$$= \chi_{P_n} - \chi_{C_{n-1}}$$
$$= k(k-1)^{n-1} - \left((k-1)^{n-1} + (-1)^{n-1}(k-1)\right)$$
$$= k(k-1)^{n-1} - (k-1)^{n-1} + (-1)^n(k-1)$$

$$= (k-1)^{n-1}(k-1) + (-1)^n (k-1)$$

$$= (k-1)^n + (-1)^n (k-1),$$ and the proof finishes. ∎

Example:



$$\chi_G = \chi_{G_1} - \chi_{G_2} - \chi_{G_3} + \chi_{G_4} - \chi_{G_5} + \chi_{G_6} + \chi_{G_7} - \chi_{G_8}$$

$$\chi_{G_1} = k\,\chi_{C_6}(k) = k\left((k-1)^6 + (k-1)\right)$$

$$\chi_{G_2} = (k-1)^6 + (k-1) = \chi_{G_3}$$

$$\chi_{G_4} = \frac{k-1}{k}\chi_{C_4} = \frac{k-1}{k}\left((k-1)^4 + (k-1)\right) = \chi_{G_7}$$

$$\chi_{G_5} = \frac{k-1}{k}\chi_{C_5} = \frac{k-1}{k}\left((k-1)^5 - (k-1)\right)$$

$$\chi_{G_6} = \chi_{C_5} = (k-1)^5 - (k-1)$$

$$\chi_{G_8} = k(k-1)^3$$

Theorem: (Roots of the Chromatic Polynomial)

For any graph on n vertices we have

a) The multiplicity of 0 as a root is the number of connected components.

b) The multiplicity of 1 as a root is the number of blocks.

c) The chromatic polynomial has no real roots larger than $n-1$.

d) The chromatic polynomial has no negative real roots.

e) The chromatic polynomial has no roots between 0 and 1.

f) $(-1)^n \chi_G'(1) \geq 0$.

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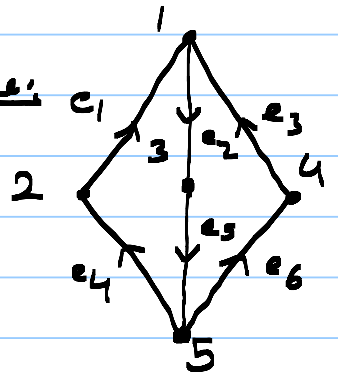
CHAPTER XIII. Matrices for Graphs

(following Chap 2. of Graphs and Matrices by R.B. Bapat)

Incidence Matrix Let G be a directed graph with vertex set $V(G)=\{1,2,\dots,n\}$ and edge set $E(G)=\{e_1,e_2,\dots,e_m\}$. The vertex-edge incidence matrix of G, denoted by $Q(G)$, is the $n\times m$-matrix defined as follows: The rows and columns of $Q(G)$ are indexed by $V(G)$ and $E(E)$, respectively. The $(i,j)^{th}$ entry of $Q(G)$ is 0 if i or e_j are not incident, and otherwise it is -1 or 1 according as e_j originates or terminates at i, respectively.

Example:



$$Q=Q(G)=\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & \end{array} \\ \left[\begin{array}{cccccc} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

Rank: Note that the sum of entries of each column is zero and thus the sum of rows of Q is zero. Hence, the rows of $Q=Q(G)$ are linearly dependent.

Lemma: If G is connected on n vertices then $\operatorname{rank} Q(G)=n-1$.

Proof: By the above paragraph $\operatorname{rank} Q(G)\le n-1$, since its rows are linearly dependent.

Now let X be a $1\times n$ vector so that $XQ=0$.

$$[x_1\ x_2 \cdots x_n]\left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array}\right]_{m\times 1}$$

$e=ij$

So $x_i - x_j = 0$ whenever we have an edge $e=ij$.

Hence, $x_i = x_j$ whenever there is a path from the vertex i to the vertex j. Since, G is connected we see that $x_i = x_j$ for all i, j. Hence, $x = (c, c, \dots, c)$, for some $c \in \mathbb{R}$. Thus $\ker(xQ)$ has dimension 1. Thus, $\operatorname{rank} Q = n-1$ since Q has n rows. ∎

<u>Theorem</u> If G is a graph with n vertices and k connected components then $\operatorname{rank} Q(G) = n-k$.

<u>Proof</u>: Let $G_1, \dots, G_k$ be the connected components of G. Then

$$Q(G) = \begin{bmatrix} Q(G_1) & & & \\ & Q(G_2) & & \\ & & \ddots & \\ & & & Q(G_k) \end{bmatrix}_{n \times m}$$

Since each G_i is connected $\operatorname{rank} Q(G_i) = n_i - 1$, where n_i is the number of vertices of G_i.
Thus, $\operatorname{rank} Q(G) = \sum_{i=1}^{k} \operatorname{rank} Q(G_i) = \sum_{i=1}^{k} (n_i - 1)$

$$= \left(\sum_{i=1}^{k} n_i\right) - k$$

$$= n - k.$$ ∎

<u>Lemma</u>: Let G be a connected graph on n vertices. Then the column space of $Q(G)$ consists of all vectors $x \in \mathbb{R}^n$ such that $\sum_i x_i = 0$.

<u>Proof</u> Let U be the column space of $Q(G)$. Then $\operatorname{rank} Q = n-1$ and thus $\operatorname{rank} U = n-1$. Also let $W = \{x \in \mathbb{R}^n \mid \sum_i x_i = 0\}$. We know that each column of Q is in W. Hence, $U \subseteq W$ and

thus $n-1 = \dim U \leq \dim W$. However, $\dim W = n-1$. (For example, consider the linear map $L: \mathbb{R}^n \to \mathbb{R}$ given by $L(x_1, \dots, x_n) = \sum x_i$. Since L is onto, $\dim \ker L + \dim \underbrace{\operatorname{im} L}_{\mathbb{R}} = n \Rightarrow \dim \ker L = n-1$.

Clearly, $\ker L = W$). It follows that $W = U$. ∎

Lemma: Let G be a graph on n vertices. Columns $J_1, \dots, J_k$ of $Q(G)$ are linearly independent if and only if the corresponding edges of G induce an acyclic subgraph.

Proof: Consider the edges $J_1, \dots, J_k$ and suppose there is a cycle in the corresponding induced subgraph. Without loss of generality we may assume that $J_1, \dots, J_p$ form a cycle $(3 \leq p \leq k)$. After relabelling the vertices if necessary, we see that the submatrix of $Q(G)$ formed by the the columns $J_1, \dots, J_p$ is of the form $\begin{bmatrix} B \\ 0 \end{bmatrix}$, where B is the $p \times p$ incidence matrix of $J_1, \dots, J_p$.

Note that B is a square matrix with column sums equal zero. Thus B is singular and the columns $J_1, \dots, J_p$ are linearly dependent. This proves the "only if" part of the lemma.

Conversely, suppose that $J_1, \dots, J_k$ induce an acyclic subgraph, that is a forest. If the forest has q components then clearly, $k = n - q$ and it is the rank of the submatrix formed by the columns $J_1, \dots, J_k$. Therefore, the columns $J_1, \dots, J_k$ are linearly independent. ??

∎

<u>Minors</u>: A matrix is said to be totally unimodular if the determinant of any square submatrix of the matrix is either 0 or ± 1.

<u>Lemma</u>: Let G be a graph with incidence matrix $Q(G)$. Then $Q(G)$ is totally unimodular.

<u>Proof</u>: Proof is by induction on the size $k \times k$ of the minor. By the definition of $Q(G)$ clearly any 1×1 minor has determinant 0 or ± 1. Now assume that any $k-1 \times k-1$ minor has determinant 0 or ± 1 $(k \geq 2)$. Let B be a $k \times k$ minor of $Q(G)$. If each column of B has a 1 and has a -1, then $\det B = 0$. Also if B has a zero column then $\det B = 0$. Now assume that B has a column with only one nonzero entry, which must be ± 1. Expand the determinant using this column. By the induction hypothesis we see that $\det B = \pm 1 \cdot \det B' = \pm 1 \cdot 0 = 0$, where B' is the submatrix of B obtained by deleting the row and the column of the only nonzero entry of a column of B. ▬

<u>Lemma</u>: Let G be a tree on n vertices. Then any submatrix of $Q(G)$ of order $n-1$ is nonsingular.

<u>Proof</u>
$n = \#$ of vertices of G
$e = \#$ of edges of $G = n-1$.

$Q(G) = n \times (n-1)$ matrix.

Consider the submatrix X of $Q(G)$ formed by the rows $1, \dots, n-1$. If we add all the

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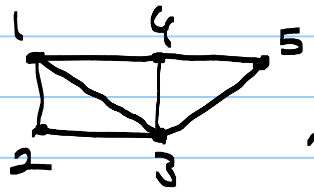
rows of X to the last row of X, then the last row of X becomes the negative of the last row of Q. So, if Y is the submatrix of Q formed by the rows $1, 2, \ldots, n-2, n$, then $\det X = -\det Y$. Thus, if $\det X = 0$ then $\det Y = 0$. This indeed implies that any $(n-1)\times(n-1)$ submatrix of $Q(G)$ must be singular. In fact, one can see that if any $(n-1)\times(n-1)$ submatrix is singular then all $(n-1)\times(n-1)$ minors are singular. However, this is a contradiction since $\operatorname{rank} Q = n-1$. Hence, all $(n-1)\times(n-1)$ minors are non singular. ∎

CHAPTER 3: Adjacency Matrix:

Let G be a graph with $V(G) = \{1, 2, \ldots, n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix of G, denoted as $A(G)$, is defined as follows: The rows and column of $A(G)$ are indexed by $V(G)$. If $i \neq j$ then the (i,j)th entry of $A(G)$ is zero for non-adjacent vertices i and j and is one for adjacent vertices i and j. We'll often write A for $A(G)$.

__Example:__ G:



$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Clearly, $A = A(G)$ is a symmetric matrix with zero elements on the diagonal. Since A is symmetric A is diagonalizable. For $i \neq j$ the principal matrix formed by the rows and columns i and j is the zero matrix if $i \nsim j$

and is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if $i \sim j$. Thus the sum of the 2×2 principal minors of A is equal to $-|E(G)|$.

Consider the principal submatrix of A formed by three distinct rows and columns i, j, k. It can be seen that the submatrix is non-singular only when i, j, k are adjacent to each other (i.e., if they constitute a triangle). In this case the matrix is indeed

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The determinant of this submatrix is 2. Thus the sum of the principal 3×3 minors of A is equal twice the number of triangles in G.

Lemma: The $(i,j)^{th}$ entry of A^k is the number of walks of length k from i to j.

Proof: Induction on k. For $k=1$ the result is clear from the definition of A. Now assume the result for k and let's consider the $(i,j)^{th}$ entry of A^{k+1}.

$$A^{k+1} = A^k A = \begin{bmatrix} & \vdots & \\ b_{i1} \ b_{i2} & \cdots & b_{in} \\ & \vdots & \end{bmatrix} \begin{bmatrix} & a_{j1} & \\ & a_{j2} & \\ \cdots & \vdots & \cdots \\ & a_{jn} & \end{bmatrix}$$

$$(A^{k+1})_{(i,j)} = b_{i1} a_{j1} + b_{i2} a_{j2} + \cdots + b_{in} a_{jn}.$$

$$\begin{aligned} b_{i1} a_{j1} &= (\#\text{ of distinct paths from } i \text{ to } 1 \text{ of length } k) \\ &\quad \times (\#\text{ of } \quad '' \quad '' \quad '' \quad 1 \text{ to } j \text{ of length } 1). \\ &= \#\text{ of distinct paths from } i \text{ to } j \text{ of length } k+1 \\ &\quad \text{where final edge is } e = 1j. \end{aligned}$$

Similarly, $b_{i2} a_{j2}$ is the # of distinct paths from i to j of length $k+1$ whose final edge is $e = 2j$.

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Hence $(A^{k+1})_{(i,j)}$ is the number of all distinct paths from i to j of length $k+1$. ∎

<u>Corollary</u> The (i,i)th entry of A^k is the number of k-cycles starting and ending at the vertex i.

<u>Definition:</u> The distance $d(i,j)$ between the vertices i and j is defined to be the minimum order of an (i,j)-path. We set $d(i,i)=0$. The maximum value of $d(i,j)$ is the diameter of G.

<u>Lemma:</u> Let G be a connected graph with vertices $\{1,2,\dots,n\}$ and let A be the adjacency matrix of G. If i and j are vertices of G with $d(i,j)=m$, then the matrices $I, A, A^2, \dots, A^m$ are linearly independent.

<u>Proof</u> We may assume that $i\neq j$. So there is no path from i to j of length less than m. Thus the (i,j)-element of $I, A, \dots, A^{m-1}$ is zero, whereas the (i,j)-element of A^m is non zero. Hence, $I, \dots, A^m$ are linearly independent. ∎

<u>Corollary</u> Let G be a connected graph with k distinct eigenvalues and let d be the diameter of G. Then $k>d$.

<u>Proof</u> Let A be the adjacency graph of G. Then the matrices $I, A, \dots, A^d$ are linearly independent. Hence, the degree of the minimal polynomial of A has to be at least $d+1$. On the other hand, since A is symmetric A is diagonalizable. Hence the

roots of the minimal polynomial has to be distinct. So A has at least $d+1$ distinct eigenvalues. Hence, $k > d$. ∎

Eigenvalues of some graphs:

Let G be a graph with adjacency matrix A. Eigenvalues of A will be called also as the eigenvalues of the graph G.

Theorem: i) For any positive integer n, the eigenvalues of K_n are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$.

ii) For any positive integers p and q, the eigenvalues of $K_{p,q}$ are $\sqrt{pq}$, $-\sqrt{pq}$ and 0 with multiplicity $p+q-2 = n-2$

Proof: i) Let J_n be the matrix of all ones. Then $A(G) = A(K_n) = J_n - I_n$. Since rank $J_n = 1$ zero is an eigenvalue of J_n with multiplicity $n-1$. Also $[1\,1 \cdots 1]^T$ is an eigenvector of J_n with eigenvalue n. Let P be the matrix so that

$$P J_n P^{-1} = \operatorname{diag}(n, 0, \ldots, 0).$$

Then
$$\begin{aligned} PAP^{-1} = P(J_n - I_n)P^{-1} &= PJ_nP^{-1} - PI_nP^{-1} \\ &= \operatorname{diag}(n, 0, \ldots, 0) - I_n \\ &= \operatorname{diag}(n-1, -1, \ldots, -1). \end{aligned}$$

This finishes the proof of (i).

ii) Note that $A(K_{p,q}) = \left[\begin{array}{c|c} 0 & J_{p,q} \\ \hline J_{q,p} & 0 \end{array}\right]$, where $J_{p,q}$ and $J_{q,p}$ are matrices of all ones.

So, rank $A = 2$ and hence A has only two nonzero eigenvalues. Also $\operatorname{tr} A = 0$ and thus the nonzero eigenvalues of A are of the form λ and $-\lambda$, for some $\lambda \neq 0$. We know that the sum of 2×2 principal minors of $A = A(K_{p,q})$ is negative of the number of edges, that is $-pq$. The sum also equals the sum of products of the eigenvalues, taken 2 at a time, which is $-\lambda^2$. Thus $-\lambda^2 = -pq \Rightarrow \lambda = \sqrt{pq}$ and the result follows. ■

<u>Remark:</u> The last line of the above proof uses the following fact from linear algebra: If A and B are similar matrices then the sum of $k\times k$ principal minors of A and B are the same.

$$A, \quad B = PAP^{-1}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \longmapsto \begin{bmatrix} & & \tfrac{1}{\lambda} a_{1i} & & \\ & & \vdots & & \\ \lambda a_{i1} & \cdots \lambda a_{i\,i-1} & a_{ii} & \cdots & \lambda a_{in} \\ & & \vdots & & \\ & & \tfrac{1}{\lambda} a_{ni} & & \end{bmatrix}$$

So a 2×2 principal minor has the form

$$\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \longmapsto \begin{bmatrix} a_{ii} & \tfrac{1}{\lambda} a_{ij} \\ \lambda a_{ji} & a_{jj} \end{bmatrix} \quad \underline{\text{OR}}$$

$$\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix} \longmapsto \begin{bmatrix} a_{ii} & \lambda a_{ij} \\ \tfrac{1}{\lambda} a_{ji} & a_{jj} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & b \\ c+\lambda a & d+\lambda b \end{bmatrix} \longrightarrow \begin{bmatrix} a-\lambda b & b \\ c+\lambda a-\lambda d-\lambda^2 b & d+\lambda b \end{bmatrix}$$

det $\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$ det

$$ad - bc \overset{?}{=} (a-\lambda b)(d+\lambda b) - b(c+\lambda a - \lambda d - \lambda^2 b)$$

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Next we'll compute the eigenvalues of C_n. First we introduce the following matrix

$$Q_n = \begin{bmatrix} 0 & 1 & 0 & - & - & 0 \\ 0 & 0 & 1 & \cdot & - & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & \ddots & 1 \\ 1 & 0 & - & \cdots & \cdots & 0 \end{bmatrix}.$$

Note that $e_i Q_n = e_{i+1}$, $i=1,\dots,n-1$ and $e_n Q_n = e_1$, where $e_i = (0 \cdots 1 \cdots 0)$ (ith place)

In particular, $Q_n^n = I_n$. Note that the characteristic polynomial of Q_n, $\det(\lambda I_n - Q_n)$ is $(-1)^n(\lambda^n - 1) = 0$

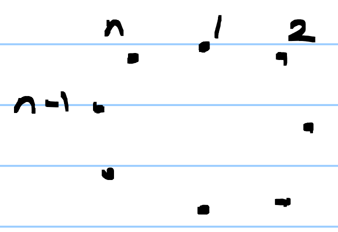
Corollary The eigenvalues of the matrix Q_n are the nth root of unities, namely, $1, \omega, \omega^2, \dots, \omega^{n-1}$, where $\omega = e^{2\pi i/n} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Theorem For $n \geq 2$, the eigenvalues of the graph C_n are $2\cos\frac{2\pi k}{n}$, $k = 1, 2, \dots, n$.

Proof First note that $A(C_n) = Q_n + Q_n^{n-1} = Q_n + Q_n^{-1}$.

$$Q = \begin{bmatrix} 0 & 1 & 0 & - & - & 0 \\ 0 & 0 & 1 & & - & 0 \\ \vdots & & & \ddots & & \\ & & & & \ddots & 1 \\ 1 & - & - & - & - & 0 \end{bmatrix}, \quad Q_n^{n-1} = Q_n^{-1} = \begin{bmatrix} 0 & \cdot & 0 & & 1 \\ 1 & & 0 & & \\ \vdots & & 1 & \cdot & \\ 0 & & \cdots & 1 & 0 \end{bmatrix}$$

$e_i Q_n^{-1} = e_{i-1}$, $i = 2, \dots, n$, $e_1 Q_n^{-1} = e_n$.



$$\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 1 & & & 1 & 0 \end{bmatrix} = A(C_n).$$

$A(C_n) = Q_n + Q_n^{n-1} = P(Q_n)$, where $P(x)$ is the polynomial $P(x) = x + x^{n-1}$. If λ is an eigenvalue of a matrix B with eigenvector $v \in \mathbb{R}^n$ then v is an eigenvector for $P(B)$ with eigenvalue $P(\lambda)$. $P(x) = p_0 + p_1 x + \dots + p_r x^r$, $P(B) = p_0 I_n + p_1 B + p_2 B^2 + \dots + p_r B^r$.

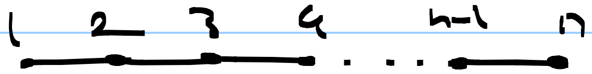
$$P(B)(v) = p_0 I_n(v) + p_1 B(v) + p_2 B^2(v) + \cdots + p_n B^n(v)$$
$$= p_0 v + p_1 \lambda v + p_2 \lambda^2 v + \cdots + p_n \lambda^n v$$
$$= (p_0 + p_1 \lambda + \cdots + p_n \lambda^n) v = P(\lambda) v.$$

Hence, the eigenvalues of $A(C_n) = Q_n + Q_n^{n-1} = Q_n + Q_n^{-1}$ are $\omega^k + \omega^{n-k} = \omega^k + \omega^{-k}$ since $\omega^n = 1$, $k = 1, \dots, n$.

However, $\omega^k + \omega^{-k} = \left(\cos\frac{2\pi k}{n} + \cancel{i \sin\frac{2\pi k}{n}}\right) + \left(\cos\frac{2\pi k}{n} - \cancel{i \sin\frac{2\pi k}{n}}\right)$

$$= 2\cos\frac{2\pi k}{n}, \quad k = 1, 2, \dots, n.$$

<u>Theorem</u> For $n \geq 2$, the eigenvalues of P_n are
$$2\cos\frac{\pi k}{n+1}, \quad k = 1, 2, \dots, n.$$

<u>Proof</u> P_n: 

$$A(P_n) = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 1 & 0 \end{bmatrix} \qquad A(P_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Let λ be an eigenvalue of $A(P_n)$ with x as the corresponding eigenvector. By symmetry $(-x_n, \dots, -x_1)$ is also an eigenvector, where $x = (x_1, \dots, x_n)$. Then we can easily see that $(x_1, \dots, x_n, 0, -x_n, \dots, -x_1, 0)$ and $(0, x_1, \dots, x_n, 0, -x_n, \dots, -x_1)$ are two linearly independent eigenvectors of $A(C_{2n+2})$, for the same eigenvalue.

In particular, each eigenvalue of P_n is an eigenvalue of C_{2n+2}, with multiplicity 2.

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We know that the eigenvalues of C_{2n+2} are $2\cos\frac{2\pi k}{2n+2} = 2\cos\frac{\pi k}{n+1}$, $k=1,2,\dots,2n+2$. Of these eigenvalues that appear twice, are $2\cos\frac{\pi k}{n+1}$, $k=1,2,\dots,n$, which must be the eigenvalues of P_n. ∎

Determinants: Let G be any graph with vertex set $V(G)=\{1,2,\dots,n\}$ and edge set $E(G)=\{e_1,e_2,\dots,e_m\}$. A subgraph H of G is called an elementary subgraph if each component of H is either an edge or a cycle. Denote by $c(H)$ and $c_1(H)$ the number of components in the subgraph H, which are cycles and edges, respectively.

Theorem If $A=A(G)$ is the adjacency matrix of a graph G then

$$\det A = \sum (-1)^{n-c_1(H)-c(H)}\, 2^{c(H)},$$

where the summation is over all spanning elementary subgraph H of G.

Proof: We know from linear algebra that

$$\det A = \sum_{\pi} \operatorname{sgn}(\pi)\, a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)},$$ where the

summation is over all permutations of $\{1,2,\dots,n\}$. Consider a term $a_{1\pi(1)}\cdots a_{n\pi(n)}$, which is non-zero. Since π admits a cycle decomposition such a term will correspond to some 2-cycles of π, which designate an edge joining i and j in G, as well as some cycles of higher order, which correspond to cycles in G (note that $\pi(i)\neq i$

for any i, since $a_{i\pi(i)} \neq 0$ and $a_{ii} = 0$.)

Thus, each nonzero term in the summation arises from an elementary subgraph of G with vertex set $V(G)$. Moreover, the sign of π is (-1) raised to n minus the number of cycles in the decomposition of π, which is the same as $(-1)^{n-c_1(H)-c(H)}$.

Finally, each spanning elementary subgraph gives rise to $2^{c(H)}$ terms in the summation, since each cycle can be associated to a cyclic permutation in two ways. These finish the proof.

<u>Theorem.</u> Let G be a graph as above. If $A = A(G)$ and $\phi_\lambda(A) = \det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \dots + c_n$ is the characteristic polynomial of A then

$$c_k = \sum (-1)^{c_1(H)+c(H)} 2^{c(H)},$$ where the

summation is over all the elementary subgraphs H of G, with k vertices $k = 1, \dots, n$.

<u>Proof.</u> Observe that c_k is $(-1)^k$ times the sum of the principal minors of A of order k, $k = 1, \dots, n$. By the above theorem

$$c_k = (-1)^k \sum (-1)^{k-c_1(H)-c(H)} 2^{c(H)},$$ where the

summation is over all the elementary subgraphs H of G with k vertices. This finishes the proof. ∎

Remark: $\Phi_A(\lambda) = \det(\lambda I - A)$

$$= \sum_{\pi} \operatorname{sgn}(\pi)\, (\lambda I - A)_{1\pi(1)} \cdots (\lambda I - A)_{n\pi(n)}$$

$$= \sum_{\pi} \operatorname{sgn}(\pi)\, (\delta_{1\pi(1)}\lambda - a_{1\pi(1)}) \cdots (\delta_{n\pi(n)}\lambda - a_{n\pi(n)})$$

$$= \lambda^n + c_1\lambda^{n-1} + \cdots + c_k\lambda^{n-k} + \cdots + c_n$$

$$c_k\lambda^{n-k} = \sum_{\substack{\pi \in S_n \\ \pi(i)=i \text{ for} \\ \text{at least } n-k \\ \text{elements of } \{1,\dots,n\}}} \operatorname{sgn}(\pi)\, (-1)^k\, a_{j_1\pi(j_1)} \cdots a_{j_k\pi(j_k)}$$

$n=7$, $k=3$, $c_3\lambda^4 = \lambda^4(-1)^3 \left(\sum_{\pi \in S_{123}} a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} \operatorname{sgn}(\pi) \right)$

$j_1 = 1$, $j_2 = 2$, $j_3 = 3$

$\pi(i) = i$, $i = 4, 5, 6, 7$.

$$= \lambda^4(-1)^3 \left(\sum_{\pi \in S_{123}} a_{1\pi(1)} a_{2\pi(2)} a_{3\pi(3)} \operatorname{sgn}\pi \right)$$

$$+ \lambda^4(-1)^3 \left(\sum_{\pi \in S_{124}} a_{1\pi(1)} a_{2\pi(2)} a_{4\pi(4)} \operatorname{sgn}\pi \right)$$

$$\vdots$$

$$= \lambda^4 (-1)^k (\text{sum of all principal } k\text{-minors of } A)$$

Corollary Assume the above set up. Suppose $c_3 = c_5 = \cdots = c_{2k-1} = 0$. Then G has no odd cycle of length i, $3 \le i \le 2k-1$. Furthermore, the number of $(2k+1)$-cycles in G is $-\frac{1}{2} c_{2k+1}$.

Proof: Since $c_3 = 0$, there are no triangles in G. Thus, any elementary subgraph of G with 5 vertices must only comprise of a 5-cycle. In other words, we do not have a 5-cycle of the form $v_1 v_2 v_3 v_2 v_1$, and so on. However $c_5 = 0$ also and hence, there are no 5-cycles in G. Continuing this way we see that G has no odd cycles of order upto $2k-1$. Hence, any elementary subgraph of G with $2k+1$ vertices must be a $2k+1$-cycle. Moreover, we have

$$c_{2k+1} = \sum (-1)^{c_1(H) + c(H)} \, 2^{c(H)},$$

where the summation is over all $(2k+1)$-cycles H in G. For any $(2k+1)$-cycle H, $c_1(H) = 0$ and $c(H) = 1$. Hence, c_{2k+1} is (-2) the number of $(2k+1)$-cycles in G. That completes the proof. ∎

Corollary If $c_{2k+1} = 0$, for $k = 0, 1, 2, \ldots$, then G is bipartite.

Bounds:

Theorem: Let G be a graph with n vertices, m edges and $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of G. Then $\lambda_1 \leq \left(\frac{2m(n-1)}{n}\right)^{1/2}$.

Proof: We have $\sum \lambda_i = tr(A) = a_{11} + \cdots + a_{nn} = 0$ and $\sum_{i=1}^{n} \lambda_i^2 = 2m$. Thus $\lambda_1 = -\sum_{i=2}^{n} \lambda_i$ and hence,

$$\lambda_1 \leq \sum_{i=2}^{n} |\lambda_i|.$$

By the Cauchy-Schwarz inequality

$$\frac{\lambda_1^2}{n-1} \leq \frac{1}{n-1}\left(\sum_{i=2}^{n}|\lambda_i|\right)^2 \leq \frac{1}{n-1}\left(\sum_{i=2}^{n} 1^2\right)\left(\sum_{i=2}^{n}\lambda_i^2\right)$$
$$= \frac{1}{n-1}(n-1)\sum_{i=2}^{n}\lambda_i^2 = 2m - \lambda_1^2$$

Hence, $2m \geq \lambda_1^2 + \frac{\lambda_1^2}{n-1} = \left(\frac{n}{n-1}\right)\lambda_1^2$.

$$\Rightarrow \lambda_1^2 \leq \frac{2m(n-1)}{n}.$$

<u>Notation:</u> For a symmetric matrix B we denote its eigenvalues as $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B)$. Similarly, for a graph G, $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ will denote the eigenvalues of $A(G)$.

<u>Lemma:</u> Let G be a graph with n vertices and let H be an induced subgraph of H with p vertices. Then $\lambda_1(G) \geq \lambda_1(H)$ and $\lambda_n(G) \leq \lambda_p(H)$.

<u>Proof:</u> Note that $A(H)$ is a principal submatrix of $A(G)$. The result follows from the interlacing inequalities of a matrix and of its principal submatrix.

<u>Fact:</u> Let A be a symmetric $n\times n$ matrix and let B be a principal submatrix of A of order $n-1$. If $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_{n-1}$ are eigenvalues of A and B then
$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

<u>Proof:</u>

$$A = \left[\begin{array}{cccc|c} \multicolumn{4}{c}{B} & \\ a_{11} & a_{12} & \cdots & a_{1\,n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,n-1} & a_{2n} \\ \vdots & & & & \\ a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n-1} & a_{n-1\,n} \\ \hline a_{n1} & a_{n2} & & a_{n\,n-1} & a_{nn} \end{array}\right]$$

Let P be an $(n-1)\times(n-1)$ orthogonal matrix so that $PBP^T = \mathrm{diag}(\mu_1, \ldots, \mu_{n-1})$

Let $Q = \left[\begin{array}{c|c} P & 0 \\ \hline 0 & 1 \end{array}\right]$. Then $QAQ^T = \left[\begin{array}{ccc|c} \mu_1 & & 0 & * \\ & \ddots & & * \\ 0 & & \mu_n & \vdots \\ \hline * & * & \cdots & * \end{array}\right]$

which is still symmetric. Now we can choose symmetric row and column operations

$$RQAQ^TR^T = \begin{bmatrix} \mu_1 & & & 0 \\ & \ddots & & \\ & & \mu_{n-1} & \\ 0 & & & \tilde{\lambda} \end{bmatrix}.$$

Assume that

$$\begin{array}{ccccccccccc} \mu_1 & \geq \cdots \geq & \mu_i & \geq & \lambda & \geq & \mu_{i+1} & \geq \cdots \geq & \mu_{n-1} \\ \| & & \| & & \| & & \| & & \| \\ \lambda_1 & & \lambda_i & & \lambda_{i+1} & & \lambda_{i+2} & & \lambda_n \end{array}$$

Then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 = \mu_2 \geq \cdots \geq \lambda_i = \mu_i \geq \lambda_{i+1} \geq \mu_{i+1} = \lambda_{i+2}$$

$$\geq \mu_{i+2} = \lambda_{i+3} \geq \cdots \geq \mu_{n-1} = \lambda_n.$$

This finishes the proof in this special case. However, the other cases are similar. ■

Lemma: For any graph G, $\delta(G) \leq \lambda_1(G) \leq \Delta(G)$.

Proof: Let $A = A(G)$ and x an eigenvector associated to the eigenvalue $\lambda_1(G)$: $Ax = \lambda_1(G)x$.

So
$$\lambda_1(G)x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n$$
$$= \sum_{i \sim j} x_j, \quad \text{for any } i = 1, 2, \ldots, n.$$

Let $x_k > 0$ be the maximum coordinate of x. Then by the above equation

$$\lambda_1(G)x_k = \sum_{k \sim j} x_j \leq \Delta(G)x_k \quad \text{and thus}$$

$\lambda_1(G) \leq \Delta(G)$.

For the lower bound, first recall the extremal representation

$$\lambda_1(A) = \max_{\|x\|=1} \{x^T A x\} = \max_{x \neq 0} \left\{ \frac{x^T A x}{x^T x} \right\}.$$

In particular, taking $x = \mathbb{1} = [1\ 1 \dots 1]^T$, we get

$$\lambda_1(A) \geq \frac{\mathbb{1}^T A \mathbb{1}}{\mathbb{1}^T \mathbb{1}} = \frac{2m}{n},$$ where m is the number of edges in G. If $d_1, \dots, d_n$ is the valency sequence of G, then

$2m = d_1 + \dots + d_n \geq n\delta(G)$ and hence, $\lambda_1(G) \geq \delta(G)$ ∎

<u>Theorem</u> For any graph G, $\chi(G) \leq 1 + \lambda_1(G)$.

<u>Proof:</u> The result is trivial if $\chi(G) = 1$. Let $\chi(G) = p \geq 2$. Let H be an induced subgraph of G such that $\chi(H) = p$ and furthermore, suppose H is minimal with respect to the number of vertices. In other words, $\chi(H \setminus \{i\}) < p$ for any vertex i of H.

<u>Claim:</u> $\delta(H) \geq p-1$.

Note that claim implies that

$\lambda_1(G) \geq \lambda_1(H) \geq \delta(H) \geq p-1$ so that

$1 + \lambda(G) \geq p = \chi(G)$.

<u>Proof of the claim:</u> Assume on the contrary that there is a vertex i of H with $d(i) < p-1$. Since $\chi(H \setminus \{i\}) < p$ we may properly color $H \setminus \{i\}$ with $p-1$ colors. Since $d(i) < p-1$ we may color the vertex i so that H is properly colored with

$p-1$ colors, a contradiction to the choice of H. Hence, $\delta(i) \geq p-1$ for any vertex i of H. Thus $\delta(H) \geq p-1$, finishing the proof. ■

We'll finish this section with the following result, whose proof is omitted.

<u>Theorem</u>: Let G be a graph with n vertices and with at least one edge. Then

$$\chi(G) \geq 1 - \frac{\lambda_1(G)}{\lambda_n(G)}.$$

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Energy of a Graph:

For any graph G, the energy $\mathcal{E}(G)$ of G is defined to be the sum of absolute values of the eigenvalues of $A(G)$:

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$$

Definition: Let A and B be matrices of size $m \times n$ and $p \times q$, respectively. The Kronecker product of A and B, denoted as $A \otimes B$, is the $mp \times nq$ block matrix $[a_{ij}B]$:

$a_{ij}B$ is $p \times q$ matrix

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}_{mp \times nq} = A \otimes B.$$

Remark: Note that $(A \otimes B).(C \otimes D) = (AC \otimes BD)$, provided that AC and BD are defined.

Lemma: Let A and B be symmetric matrices of order $m \times m$ and $n \times n$, respectively. If $\lambda_1, \dots, \lambda_m$ and $\mu_1, \dots, \mu_n$ are eigenvalues of A and B, respectively, then the eigenvalues of $A \otimes I_n + I_m \otimes B$ are given by $\lambda_i + \mu_j$, $i = 1, \dots, m$, $j = 1, \dots, n$.

Proof. Since A and B are symmetric they are diagonalizable. Let P and Q be orthogonal matrices such that

$$P^T A P = \operatorname{diag}(\lambda_1, \dots, \lambda_m) \text{ and } Q^T B Q = \operatorname{diag}(\mu_1, \dots, \mu_n).$$

Then, we have

$$(P^T \otimes Q^T) \cdot (A \otimes I_n + I_m \otimes B)(P \otimes Q) = P^T A P \otimes Q^T I_n Q + P^T I_m P \otimes Q^T B Q$$

$$= \mathrm{diag}(\lambda_1,\dots,\lambda_m) \otimes I_n \otimes I_m \otimes \mathrm{diag}(\mu_1,\dots,\mu_n).$$

Now it is easy to see that this finishes the proof.

$$\mathrm{diag}(\lambda_1,\dots,\lambda_m) \otimes I_n = \begin{bmatrix} \lambda_1 I_n & 0 & 0 & 0 \\ 0 & \lambda_2 I_n & & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_m I_n \end{bmatrix}$$

$$I_m \otimes \mathrm{diag}(\mu_1,\dots,\mu_n) = \begin{bmatrix} \mathrm{diag}(\mu_1,\dots,\mu_n) & 0 & \cdots & 0 \\ 0 & \mathrm{diag}(\mu_1,\dots,\mu_n) & & \\ \vdots & & \ddots & \\ 0 & & & \mathrm{diag}(\mu_1,\dots,\mu_n) \end{bmatrix}$$

$$+$$

$$\begin{matrix} \lambda_1+\mu_1 & & & & & & & \\ & \lambda_1+\mu_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_1+\mu_n & & & & \\ & & & & \lambda_2+\mu_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2+\mu_n & \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_m+\mu_1 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & \lambda_m+\mu_n \end{matrix}$$

Lemma: Let A and B be as in the above lemma. Then the eigenvalues of $A \otimes B$ are given by $\lambda_i \mu_j$ for all $i = 1,\dots,m$, $j = 1,\dots,n$.

Proof: Let P and Q be as in the proof of the above lemma so that

$$P^T A P = \mathrm{diag}(\lambda_1,\dots,\lambda_m) \text{ and } Q^T B Q = \mathrm{diag}(\mu_1,\dots,\mu_n).$$

Then

$$(P^T \otimes Q^T)(A \otimes B)(P \otimes Q) = P^T A P \otimes Q^T B Q$$
$$= \mathrm{diag}(\lambda_1,\dots,\lambda_m) \otimes \mathrm{diag}(\mu_1,\dots,\mu_n)$$

$$= \operatorname{diag}(\lambda_i \mu_j)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$$

Recall that for any two graphs G and H the Cartesian product $G \times H$ is defined as follows:

$V(G \times H) = V(G) \times V(H)$ and $(u,v) \sim (u',v')$ if and only if either ($u = u'$ and $v \sim v'$) or ($u \sim u'$ and $v = v'$).

Note that $A(G \times H) = A(G) \otimes I_n + I_m \otimes A(H)$, where $|V(G)| = m$ and $|V(H)| = n$.

The first lemma has the following immediate

Corollary If $\lambda_1, \dots, \lambda_m$ are eigenvalues of G (of $A(G)$) and $\mu_1, \dots, \mu_n$ are eigenvalues of H, then $\lambda_i + \mu_j$ are eigenvalues of $G \times H$, $i = 1, \dots, m$, $j = 1, \dots, n$.

Theorem: Let G be a graph with n vertices. If the energy $\mathcal{E}(G)$ of G is a rational number then it must be an even integer.

Proof: Let $\lambda_1, \dots, \lambda_k$ be the positive eigenvalues of G. Since $0 = tr(A) = \lambda_1 + \dots + \lambda_n$ the sum of the positive eigenvalues is equal to the negative of the sum of the negative eigenvalues.

By the above lemma $\lambda_1 + \dots + \lambda_k$ is an eigenvalue of $G^k = G \times G \times \dots \times G$. The characteristic polynomial of the adjacency matrix of G^k is a monic polynomial with integer coefficient.

Hence, any rational root of the characteristic polynomial is an integer. Hence, $\lambda_1 + \cdots + \lambda_k$ is an integer.

So, $\mathcal{E}(G) = |\lambda_1| + \cdots + |\lambda_n|$

$$= 2(|\lambda_1| + \cdots + |\lambda_k|)$$ is an even integer.

$$= 2(\lambda_1 + \cdots + \lambda_k)$$

∎