METU MATH. DEPT. – Summer 2021
Intr. to Differential Equations

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Textbook: Boyce – Di Prima, Elementary Diff. Equations and Boundary Value Problems, 9th Ed.

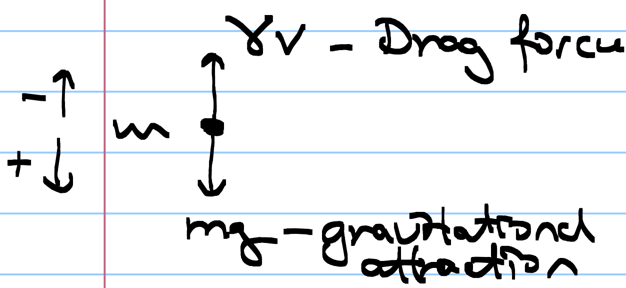
Video 1

§1.1. Some Basic Mathematical Models; Direction Fields

Many of the principles or laws governing the nature are statements or relations involving rates at which things happen. Expressing rates at which things happen in mathematical writing can be done using derivatives. Relations and equations involving some quantities and their derivatives with respect to other variables (quantities) are called differential equations.

Expressing natural processes in mathematical terms using (differential) equations or relations is called mathematical modelling.

Example 1 A falling object.



Total force $F = mg - \gamma v$

Newton's Second Law of Motion: $F = ma$

$\Rightarrow mg - \gamma v = ma$, $v = v(t)$ velocity of the object at time t

$a = a(t)$ acceleration of the object at time t.

We know that $a(t) = \frac{dv}{dt}$.

Hence, our equation becomes

$$mg - \gamma v = m\frac{dv}{dt}.$$

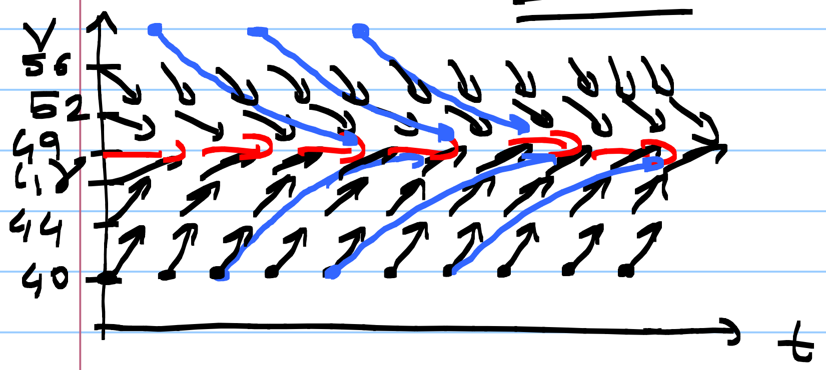
This is a differential equation in $v = v(t)$ and its derivative dv/dt.

$g = 9.8$ kg-m/sec², $\gamma = 2$kg/sec, $m = 10$kg

$$\frac{dv}{dt}10 = 10 \cdot 9.8 - 2v$$ (F = $v\gamma$, kg-m/sec² = m/sec · γ ⇒ γ = kg/sec!)

$$\Rightarrow \frac{dv}{dt} = 9.8 - \frac{v}{5}$$

Direction Field: $v = v(t)$



$\frac{dv}{dt} = 9.8 - \frac{v}{5}$

$v = 40$, $\frac{dv}{dt} = 1.8$

$v = 44$, $\frac{dv}{dt} = 1$

$v = 48$, $\frac{dv}{dt} = 0.2$

$v = 49$, $\frac{dv}{dt} = 0$, $v = 52$, $\frac{dv}{dt} = -0.6$

$v = 56$, $\frac{dv}{dt} = -1.4$

$v = 49$m/sec ⇒ $a = 0$. Equilibrium velocity. This is also called the limiting velocity.

This equilibrium is called stable, because if you move from the equilibrium then a trajectory takes you back to the equilibrium.

Video 2

Example 2: Field Mice and Owl

$p(t)$ = mouse population in a field at time t

In the absence of predators the mouse population increases at a rate proportional to the population, with proportionality constant, say r.

$$\frac{dp}{dt} = rp$$

Now assume that several owls live in this field and they hunt 15 mice per day.

t: time in months, say $r = 0.5$

$$\frac{dp}{dt} = 0.5p - 15 \times 30$$

$$\Rightarrow \frac{dp}{dt} = 0.5p - 450$$

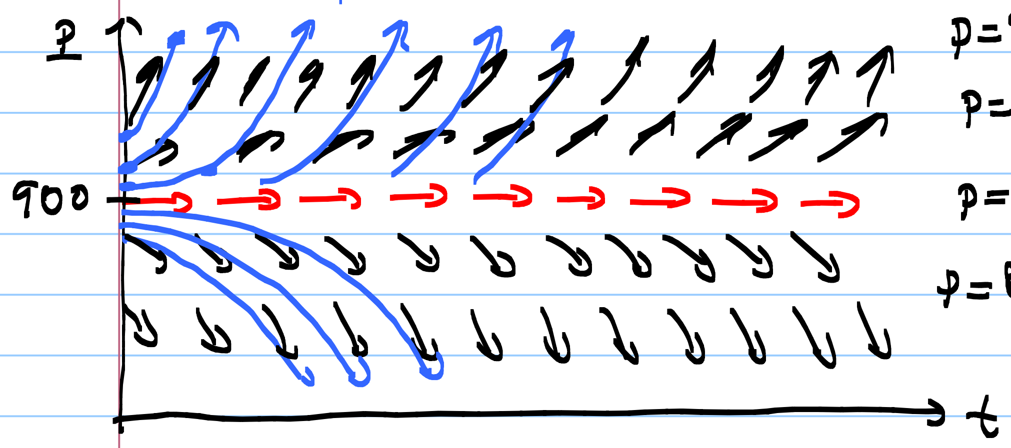
$p = 1000, \frac{dp}{dt} = 50$

$p = 950, \frac{dp}{dt} = 25$

$p = 900, \frac{dp}{dt} = 0$

$p = 850, \frac{dp}{dt} = -25$

$p = 800, \frac{dp}{dt} = -50$



$p = 900$ is the equilibrium $\left(\frac{dp}{dt} = 0 \text{ when } p = 0\right)$ and it is unstable.

We see that unlike the first example this model is not realistic.

§1.2. Solutions of Some Differential equations

Example 1 $\frac{dv}{dt} = 9.8 - \frac{v}{5}$, $v = v(t)$

(First order O.D.E.)

$$\frac{dv/dt}{v-49} = -\frac{1}{5} \Rightarrow \frac{d}{dt}\left(\ln|v-49|\right) = -1/5$$

$$\Rightarrow \ln|v-49| = \int\left(-1/5\right)dt$$

$$= \frac{-t}{5} + c' \qquad (c' \in \mathbb{R})$$

$$\Rightarrow |v-49| = e^{-t/5+c} = e^{c'} e^{-t/5}$$

$$\Rightarrow v - 49 = \pm e^{c'} e^{-t/5}$$

$$\Rightarrow v = 49 + c e^{-t/5}, \quad c \in \mathbb{R}\setminus\{0\}.$$

However, $c = 0$ gives $v = v(t) = 49$ the equilibrium solution.

Any choice of $c \in \mathbb{R}$ gives us a solution. We'll see later that these are all the solutions.

Example 2 $\frac{dp}{dt} = 0.5p - 450$ (First order O.D.E.)

$$\Rightarrow \frac{dp/dt}{p-900} = \frac{1}{2} \Rightarrow \frac{d}{dt}\left(\ln|p-900|\right) = \frac{1}{2}$$

$$\Rightarrow \ln|p-900| = \int \frac{1}{2}\, dt = \frac{t}{2} + c', \quad c' \in \mathbb{R}$$

$$\Rightarrow |P-900| = e^{t/2 + C'} = e^{C'} e^{t/2}$$

$$\Rightarrow P - 900 = \pm e^{C'} e^{t/2} = C e^{t/2}, \quad C \neq 0$$

$$P(t) = 900 + C e^{t/2}, \quad C \in \mathbb{R} \setminus \{0\}.$$

Indeed, $C = 0$ gives us the equilibrium solution $P(t) = 900$.

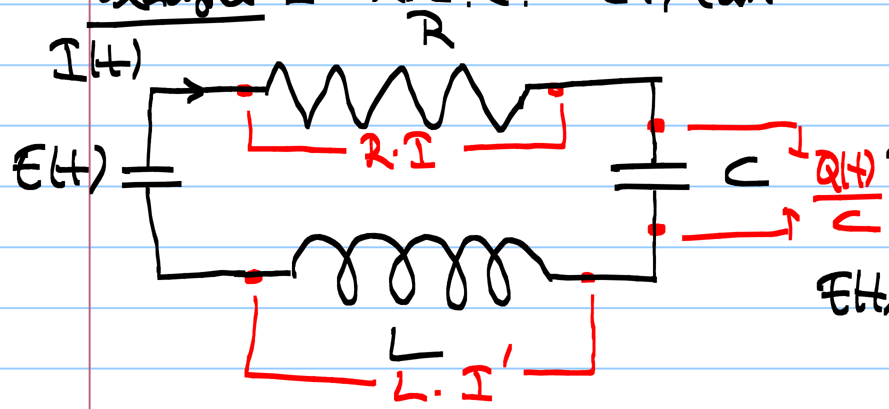
In fact, we'll see that $P(t) = 900 + Ce^{t/2}$, $C \in \mathbb{R}$, is the list of all solutions.

§1.3. Classification of Differential Equations

If a differential equation involves derivatives with respect to a single variable then it is called an Ordinary Differential Equation (O.D.E.). Otherwise, the equation is called a Partial Differential Equation (P.D.E.)

The order of highest derivative term in the equation is called the order of the equation.

Example 1 R.L.C. Circuit



$I(t)$: current at time t

$E(t)$: potential difference applied to the circuit

$Q(t)$: charge in the capacitor C.

Video 3

$$E(t) = R \cdot I + \frac{Q(t)}{C} + L \cdot \frac{dI}{dt}$$

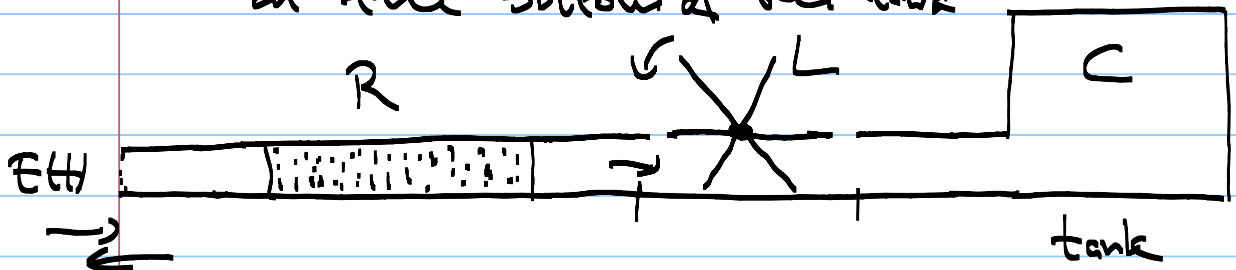
$$I(t) = \frac{dQ}{dt}, \quad \frac{dI}{dt} = \frac{d^2Q}{dt}$$

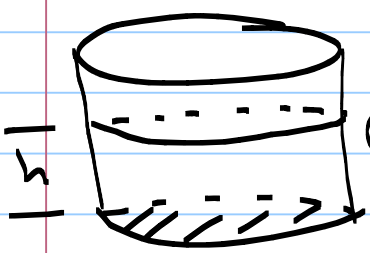
$$\Rightarrow L \cdot \frac{d^2Q}{dt^2} + R \cdot \frac{dQ}{dt} + \frac{1}{C} Q(t) = E(t)$$

2^{nd} order O.D.E.

Example 2 Water flowing in a pipe.

$p(t)$: the pressure of the water at time t at the bottom of the tank





$Q(t)$: amount of water in the tank at time t

$p(t) = h(t)$ height $= \frac{Q(t)}{C}$, C: cross sectional area

$E(t)$: pressure at the beginning of the pipe.

$$E(t) = RI(t) + LI'(t) + \frac{Q(t)}{C} \qquad I = Q'$$
$$= LQ''(t) + RQ' + \frac{1}{C}Q \qquad I' = Q''$$

Linear and Nonlinear Equations

If the equation involves no terms which is in form of product of dependent variables or their derivatives then the equation is called linear. Otherwise, the equation is called nonlinear.

Examples 1) $LQ'' + RQ' + \frac{Q}{C} = E(t)$ is linear.

2) $\underline{\underline{y^2}} + y' = 0$, $y = y(t)$ is non linear.

$y \cdot y' + 3y = e^{t}$ is also non linear $(y = y(t))$

3) $u_{xx} + \underbrace{u_x \cdot u_y}_{\text{non linear}} + 2u = xy$, $u = u(x,y)$

This is a 2^{nd} order non linear P.D.E.

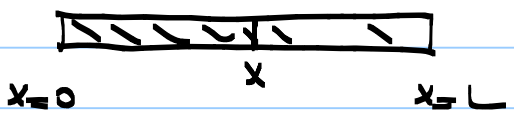
As in the case of algebraic equations linear differential equations have a well established general theory. However, for nonlinear differential equations we don't have a general theory.

An n^{th} order linear O.D.E. has the following form:

$y = y(t)$ t: Independent variable
y: dependent variable

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_{n-1}(t)y' + a_n(t)y = g(t)$$

Heat Equation:



$u(x,t)$: the temp. at x at time t
Then $u(x,t)$ obeys so called the heat equation

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}$$

This is a linear 2nd order P.D.E.

Wave Equation:

$u(x,t)$: the amount of displacement at x at time t.

$u(x,t)$ obeys so called the Wave equation

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$ A 2nd order linear P.D.E

§2.1. Linear Equations; Method of Integrating Factors:

We'll consider 1st order linear equations.

$y = y(t)$, $a_1(t)y'(t) + a_0(t)y(t) = g(t)$

$$\boxed{y' + p(t)y = q(t)}$$

Integrating factor: Let's try to find a function $\mu = \mu(t)$, called an integrating factor, so that once we multiply the equation by $\mu = \mu(t)$ the left hand side becomes the derivative of some function.

$$\left.\begin{aligned} \mu y' + \mu y p &= \mu q \\ \frac{d}{dt}(\mu y) &= \mu q \end{aligned}\right\} \Rightarrow \mu' y + \cancel{\mu y'} = \cancel{\mu y'} + \mu y p$$

$$\Rightarrow \mu' \cancel{y} = \mu \cancel{y} p$$

$$\frac{d\mu/dt}{\mu} = p \Rightarrow \frac{d}{dt}\ln|\mu(t)| = p(t)$$

$$\Rightarrow \ln|\mu(t)| = \int p(t)\,dt$$

$$\Rightarrow |\mu(t)| = e^{\int p(t)\,dt} \quad \Rightarrow \mu(t) = \pm e^{\int p(t)\,dt}.$$

Note using $\mu = \mu(t)$ we obtain

$$\frac{d}{dt}(\mu y) = \mu q \Rightarrow \mu y = \int \mu(t) q(t)\,dt$$

$$y = \frac{1}{\mu}\int \mu(t)\, q(t)\, dt = e^{-\int p(t)\,dt} \int e^{\int p(t)\,dt} q(t)\, dt$$

$\mu = \mu(t) = e^{\int p(t)\,dt}$ is called an integrating factor for the linear equation.

Example: Solve the equation

$$\frac{dy}{dt} + \frac{1}{2} y = \frac{1}{2} e^{t/3}, \quad p(t) = \frac{1}{2}, \quad q(t) = \frac{e^{t/3}}{2}.$$

Integrating factor $\mu = \mu(t) = e^{\int p(t)\,dt} = e^{t/2 + C}$

Just let $C = 0$. $\mu = e^{t/2}$

Multiply the equation by $\mu = \mu(t) = e^{t/2}$ to get

$$e^{t/2} y' + e^{t/2} \frac{1}{2} y = e^{t/2} \frac{1}{2} \cdot e^{t/3}$$

$$\int \frac{d}{dt}\left(e^{t/2} y\right) = \int \frac{1}{2} e^{5t/6}$$

$$e^{t/2} y = \frac{1}{2} \frac{6}{5} e^{5t/6} + C$$

$$\boxed{y = y(t) = \frac{3}{5} e^{t/3} + C e^{-t/2}, \quad C \in \mathbb{R}}$$

Ex 16/p. 40 Solve the Initial Value Problem

$y' + (3/t) y = \cos t / t^3$, $y(\pi) = 0$, $t > 0$.

Video 4

Solution: $y' + \frac{3}{t} y = \frac{\cos t}{t^3}, \quad t > 0.$

$p(t) = \frac{3}{t}, \quad q(t) = \frac{\cos t}{t^3}$

Integrating factor: $\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{3}{t}\,dt} = e^{3\ln t + c}$

$\Rightarrow \mu(t) = e^{c} e^{\ln t^3} = e^{c} t^3$. Just let $C=0$ so that $\mu(t) = t^3$.

Multiply the equation by t^3:

$$t^3 y' + t^3 \frac{3}{t} y = t^3 \cdot \cos t / t^3$$

$$\frac{d}{dt}(t^3 y) = \cos t \Rightarrow t^3 y = \int \cos t\,dt$$

$$t^3 y = \sin t + C, \quad C \in \mathbb{R}$$

$$\boxed{y = \frac{\sin t}{t^3} + \frac{C}{t^3}, \quad C \in \mathbb{R}.}$$

To solve the initial value problem we need to use the initial condition $y(\pi) = 0$.

$$0 = y(\pi) = \frac{\sin \pi}{\pi^3} + \frac{C}{\pi^3} = \frac{C}{\pi^3} \Rightarrow C = 0.$$

Hence, the solution of the I.V.P. $y = \frac{\sin t}{t^3}$.

If the initial condition were $y(\pi) = 1$, then we would have

$$1 = y(\pi) = \frac{\sin \pi}{\pi^3} + \frac{C}{\pi^3} = \frac{C}{\pi^3} \Rightarrow C = \pi^3.$$

Hence, the solution of the I.V.P. would be $y = (\sin t + \pi^3)/t^3$.

§ 2.2. Seperable Equations:

$y = y(x)$, $\quad \frac{dy}{dx} = f(x, y)$ General form of 1^{st} order equations.

If $f(x,y) = -p(x)y + q(x)$ (linear in y), then $y' = -p(x)y + q(x) \Rightarrow y' + p(x)y = q(x)$, just the general form of linear equations.

If the function $f(x,y)$ has the form

$f(x,y) = \frac{-M(x)}{N(y)}$ then the equation is called seperable.

In this case the equation becomes

$$\frac{dy}{dx} = f(x,y) = \frac{-M(x)}{N(y)} \Rightarrow \frac{dy}{dx} N(y) = -M(x)$$

or $M(x)\,dx + N(y)\,dy = 0$.

$$N(y) \frac{dy}{dx} = -M(x)$$

$$N(y)\, y'(x) = -M(x)$$

$$\Rightarrow \int N(y)\, y'(x)\, dx = -\int M(x)\, dx$$

Let $\tilde{N}(y)$ and $\tilde{M}(x)$ be antiderivatives for $N(y)$ and $M(x)$. Then

$$\frac{d\tilde{N}(y)}{dx} = \frac{d\tilde{N}(y)}{dy} \frac{dy}{dx} = N(y)\, y'$$

$$-\tilde{M}(x)+c_1=\int \frac{d\tilde{N}(y)}{dx}\,dx=\tilde{N}(y)+c_2$$

$$\Rightarrow \tilde{N}(y)+\tilde{M}(x)=c_1-c_2=C$$

$$\boxed{\tilde{N}(y)+\tilde{M}(x)=C,\quad C\in\mathbb{R}}$$

Examples: 1) Solve the equation

$$y'=\frac{3x^2-e^x}{2y-4}$$

Solution (17, p. 48) $y'=f(x,y)=(3x^2-e^x)\,\frac{1}{2y-4}$.

$$\frac{dy}{dx}=\frac{3x^2-e^x}{2y-4}\Rightarrow (3x^2-e^x)\,dx=(2y-4)\,dy$$

$$3x^2-e^x=(2y-4)\frac{dy}{dx}$$

$$\int(3x^2-e^x)\,dx=\int(2y-4)\frac{dy}{\cancel{dx}}\cancel{dx}$$

$$x^3-e^x+c_1=y^2-4y+c_2$$

$$\boxed{y^2-4y-x^3+e^x=c,\quad c\in\mathbb{R}}$$

Example (23, p. 49) Solve the I.V.P.

$$y'=2y^2+xy^2;\quad \underline{y(0)=1}.$$

Solution: $y'=(2+x)y^2\Rightarrow \frac{dy}{dx}=(2+x)y^2$

$$\Rightarrow (2+x)\,dx=\frac{dy}{y^2}$$

$$\int (2+x)\,dx = \int \frac{dy}{y^2} \Rightarrow 2x + \frac{x^2}{2} + c_1 = -\frac{1}{y} + c_2$$

$$\Rightarrow \frac{1}{y} + 2x + \frac{x^2}{2} = c, \quad c \in \mathbb{R}.$$

$$y(0) = 1 \Rightarrow \frac{1}{1} + 2.0 + \frac{0^2}{2} = c \Rightarrow c = 1.$$

$$\frac{1}{y} + 2x + \frac{x^2}{2} = 1 \Rightarrow \frac{1}{y} = 1 - \frac{x^2}{2} - 2x$$

$$= \frac{2 - x^2 - 4x}{2}$$

$$\boxed{y = \frac{2}{2 - 4x - x^2}}$$

Homogeneous Equations:

$y' = \frac{dy}{dx} = \frac{ay + bx}{cy + dx}$ To solve this equation we make a variable change.

We replace the dependent variable $y = y(x)$ with $v = v(x) = \frac{y}{x}$.

So $xv = y$ and $y' = (xv)' = v + x.v'$

$$\Rightarrow v + xv' = \frac{ay + bx}{cy + dx} = \frac{a\,y/x + b}{c\,y/x + d}$$

$$v + xv' = \frac{av + b}{cv + d} \Rightarrow xv' = \frac{av + b}{cv + d} - v$$

so that it is seperable.

Example (30, p. 50) Solve the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}.$$

Video 5

A function $f(x,y)$ is called homogeneous if $f(x,y) = g(y/x)$ for some function g.

In this case, the differential equation $\frac{dy}{dx} = f(x,y) = g(y/x)$ is called homogeneous.

Back to the Example: $\frac{dy}{dx} = \frac{y-4x}{x-y}$.

Solution: $f(x,y) = \frac{y-4x}{x-y} = \frac{y/x - 4}{1 - y/x} = g(y/x)$,

where $g(v) = \frac{v-4}{1-v}$, $v = y/x$.

$$\frac{dy}{dx} = \frac{v-4}{1-v}, \qquad v = \frac{y}{x} \Rightarrow y = xv$$

$$\Rightarrow y' = 1 \cdot v + x \cdot v'$$

$$\Rightarrow v + xv' = \frac{v-4}{1-v} \Rightarrow xv' = \frac{v-4}{1-v} - v$$

$$\Rightarrow xv' = \frac{v-4-v+v^2}{1-v}$$

$$\Rightarrow xv' = \frac{v^2-4}{1-v} \Rightarrow x\frac{dv}{dx} = \frac{v^2-4}{1-v}$$

$$\int \frac{1-v}{v^2-4}\,dv = \int \frac{dx}{x} \Rightarrow \int \left(\frac{A}{v-2} + \frac{B}{v+2}\right) dv = \ln|x| + C$$

$A(v+2) + B(v-2) = 1-v$

$(A+B)v + (2A-2B) = 1-v$

$2/\ A+B = -1 \Rightarrow B = -3/4$

$1/\ 2A-2B = 1$

$4A = -1 \Rightarrow A = -1/4$

$$-\int \left(\frac{1/4}{v-2} + \frac{3/4}{v+2}\right) dv = \ln|x| + C$$

$$-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| + C$$

$$\Rightarrow \ln|v-2| + 3\ln|v+2| + 4\ln|x| = C$$

$$\ln|(v-2)(v+2)^3 x^4| = C$$

$$\Rightarrow (v-2)(v+2)^3 x^4 = C$$

$$\Rightarrow \left(\frac{y}{x}-2\right)\left(\frac{y}{x}+2\right)^3 x^4 = C$$

$$\Rightarrow \boxed{(y-2x)(y+2x)^3 = C.}$$

Remark: For a homogeneous equation $y' = \dfrac{ay+bx}{cy+dx}$

If the variable change $v = y/x$ does not yield "computable" integrals you may try the variable change $v = \dfrac{x}{y}$, where we regard

$x = x(y)$ and $v = v(y)$.

§ 2.3. Modelling with First Order Equations

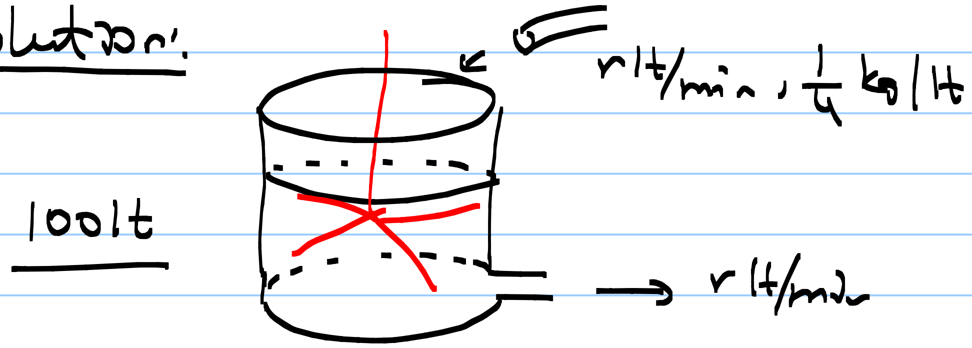
I'll explain construction models on some examples.

Example: (Mixing Problem)

At time $t=0$ a tank containing Q_0 kg of salt dissolved in 100 lt of water.
Assume that the water containing $1/4$ kg/lt salt is entering the tank at a rate of r lt/min, and the well-stirred mixture is draining from the tank at the same rate. Determine the amount of salt in the tank at any time.

Determine the limiting salt concentration of the water in the tank.

Solution:



t: time in minutes

$Q(t)$ = amount of salt in the tank at time t (kg)

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$
$$= r \cdot \frac{1}{4} - r \cdot c(t), \quad c(t): \text{salt concentration in the tank}$$

$$c(t) = \frac{Q(t)}{\text{Volume}} = \frac{Q(t)}{100}$$

$$\frac{dQ}{dt} = r\frac{1}{4} - \frac{r}{100}Q \Rightarrow \frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4},$$

Linear Equation.

$Q = Q(t)$. $\quad P(t) = \frac{r}{100}$

Integrating factor $\mu = \mu(t) = e^{\int p(t)\,dt} = e^{\frac{rt}{100}}$.

$$\Rightarrow e^{rt/100} Q' + e^{rt/100}\frac{r}{100}Q = \frac{r}{4}e^{rt/100}$$

$$\Rightarrow \int \frac{d}{dt}\left(Q e^{rt/100}\right)dt = \int \frac{r}{4} e^{rt/100}\,dt$$

$$Q e^{rt/100} = \frac{r}{4} e^{rt/100} \cdot \frac{100}{r} + C, \quad C \in \mathbb{R}.$$

$$Q(t) = 25 + C\, e^{-rt/100}, \quad C \in \mathbb{R}.$$

$$Q_0 = Q(0) = 25 + C \cdot e^{0} = 25 + C$$

$$\Rightarrow C = Q_0 - 25.$$

Hence, $Q(t) = 25 + (Q_0 - 25)\, e^{-rt/100}$ kg

$$\lim_{t \to \infty} Q(t) = \lim_{t \to \infty} 25 + (Q_0 - 25)\, e^{-rt/100}$$
$$= 25 + 0 \text{ kg}.$$

Limiting salt concentration is $25/100 = 1/4$ kg/lt.

<u>Example</u>: (Newton's Law of Cooling)

A dead body is found at midnight in Ankara. The CSI people took liver temperatures at 1 a.m and 2 a.m and measured 35°C and 30°C, respectively. The outside temperature is 20°C. Determine the time of murder.

<u>Solution</u>: $T(t)$: liver temperature at time t.

$T(1) = 35$, $T(2) = 30$, t: time

Newton's law of cooling: Temperature of an object changes at a rate proportional to the difference of the temperature of the surrounding and the object.

$$\frac{dT}{dt} = r(20 - T)$$

(Separable Eqn.)

Video 6

$$\Rightarrow \int \frac{dT}{20-T} = \int r\,dt \Rightarrow -\ln|20-T| = rt + c$$

$$|20-T| = e^{-rt+c} \Rightarrow 20-T = \pm e^{c} e^{-rt}$$

$$\Rightarrow T(t) = 20 \pm e^{c} e^{-rt}$$

$$\Rightarrow T(t) = 20 + c\,e^{-rt}, \quad c \in \mathbb{R},\ r \in \mathbb{R}.$$

$T(1) = 35$ and $T(2) = 30$

$$20 + c\,e^{-r} = 35 \Rightarrow c\,e^{-r} = 15$$
$$20 + c\,e^{-2r} = 30 \Rightarrow 20 + c\,e^{-r} e^{-r} = 30$$

$$\Rightarrow 20 + 15 e^{-r} = 30 \Rightarrow 15 e^{-r} = 10$$
$$e^{-r} = 10/15 = 2/3, \quad r = -\ln 2/3 < 0.$$

$$c\,e^{-r} = 15 \Rightarrow c \cdot \frac{2}{3} = 15 \Rightarrow c = \frac{45}{2}.$$

So, $T(t) = 20 + c\,e^{-rt}$.

$$T(t) = 20 + \frac{45}{2} e^{t \ln \frac{2}{3}} = 20 + \frac{45}{2}\left(\frac{2}{3}\right)^t$$

Check: $T(1) = 20 + \frac{45}{2}\left(\frac{2}{3}\right)^1 = 35$ ✓

$$T(2) = 20 + \frac{45}{2}\left(\frac{2}{3}\right)^2 = 20 + \frac{\cancel{45}^{5}}{\cancel{2}} \cdot \frac{\cancel{4}^{2}}{\cancel{9}} = 30\ ✓$$

If we take limit as $t \to \infty$, $\lim_{t \to \infty} T(t) = 20.$

$$T(t) = 20 + \frac{45}{2}\left(\frac{2}{3}\right)^t$$

At the time of murder say at t_0 the temperature must be $36{,}5\,^\circ C$. Hence

$$T(t_0) = 36{,}5 \Rightarrow 20 + \frac{45}{2}\left(\frac{2}{3}\right)^{t_0} = 36{,}5$$

$$\Rightarrow \left(\frac{2}{3}\right)^{t_0} = \frac{16{,}5 \times 2}{45} = \frac{33}{45} = \frac{11}{15}$$

$$t_0 \ln\frac{2}{3} = \ln 11/15 \Rightarrow t_0 = \frac{\ln 11/15}{\ln 2/3} \simeq 0.76$$

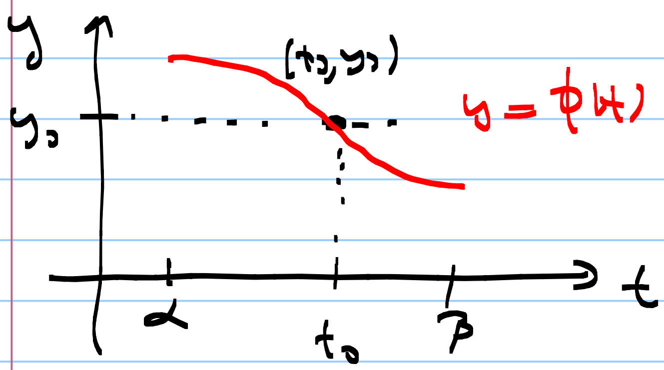
$$\simeq 00:45$$

§2.4. Differences Between Linear and Nonlinear Equations:

Theorem: Consider the 1^{st} order linear O.D.E. $y' + p(t)y = q(t)$. If $p(t)$ and $q(t)$ are continuous on some open interval $I: \alpha < t < \beta$, containing a point t_0 then for any $y_0 \in \mathbb{R}$ the initial value problem

$$\begin{cases} y' + p(t)y = q(t) \\ y(t_0) = y_0 \end{cases}$$

has a unique solution $y = \phi(t)$ defined on the interval $I = (\alpha, \beta)$.



$\phi'(t) + p(t)\phi(t) = q(t)$ $\forall t \in (\alpha, \beta)$ and $\phi(t_0) = y_0$.

This theorem is called the Existence-Uniqueness Theorem for 1^{st} order linear Equations.

Recall that we proved the existence part of the above theorem using integrating factors. However, we did not say anything about the uniqueness statement.

The story for non linear equation is quite different.

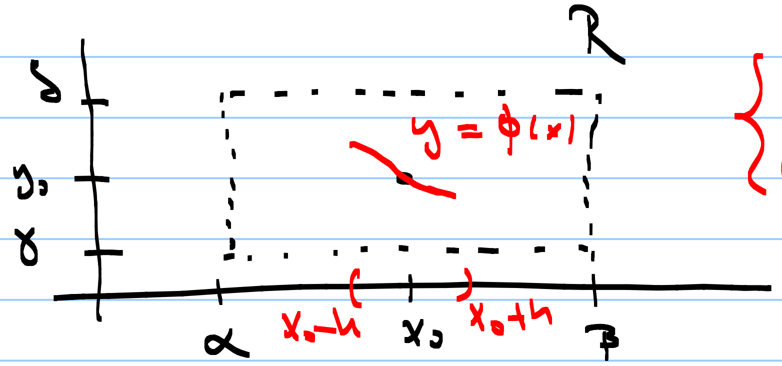
Theorem (Existence–Uniqueness Thm for 1st O.D.E.'s)
Consider the initial value problem below

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

Assume there is a rectangle $R: \alpha < x < \beta,\ \gamma < y < \delta$

so that $(x_0, y_0) \in R$ and both $f(x,y)$ and $\partial f / \partial y$ are continuous on the rectangle R.

Then there is a unique solution $y = \phi(x)$ to the above I.V.P. define on some interval $(x_0 - h, x_0 + h)$ for some $h > 0$.



$$\begin{cases} \phi'(x) = f(x, \phi(x)) \\ \phi(x_0) = y_0 \end{cases}$$

Sketch of the proof: $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ $\quad \alpha < x < \beta$, $\gamma < y < \delta$

Convert this to an integral equation (you may watch my Math 349 Videos).

$$y(x) - y(x_0) = \int_{x_0}^{x} y'(t)\,dt = \int_{x_0}^{x} f(t, y(t))\,dt$$

$$\Rightarrow \boxed{y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt.}$$

Next we solve the Integral Equation

Form a sequence of functions:

$y = \phi_0(x) = y_0$ the constant function.

Let $y = \phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt$

Similarly, let $y = \phi_n(x) = \int_{x_0}^{x} f(t, \phi_{n-1}(t))\,dt$

So we obtain a sequence of functions $(\phi_n(x))$.

Finally, we show that this sequence $(\phi_n(x))$ converges to a solution and one proves that the solution is unique.

Let $\phi(x) = \lim_{n\to\infty} \phi_n(x)$. (Picard Iteration)

$$\phi_n(x) = y_0 + \int_{x_1}^{x} f(t, \phi_{n-1}(t))\,dt$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt.$$

Here, $\phi(x)$ solves the Integral equation.

This finishes the proof. ∎

Video 7

Example 1: Solve the I.V.P.

$$\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2 \end{cases} \quad \text{and} \quad \begin{cases} ty' + 2y = 4t^2 \\ y(-1) = 2 \end{cases}$$

$0 < t < \infty \qquad\qquad -\infty < t < 0$

Solution: $y' + \frac{2}{t} y = 4t$, $p(t) = \frac{2}{t}$, $q(t) = 4t$

Integrating factor: $\mu(t) = e^{\int p(t)\,dt}$
$= e^{\int \frac{2dt}{t}}$
$= t^2$

$t^2 \cdot y' + t^2 \cdot \frac{2}{t} y = t^2 \cdot 4t \Rightarrow t^2 y' + 2ty = 4t^3$

$\Rightarrow \frac{d}{dt}(t^2 y) = 4t^3 \Rightarrow t^2 y = \int 4t^3 dt$

$\Rightarrow t^2 y = t^4 + C \Rightarrow y = t^2 + \frac{C}{t^2}$.

1) $y(1) = 2$

$2 = 1^2 + \frac{C}{1^2} \Rightarrow C = 1$

Hence the unique solution is

$y = t^2 + \frac{1}{t^2}$, $0 < t < \infty$.

2) $y(-1) = 2$

$2 = (-1)^2 + \frac{C}{(-1)^2} \Rightarrow C = 1$

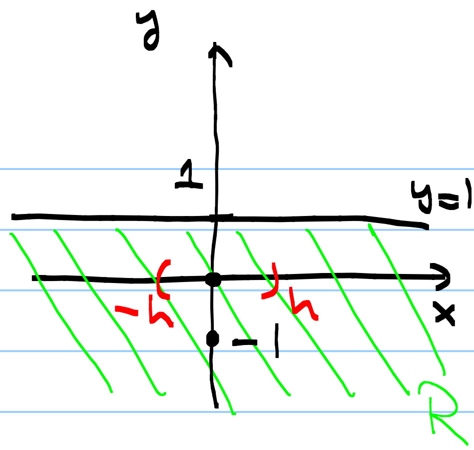
$y = t^2 + \frac{1}{t^2}$

$-\infty < t < 0$

Example 2 Solve the I.V.P.

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1.$$

$$\frac{dy}{dx} = f(x, y), \quad f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}.$$



$$\frac{\partial f}{\partial y} = -\frac{3x^2+4x+2}{2(y-1)^2}$$

R: $-\infty < x < \infty, \; -\infty < y < 1$

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \Rightarrow \int (3x^2+4x+2)\,dx = \int 2(y-1)\,dy$$

$$\Rightarrow x^3+2x^2+2x+c_1 = (y-1)^2 + c_2$$

$$\Rightarrow (y-1)^2 = x^3+2x^2+2x+C$$

$$y = 1 \pm \sqrt{x^3+2x^2+2x+C}\,, \quad y(0) = -1.$$

$$\Rightarrow y = 1-\sqrt{x^3+2x^2+2x+C} \Rightarrow -1 = 1-\sqrt{C} \Rightarrow -\sqrt{C} = -2$$

$$\Rightarrow C = 4.$$

Hence, $y = 1-\sqrt{x^3+2x^2+2x+4}$ is the unique solution.

Note that $x=-3$ makes $x^3+2x^2+2x+4 = -27+18-6+4 = -11 < 0$.

Thus the interval $(-h,h)$ should be contained in $(-3,\infty)$. The solution is defined on an interval of the form $(-2,\infty)$, where $a^3+2a^2+2a+4=0$,

Example 3. Solve the I.V.P. $\begin{cases} y' = y^2 \\ y(0) = y_0 \end{cases}$.

Solution: $\dfrac{dy}{dx} = y^2 \Rightarrow \dfrac{dy}{y^2} = dx \Rightarrow -\dfrac{1}{y} = x + c$

$$\Rightarrow y = \frac{-1}{x+c}. \quad y_0 = y(0) \Rightarrow y_0 = \frac{-1}{c}$$

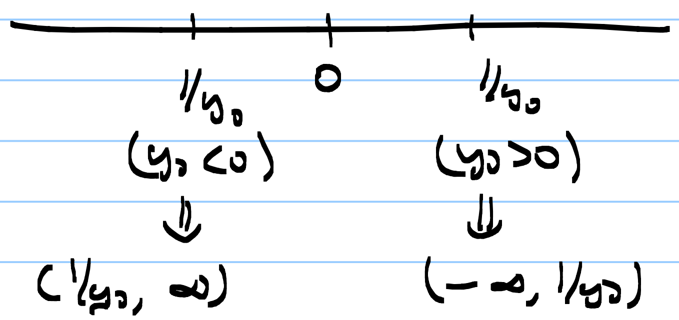
$$\Rightarrow c = -1/y_0.$$

Hence, the unique solution is $y = \dfrac{-1}{x + (-1/y_0)}$

$$\Rightarrow y = \frac{-y_0}{y_0 x - 1} = \frac{y_0}{1 - y_0 x}.$$

Note that $f(x,y) = y^2$, $\frac{\partial f}{\partial y} = 2y$, which are continuous on the whole plane. So we may take the rectangle R to be the whole plane. $R = \mathbb{R}^2 : -\infty < x < \infty,\ -\infty < y < \infty$.

On the other hand solution is defined only on the interval



§ 2.6. Exact Equations and Integrating factors:

An equation of the form
$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$
is called called if there is a function $\psi = \psi(x,y)$ so that $\psi_x = M$ and $\psi_y = N$.

Note that if such ψ exists then our equation becomes
$$\psi_x(x,y) + \psi_y(x,y)\frac{dy}{dx} = 0.$$ Since $y = y(x)$

this becomes $\psi_x(x, y(x)) + \psi_y(x,y)\frac{dy}{dx} = 0.$

Video 8

The left hand side is nothing but the derivative $\frac{d}{dx}\left(\psi(x,y(x))\right)$.

Hence, the equation becomes

$\frac{d}{dx}\left(\psi(x,y(x))\right) = 0$. Thus $\psi(x,y(x)) = C$, for some $C \in \mathbb{R}$.

So the solution is $\boxed{\psi(x,y) = C,\ C \in \mathbb{R}.}$

<u>Example:</u> Solve the equation

$(2x+y^2)\,dx + 2xy\,dy = 0.$

<u>Solution:</u> $2x+y^2+2xy\frac{dy}{dx} = 0.$

$M(x,y) = 2x+y^2, \quad N(x,y) = 2xy.$

Let's check if it is exact: Look for a function $\psi = \psi(x,y)$ satisfying

$\psi_x = M = 2x+y^2$ and $\psi_y = N = 2xy.$

$\Downarrow$

$\psi = x^2 + xy^2 + h(y) \implies \psi_y = 0 + 2xy + h'(y)$

$\implies 2xy + h'(y) = 2xy$

$\implies h'(y) = 0.$

$\implies h(y) = C.$

Hence, $\psi = \psi(x,y) = x^2 + xy^2 + C$ so that the equation is exact. In particular, the general solution is $\psi(x,y) = C'$.

$$\Rightarrow x^2 + xy^2 + C = C' \Rightarrow \boxed{\begin{array}{l} x^2 + xy^2 = C \\ C \in \mathbb{R} \end{array}}$$

<u>Question</u>: How to check whether an equation is exact or not?

$M(x,y) + N(x,y)\frac{dy}{dx} = 0$. If it is exact then

then is $\psi = \psi(x,y)$ with $\psi_x = M$ and $\psi_y = N$.
If M and N are continuously differentiable then
$\psi_{xy} = M_y$ and $\psi_{yx} = N_x$.

Hence, we get $M_y = \psi_{xy} = \psi_{yx} = N_x$.

Thus if $M_y \neq N_x$ then the equation cannot be exact.

<u>Example</u>: Decide if the equation

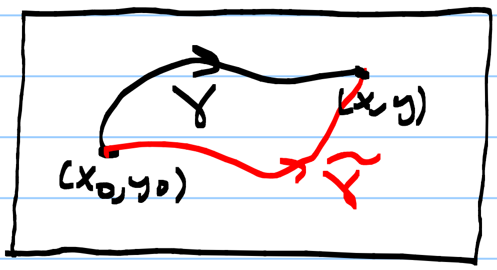
$$2xy + (x+y)\frac{dy}{dx} = 0$$ is exact.

<u>Solution</u>: $M(x,y) = 2xy$, $N(x,y) = x+y$.

$M_y = 2x$ and $N_x = 1$. Since $M_y \neq N_x$ as functions the equation is <u>not</u> exact.

<u>Theorem</u>: Let $M, N, M_y = \frac{\partial M}{\partial y}$ and $N_x = \frac{\partial N}{\partial x}$ be continuous in the rectangle $R: \alpha < x < \beta, \gamma < y < \delta$.
Then the equation
$M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact if and only if $M_y = N_x$ on R.

Sketch of Proof: $M_y = N_x$ on R.

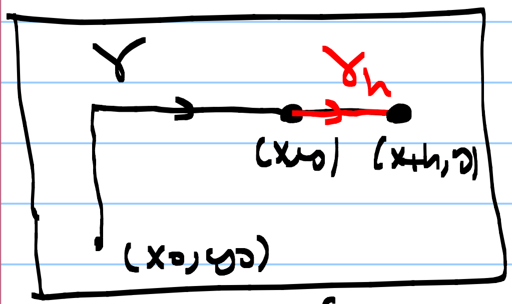


$$\psi(x,y) = \int_\gamma M(x,y)\,dx + N(x,y)\,dy$$

$\psi(x,y)$ is well defined because $M_y = N_x$ so that the integral is path independent.

must check: $\psi_x = M$ and $\psi_y = N$.

$$\psi(x,y) = \int_\gamma M(x,y)\,dx + N(x,y)\,dy$$



$$\psi_x(x,y) = \lim_{h\to 0} \frac{\psi(x+h,y)-\psi(x,y)}{h}$$

$$\psi_x(x,y) = \lim_{h\to 0} \frac{\int_{\cancel{\gamma}+\gamma_h} M\,dx + N\,dy \cancel{- \int_\gamma M\,dx + N\,dy}}{h}$$

$$= \lim_{h\to 0} \frac{\int_{\gamma_h} M\,dx + N\,dy}{h}$$

$\gamma_h(t) = (x+t, y)$

$x = x(t) = x+t$

$y = y(t) = y$

$dx = dt,\ dy = 0$

$$= \lim_{h\to 0} \frac{\int_0^h M(x+t,y)\,dt}{h}$$

If $\tilde{M}(t)$ is an antiderivative for $M(x+t,y)$ with respect to t ($\tilde{M}'(t) = M(x+h,y)$.), then

$$= \lim_{h \to 0} \frac{\widetilde{M}(h) - \widetilde{M}(0)}{h}$$

$$= \widetilde{M}'(0) = M(x+h, y)\Big|_{h=0} = M(x,y).$$

Hence, $\psi_x(x,y) = M(x,y)$.

Similarly, one can show that $\psi_y(x,y) = N(x,y)$. Thus the equation is exact. ∎

<u>Remark:</u> The region R in the statement of the theorem could be replaced by any simply connected region.

On the other hand, the theorem fails if R is <u>not</u> simply connected.

<u>Example:</u> $\theta = \theta(x,y) = \tan^{-1}\frac{y}{x}$.

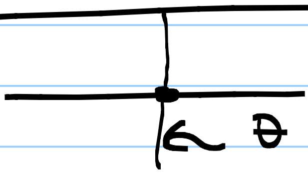
$$d\theta = \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$M = M(x,y) = \frac{-y}{x^2+y^2}, \quad N = N(x,y) = \frac{x}{x^2+y^2}$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \Rightarrow \frac{y}{x^2+y^2} - \frac{x}{x^2+y^2}\frac{dy}{dx} = 0.$$

Note that the equation is defined on $\mathbb{R}^2 \setminus \{(0,0)\}$. Integrating this equation one gets the solution

$$\boxed{\theta = \theta(x,y) = \tan^{-1}\frac{y}{x} = C, \quad C \in \mathbb{R}}$$



← θ is not defined on the origin.

Video 9

Here, we do not a solution on $\mathbb{R}^2 \setminus \{(0,0)\}$.
The theorem does not apply here since the region $\mathbb{R}^2 \setminus \{(0,0)\}$ is not simply connected.

Integrating factor: Suppose that the given equation
$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$ is not exact.

Question: Can we find a function $\mu = \mu(x,y)$ so that after multiplying the equation with $\mu = \mu(x,y)$ the equation becomes exact?

If such $\mu = \mu(x,y)$ exists then we call it an integrating factor for the equation.

Want $\mu(x,y)$ so that $M(x,y)\mu(x,y) + N(x,y)\mu(x,y)\frac{dy}{dx} = 0$ is exact.

$$\frac{\partial}{\partial y}(M\mu) = \frac{\partial}{\partial x}(N\mu). \qquad \mu = \mu(x,y)\ ?$$

$$M_y \mu + M \mu_y = N_x \mu + N \mu_x. \quad \text{P.D.E. in } \mu.$$

Assumption: Assume that $\mu = \mu(x,y)$ is a function of x only or a function of y only.

Case 1 $\mu = \mu(x)$ function of x only.

Hence, $\mu_y = 0$ and $\mu_x = d\mu/dx$.

$$M_y \mu = N_x \mu + N\, d\mu/dx$$

$$\Rightarrow (M_y - N_x)\mu = N\frac{d\mu}{dx}$$

$$\Rightarrow \frac{M_y - N_x}{N} = \frac{d\mu/dx}{\mu}.$$

Note that the R.H.S. μ'/μ is a function of x only. Hence, if the L.H.S. $(M_y - N_x)/N$ is a function of x only then we can solve this equation for μ (separable eqn.) and obtain an integrating factor for the original O.D.E.

<u>Case 2</u> $\mu = \mu(y)$ function of y only.

Then $\mu_x = 0$ and $\mu_y = \frac{d\mu}{dy}$. Hence the equation for μ becomes

$$\mu_y M + \mu M_y = \underbrace{\mu_x}_{=0} N + \mu N_x$$

$$\Rightarrow \mu_y M = (N_x - M_y)\mu \Rightarrow \frac{d\mu/dy}{\mu} = \frac{N_x - M_y}{M}$$

This time the L.H.S. is a function of y only. Thus, if $(N_x - M_y)/M$ is a function of y only then we can obtain an integrating factor $\mu = \mu(y)$ for the original equation.

<u>Examples:</u> 1) Solve the equation

$$(y^2 + xy) + (3xy + x^2)y' = 0.$$

<u>Solution:</u> Let $M = y^2 + xy$ and $N = 3xy + x^2$.

Then $M_y = 2y + x$ and $N_x = 3y + 2x$. Since

$M_y \neq N_x$ as functions the equation is not exact.

Look for an integrating factor:

Case 1 $\mu = \mu(x)$ function of x only.

$M\mu + (N\mu)\frac{dy}{dx} = 0$ has to be exact.

$$(M\mu)_y = (N\mu)_x \Rightarrow M_y \mu + M \underbrace{\mu_y}_{=0} = N_x \mu + N \underbrace{\mu_x}_{= d\mu/dx}$$

$$\Rightarrow \frac{d\mu/dx}{\mu} = \frac{M_y - N_x}{N} = \frac{(2y+x)-(3y+2x)}{3xy+x^2}$$

$$= \frac{-x-y}{3xy+x^2}$$ not a function of x only.

Case 2 $\mu = \mu(y)$ function of y only.

$$\frac{d\mu/dy}{\mu} = \frac{N_x - M_y}{M} = \frac{x+y}{y^2+xy} = \frac{x+y}{y(y+x)} = \frac{1}{y},$$ a function of y only.

$\Rightarrow \frac{d\mu}{\mu} = \frac{dy}{y}$. Hence, $\mu = \mu(y) = y$ is a solution.

$$(y^3 + xy^2) + (3xy^2 + yx^2)\frac{dy}{dx} = 0. \quad (R = \mathbb{R}^2)$$

This equation must be exact!

$$(y^3 + xy^2)_y \stackrel{?}{=} (3xy^2 + yx^2)_x$$

$$\Rightarrow 3y^2 + 2xy \stackrel{?}{=} 3y^2 + 2xy \checkmark$$

Let $\psi = \psi(x,y)$ be so that $\psi_x = y^3 + xy^2$ and $\psi_y = 3xy^2 + yx^2$.

Find ψ.

$\psi_x = y^3 + xy^2 \Rightarrow \psi(x,y) = xy^3 + \frac{x^2y^2}{2} + h(y)$

$h(y) = ?$

$\underline{3xy^2} + \underline{yx^2} = \psi_y = \underline{3xy^2} + \underline{x^2y} + h'(y).$

$\Rightarrow h'(y) = 0 \Rightarrow h(y) = C.$

So, $\psi(x,y) = xy^3 + \frac{x^2y^2}{2} + C$ and the solution of the equation is just

$$\boxed{xy^3 + \frac{x^2y^2}{2} + C = 0, \quad C \in \mathbb{R}.}$$

27/p. 102 Solve the equation $y + (2xy - ye^{-2y})y' = 0$.

<u>Solution</u> $M(x,y) = y$, $N(x,y) = 2xy - ye^{-2y}$.

$M_y = 1$, $N_x = 2y$ and thus the equation is <u>not</u> exact.

Look for integrating factor:

<u>Case 1</u> $\mu = \mu(x)$.

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \Rightarrow \frac{d\mu/dx}{\mu} = \frac{1-2y}{2xy - ye^{-2y}}$$

not a function of x only.

Case 2 $\mu = \mu(y)$

$$\frac{d\mu/dy}{\mu} = \frac{N_x - M_y}{M} = \frac{2y-1}{y} = 2 - \frac{1}{y}.$$

$$\frac{d\mu}{\mu} = \left(2 - \frac{1}{y}\right) dy \Rightarrow \ln|\mu| = 2y - \ln y$$

$$\Rightarrow \mu = e^{2y} e^{-\ln y} = \frac{e^{2y}}{y}.$$

$$y + (2xy - y e^{-2y}) y' = 0 \;\Big/\; \frac{e^{2y}}{y}.$$

$$\Rightarrow \underbrace{e^{2y}}_{M} + \underbrace{(2x e^{2y} - 1)}_{N} y' = 0.$$

Check if it is exact: $M_y = 2e^{2y}$
$N_x = 2e^{2y}$

Yes, it is exact!

$\Psi = \Psi(x,y)$, $\Psi_x = M$ and $\Psi_y = N$.

$\Psi_x = e^{2y} \Rightarrow \Psi = x e^{2y} + h(y)$. $h(y) = ?$

$N = \Psi_y \Rightarrow 2x e^{2y} - 1 = 2x e^{2y} + h'(y).$

$\Rightarrow h'(y) = -1 \Rightarrow h(y) = y + C.$

Hence, $\Psi(x,y) = \boxed{x e^{2y} + y + C = 0 \quad C \in \mathbb{R}.}$

If we were given an initial condition say, $y(0) = 5$, then

$$0 \cdot e^{10} + 5 + C = 0 \Rightarrow C = -5.$$

$x e^{2y} + y - 5 = 0$ is the unique solution 4 the I.V.P.

Video 10

CHAPTER 7: Systems of First Order Linear Equations

t: independent variable

$y_1(t), \ldots, y_n(t)$ dependent variables

Example:
$$\begin{cases} y_1'' + t y_2'' - 3\cos t \, y_1' + 5 y_2' - 7y_1 + 2y_2 = e^t \cos 2t \\ 3y_1'' - 5y_2'' + e^t y_1' + \cos t \, y_2' + 8y_1 + 3y_2 = 0 \end{cases}$$

Linear 2^{nd} order system of O.D.E.'s.

In this chapter we'll consider only first order systems of equations. The general form of such a system is as follows:

$$AY' + BY = C, \text{ where } Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \; Y' = \begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix}$$

A, B, C are matrices of functions of t.

$$A = (a_{ij}(t))_{n \times n}, \quad B = (b_{ij}(t))_{n \times n}, \quad C = (c_i(t))_{n \times 1}$$

Example:
$$\begin{aligned} 3t\, y_1' - 5 y_2' + e^t y_1 - \cos t \, y_2 &= t \\ y_1' + y_2' - e^{-t} y_1 + \sin t \, y_2 &= \cos 3t \end{aligned}$$

$$\begin{bmatrix} 3t & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} + \begin{bmatrix} e^t & -\cos t \\ -e^{-t} & \sin t \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} t \\ \cos 3t \end{bmatrix}.$$

For simplicity we'll consider the case when $A = Id$.

$$\begin{aligned} y_1' &= a_{11}(t) y_1 + \cdots + a_{1n}(t) y_n + b_1(t) \\ y_2' &= a_{21}(t) y_1 + \cdots + a_{2n}(t) y_n + b_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t) y_1 + \cdots + a_{nn}(t) y_n + b_n(t) \end{aligned}$$

$$Y' = AY + B, \text{ where } Y = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \ Y' = \begin{bmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{bmatrix},$$

$A = (a_{ij}(t))_{n\times n}$, $B = (b_i(t))_{n\times 1}$

<u>Remark:</u> Any system of linear differential equations is equivalent to a first order system.

<u>Example</u> $y'' - e^t y' + y = \cos t$

The above equation is equivalent to a 1st order system:
Let $u_1 = y$, $u_2 = y'$. Then

$u_1' = y' = u_2 \Rightarrow \quad u_1' = u_2$
$u_2' = y'' = e^t y' - y + \cos t \Rightarrow u_2' = e^t u_2 - u_1 + \cos t$

$$\begin{cases} u_1' = u_2 \\ u_2' = -u_1 + e^t u_2 + \cos t \end{cases}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & e^t \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$

<u>Theorem</u> (Existence-Uniqueness Theorem)

Consider the following 1st order system of linear differential equations:

$$Y' = AY + b, \text{ where } Y = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \ A = (a_{ij}(t))_{n\times n}$$

and $b = (b_i(t))_{n\times 1}$.
If $A(t)$ and $b(t)$ are continuous on an Interval $I = (\alpha, \beta)$ then any I.V.P.

$$\begin{cases} Y' = AY + b \\ Y(t_0) = Y_0 \end{cases}, \quad \text{where } t_0 \in (\alpha, \beta) \text{ and } Y_0 = \begin{pmatrix} y_1^0 \\ \vdots \\ y_n^0 \end{pmatrix}$$

any vector has a unique solution $Y = \phi(t)$ defined on the interval $I = (\alpha, \beta)$.

We'll find the solution of the system and that will prove the existence part. For the uniqueness one has to go through the proof for 1st O.D.E.'s considered before. One has to modify that proof for equations of vector valued functions.

§7.2 and 7.3: Review of matrices and Eigen values and Eigen vectors:

$$A = A(t) = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \\ a_{m1}(t) & \cdots & a_{mn}(t) \end{pmatrix}_{m \times n} \quad \text{matrix of functions.}$$

The derivative of $A(t)$ is defined as

$$A'(t) = \begin{pmatrix} a'_{11}(t) & \cdots & a'_{1n}(t) \\ \vdots & & \vdots \\ a'_{m1}(t) & \cdots & a'_{mn}(t) \end{pmatrix}.$$

Properties: 1) $(c_1 A + c_2 B)' = c_1 A' + c_2 B'$, when $c_1, c_2 \in \mathbb{R}/\mathbb{C}$.

2) $(AB)' = A'B + AB'$.

$$C = (c_{ij}) = AB \Rightarrow c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$c'_{ij}(t) = \left(\sum_k a_{ik} b_{kj} \right)'$$

Video 11

$$\Rightarrow c'_{ij}(t) = \sum_k \left(a'_{ik}(t) b_{kj}(t) + a_{ik}(t) b'_{kj}(t)\right)$$

$$= \sum_k a'_{ik}(t) b_{kj}(t) + \sum_k a_{ik}(t) b'_{kj}(t)$$

$$= (A'B)_{ij} + (AB')_{ij}$$

$$(AB)'_{ij} = (A'B)_{ij} + (AB')_{ij}$$

$$\Rightarrow (AB)' = A'B + AB'.$$

Ex If A is an invertible matrix and if A' exists then

$$A \cdot A^{-1} = Id$$ and then taking derivatives we get

$$A' \cdot A^{-1} + A \cdot (A^{-1})' = (Id)' = 0.$$

$$A \cdot (A^{-1})' = -A' \cdot A^{-1} \Rightarrow (A^{-1})' = -A^{-1} A' \cdot A^{-1}.$$

How to solve 1^{st} order systems?

$$y' + p(t) y = q(t), \quad \mu(t) = e^{\int p(t) dt}, \quad \frac{1}{\mu(t)} = e^{-\int p(t) dt}$$

$$\Rightarrow y' \cdot \mu(t) + p(t) y \mu(t) = q(t) \mu(t)$$

$$\frac{d}{dt}(y(t)\mu(t)) = q(t)\mu(t) \Rightarrow y(t)\mu(t) = \int q(t)\mu(t) dt$$

$$\Rightarrow y(t) = e^{-\int p(t) dt} \int q(t) e^{\int p(t) dt} dt.$$

We'll just mimic this solution to solve the system

$$Y' = AY + b.$$

Note that we can write this system as

$Y' - AY = b$, which is the same as

$y' + p(t) y = q(t)$, when $p(t) = -A(t)$ and $q(t) = b(t)$.

If we can find an integrating factor as in case of single equations then we may proceed the same way.

$\mu(t) = e^{\int p(t)dt}$. $\quad \mu(t) = e^{-\int A(t) dt}$

$\int A(t)\, dt = \left(\int a_{ij}(t)\, dt\right)_{n\times n}$.

What about $e^{P(t)}$, when $P(t)$ is a matrix?

Once $\mu(t) = e^{-\int A(t)dt}$ is obtained we can proceed as follows:

$$Y' = AY + b \Rightarrow Y' - AY = b$$

$$\Rightarrow Y'\mu(t) - A\mu(t) Y = b\mu(t).$$

Again we'll have $\mu'(t) = -A(t) e^{-\int A(t)dt}$ and then

$\frac{d}{dt}(Y\mu) = b\mu. \Rightarrow Y(t)\mu(t) = \int b(t)\mu(t)\, dt$

$\Rightarrow Y(t) = (\mu(t))^{-1} \int b(t)\mu(t) dt$ will be the solution.

<u>Conclusion</u>: We need to define $e^{A(t)}$ so that

$\frac{d}{dt}(e^{A(t)}) = A'(t) e^{A(t)}$.

How to define $e^{A(t)}$? x^0

Recall that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

So we may define $e^{A(t)}$ as

$$e^{A(t)} \doteq I_n + \frac{A(t)}{1!} + \frac{A^2(t)}{2!} + \cdots + \frac{A^n(t)}{n!} + \cdots$$

This series turns out to be convergent and its derivative and integral can be computed termwise:

$$\frac{d}{dt} e^{A(t)} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{A^k(t)}{k!}\right), \text{ where } A^0(t) = I_n.$$

(The proofs of these statements require uniform convergence, which is covered in mathematical analysis courses.)

<u>Claim</u> $\frac{d}{dt}\left(e^{A(t)}\right) = A'(t)\, e^{A(t)}$, if $A(t) = tA$.

<u>Proof</u>: $\frac{d}{dt}\left(e^{A(t)}\right) = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{A^k(t)}{k!}\right)$ $\quad (A \cdot B)' = A'B + AB'$

$$= \frac{d}{dt}\left(I_n + \frac{A(t)}{1!} + \frac{A^2(t)}{2!} + \cdots + \frac{A^k(t)}{k!} + \cdots\right)$$

$$\Rightarrow \frac{d}{dt}\left(e^{A(t)}\right) = 0 + A'(t) + \frac{A'A + A \cdot A'}{2!} + \cdots$$

<u>Special Case</u>: $A(t) = tA$, for some

constant matrix A. $\quad (tA)^2 = tA \cdot tA = t^2A^2$

$$e^{A(t)} = e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = I_n + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots$$

$$\frac{d}{dt} e^{A(t)} = 0 + A + \frac{2tA^2}{2!} + \frac{3t^2A^3}{3!} + \cdots + \frac{kt^{k-1}A^k}{k!} + \cdots$$

$$= A\left(I_n + tA + \frac{t^2A^2}{2!} + \cdots + \frac{t^{k-1}A^{k-1}}{(k-1)!} + \cdots\right)$$

$$= A \cdot e^{tA}.$$

So to solve the system we need to compute e^{tA}.

§2.5. Homogeneous Linear Systems with Constant Coefficients:

$Y' = AY + b$. If $b = b(t) = 0$ then the equation is called homogeneous.

$Y' = AY$ or $Y' - AY = 0$.

Moreover, we assume $A = A(t)$ is a constant matrix.

How to solve? $\quad Y' - AY = 0, \quad p(t) = e^{-\int A\,dt}$

$$p(t) = e^{-tA} = \sum_{k=0}^{\infty} \frac{(-tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k t^k A^k}{k!}$$

Hence, we must compute A^k, for a square matrix A.

<u>Example</u> $y_1' = 3y_1 - y_2$, $y_2' = 4y_1 - 2y_2$, $Y' = AY$, where

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad Y' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}, \quad A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}.$$

$$Y' - AY = 0, \quad \mu(t) = e^{-\int A\,dt} = e^{-At}$$

$$(Y\mu(t))' = 0 \Rightarrow Y\mu(t) = C \Rightarrow Y = \mu(t)^{-1} \cdot C$$

$$\Rightarrow Y(t) = e^{tA} \cdot C$$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \quad A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}, \quad A^k = ?$$

To compute A^k we may use diagonalization or canonic forms in general.

First we diagonalize A.

So we need to find eigenvalues and eigenvectors.

$$Av = \lambda v, \quad v \in \mathbb{C}^{2\times 1}, \quad \lambda \in \mathbb{R}/\mathbb{C}. \quad v \neq 0$$

$\Rightarrow (A - \lambda I)v = 0$. Since $v \neq 0$, $\det(A - \lambda I) = 0$, called the characteristic equation.

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \underbrace{(3-\lambda)(-2-\lambda)}_{} + 4 = 0.$$

Video 12

$\Rightarrow \lambda^2 - \lambda - 2 = 0.$ Characteristic equation.

$(\lambda+1)(\lambda-2) = 0 \Rightarrow \lambda_1 = -1$ and $\lambda_2 = 2.$

So $\lambda_1 = -1$ and $\lambda_2 = 2$ are the eigenvalues.

Eigenvectors?

$Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$

$(A - \lambda_1 I)v_1 = 0 \Rightarrow (A + I)v_1 = 0$

$$\left(\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) v_1 = 0, \quad v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 4a - b = 0 \\ 4a - b = 0 \end{matrix}$$

Let $a = 1$ then $b = 4$. So $v_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_1 = -1$.

$$Av_1 = \lambda_1 v_1 \quad \left[\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix}\right]$$

For the eigenvalue $\lambda_2 = 2$ an eigenvector $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$ satisfies

$$Av_2 = \lambda_2 v_2 \Rightarrow (A - \lambda_2 I)v_2 = 0 \Rightarrow (A - 2I)v_2 = 0$$

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c - d = 0 \\ 4c - 4d = 0 \end{matrix} \quad \left(\begin{matrix} \text{let } c = 1 \\ \text{then } d = 1. \end{matrix} \right.$$

So, $v_2 = \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_2 = 2$.

$$\left[Av_2 \stackrel{?}{=} \lambda_2 v_2, \quad \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]$$

Let P be matrix whose columns are the eigenvectors. So of $Av_i = \lambda_i v_i$, $i = 1, \dots, n$.

$P = (v_1 \; v_2 \cdots v_n)_{n \times n}$. Then

$$\begin{aligned} P^{-1}AP &= P^{-1}A(v_1 \; v_2 \cdots v_n) \\ &= P^{-1}(Av_1 \; Av_2 \cdots Av_n) \\ &= P^{-1}(\lambda_1 v_1 \; \lambda_2 v_2 \cdots \lambda_n v_n) \\ &= (\lambda_1 P^{-1}v_1 \; \lambda_2 P^{-1}v_2 \cdots \lambda_n P^{-1}v_n) \end{aligned}$$

Note that $I_n = P^{-1} \cdot P = P^{-1}(v_1 \; v_2 \cdots v_n)$ so that

$I_n = (P^{-1}v_1 \; P^{-1}v_2 \cdots P^{-1}v_n)$. Then

$$\begin{aligned} P^{-1}AP &= (\lambda_1 P^{-1}v_1 \; \lambda_2 P^{-1}v_2 \cdots \lambda_n P^{-1}v_n) \\ &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \lambda_n \end{pmatrix} = D(\lambda_1, \lambda_2, \dots, \lambda_n) \end{aligned}$$

In our example we have, $P^{-1}AP = D(\lambda_1, \lambda_2) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$.

Now let's compute A^k.

$$P^{-1}AP = D(\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$(P^{-1}AP)^k = D^k = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}^k$$

$$\Rightarrow \overbrace{(P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)}^{k \text{ times}} = \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix} = D^k$$

$$P^{-1}A\underbrace{PP^{-1}}_{I_n}A\underbrace{PP^{-1}}_{I_n}\cdots$$

$$\Rightarrow P^{-1} A \cdot A \cdots A \cdot P = \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix} = D^k$$

$$P^{-1}A^kP = D^k \Rightarrow A^k = P D^k P^{-1}.$$

Now let's compute e^{tA}.

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k \cdot P D^k P^{-1}}{k!}$$

$$\Rightarrow e^{tA} = \sum_{k=0}^{\infty} \frac{P\, t^k D^k P^{-1}}{k!} = P \cdot \left(\sum_{k=0}^{\infty} \frac{t^k D^k}{k!} \right) P^{-1}$$

So, $e^{tA} = P\, e^{tD}\, P^{-1}$, where

$$e^{tD} = \sum_{k=0}^{\infty} \frac{t^k D^k}{k!} = \sum_{k=0}^{\infty} \begin{pmatrix} t^k\lambda_1^k & & & \\ & t^k\lambda_2^k & & \\ & & \ddots & \\ & & & t^k\lambda_n^k \end{pmatrix} \frac{1}{k!}$$

$$= \begin{pmatrix} \sum \frac{t^k\lambda_1^k}{k!} & & & \\ & \sum \frac{t^k\lambda_2^k}{k!} & & \\ & & \ddots & \\ & & & \sum \frac{t^k\lambda_n^k}{k!} \end{pmatrix}$$

$$= \begin{pmatrix} e^{t\lambda_1} & & & \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ & & & e^{t\lambda_n} \end{pmatrix}$$

So, $e^{tA} = P\, e^{tD}\, P^{-1} = P \begin{pmatrix} e^{t\lambda_1} & & & \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ & & & e^{t\lambda_n} \end{pmatrix} P^{-1}.$

Hence, the solution to $Y' = AY$ is the

$$Y(t) = e^{tA} C_{n \times 1} = P \begin{pmatrix} e^{t\lambda_1} & & & \\ & e^{t\lambda_2} & & \\ & & \ddots & \\ & & & e^{t\lambda_n} \end{pmatrix} P^{-1} C$$

$$\Rightarrow Y(t) = P\begin{pmatrix} e^{t\lambda_1} & & \\ & e^{t\lambda_2} & \\ & & \ddots \ e^{t\lambda_n} \end{pmatrix} C, \text{ where } C \text{ is}$$

an arbitrary $n \times 1$-vector.

For our example, we get

$$Y(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} & e^{2t} \\ 4e^{-t} & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow Y(t) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ 4c_1 e^{-t} + c_2 e^{2t} \end{pmatrix}$$

Or, $y_1 = c_1 e^{-t} + c_2 e^{2t}$

$y_2 = 4c_1 e^{-t} + c_2 e^{2t}$, $c_1, c_2 \in \mathbb{R}/\mathbb{C}$.

<u>Another Example:</u> Solve the system $Y' = AY$, where

$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$.

<u>Solution:</u> Characteristic Equation.

$$\det(A - \lambda I) = 0 \Rightarrow \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda I\right) = 0.$$

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0.$$

$$\lambda^2 - \operatorname{tr} A\, \lambda + \det A = 0.$$

$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \Rightarrow \lambda^2 - \operatorname{tr} A\, \lambda + \det A = 0$ becomes

$\lambda^2 - 5\lambda + 5 = 0.$ $\quad \lambda_{1,2} = \dfrac{5 \pm \sqrt{25-20}}{2}$

$\lambda_1 = \frac{5+\sqrt{5}}{2}$ and $\lambda_2 = \frac{5-\sqrt{5}}{2}$ Eigenvalues

$Av_1 = \lambda_1 v_1 \Rightarrow (A - \lambda_1 I) v_1 = 0.$

$$\left(\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} - \lambda_1 I \right) v_1 = 0 \Rightarrow \begin{pmatrix} 2-\lambda_1 & 1 \\ 1 & 3-\lambda_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(2-\lambda_1)a + b = 0 \Rightarrow$ let $a = 1$, then

$\cancel{a_1 + (3-\lambda_1)b = 0}$ $\quad b = -(2-\lambda)$

$$v_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1+\sqrt{5}}{2} \end{pmatrix}, \quad \lambda_1 = \frac{5+\sqrt{5}}{2}$$

$$v_2 = \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1-\sqrt{5}}{2} \end{pmatrix}, \quad \lambda_2 = \frac{5-\sqrt{5}}{2}$$

$\Big[\, Av_1 = \lambda_1 v_1 \Rightarrow \overline{Av_1} = \overline{\lambda_1 v_1}$

$\Rightarrow \bar{A}\,\bar{v}_1 = \bar{\lambda}_1 \bar{v}_1 \Rightarrow A\bar{v}_1 = \lambda_2 \bar{v}_1$ and

hence, we may take v_2 as $\bar{v}_1$. $\Big]$

Hence, $P = \begin{pmatrix} 1 & 1 \\ -\frac{1+\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} \end{pmatrix}$ and

$$y(t) = e^{tA} c = P \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}$$

$$\Rightarrow y(t) = \begin{pmatrix} 1 & 1 \\ -\frac{1+\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\lambda_1 = \frac{5+\sqrt{5}}{2}, \quad \lambda_2 = \frac{5-\sqrt{5}}{2}. \qquad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$

Video 13

Last time: $Y' = AY$ constant coefficient homogeneous 1^{st} order system

$$Y = Y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}, \quad Y' = Y'(t) = \begin{bmatrix} y_1'(t) \\ \vdots \\ y_n'(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

$$\begin{cases} y_1' = a_{11}y_1 + \cdots + a_{1n}y_n \\ \vdots \\ y_n' = a_{n1}y_1 + \cdots + a_{nn}y_n \end{cases}$$

How to solve? $Y' = AY$ $\qquad y' = ay \Rightarrow \frac{dy}{dt} = ay$

$$\Rightarrow \frac{dy}{y} = a\,dt \Rightarrow \ln|y| = at + c \Rightarrow y = Ce^{at}$$

So for the system $Y' = AY$ we look for a solution of the form $Y(t) = e^{tA}C$, where C is an arbitrary $n\times 1$-vector and

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = I_n + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots$$

We use diagonalization / Jordan form of A to compute A^k, for any k.

Example: (p. 405 problem 11). Find the general solution of the system

$$X' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} X$$

Solution: $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

Diagonalize A: characteristic equation $\det(A-\lambda I)=0$.

$$0=\begin{vmatrix}1-\lambda & 1 & 2\\ 1 & 2-\lambda & 1\\ 2 & 1 & 1-\lambda\end{vmatrix}=(1-\lambda)\left[(2-\lambda)(1-\lambda)-1\right]$$
$$-1\cdot\left[(1-\lambda)-2\right]+2\left[1-2(2-\lambda)\right]$$

$$0=(1-\lambda)(\lambda^2-3\lambda+1)+\lambda+1+4\lambda-6$$
$$=(1-\lambda)(\lambda^2-3\lambda+1)+5\lambda-5$$
$$=(1-\lambda)(\lambda^2-3\lambda+1-5)$$
$$=(1-\lambda)(\lambda^2-3\lambda-4)$$
$$=(1-\lambda)(\lambda+1)(\lambda-4)$$

So, $\lambda_1=-1$, $\lambda_2=1$ and $\lambda_3=4$ are the eigenvalues.

$\lambda_1=-1$: $Av_1=\lambda_1 v_1 \Rightarrow Av_1=-v_1 \Rightarrow (A+I)v_1=0.$

$$\begin{pmatrix}2 & 1 & 2\\ 1 & 3 & 1\\ 2 & 1 & 2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow \begin{matrix}2a+b+2c=0\\ a+3b+c=0\end{matrix}$$

Let $a=1$. Then
$$\begin{aligned} b+2c&=-2 \quad /\cdot 1\\ +\quad 3b+c&=-1 \quad /\cdot -2\\ \hline -5b+0.c&=0 \Rightarrow b=0.\end{aligned}$$

$$v_1=\begin{pmatrix}a\\ b\\ c\end{pmatrix}=\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$$

Hence, $c=-1$.

$\lambda_2=1$: $Av_2=\lambda_2 v_2=v_2 \Rightarrow (A-I)v_2=0.$

$$\begin{pmatrix}0 & 1 & 2\\ 1 & 1 & 1\\ 2 & 1 & 0\end{pmatrix}\begin{pmatrix}d\\ e\\ f\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\Rightarrow\begin{pmatrix}0 & 1 & 2\\ 1 & 1 & 1\\ 0 & -1 & -2\end{pmatrix}$$

$$\Rightarrow \begin{matrix}e+2f=0\\ d+e+f=0\\ \cancel{-e-2f=0}\end{matrix}$$

Let $d=1$, then
$$\begin{aligned} e+2f&=0\\ -\quad e+f&=-1\\ \hline \end{aligned}$$
$f=1$, $e=-2$.

So, $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$\underline{\lambda_3 = 4}$: $A v_3 = \lambda_3 v_3 = 4 v_3 \Rightarrow (A - 4I) v_3 = 0$.

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} g \\ h \\ j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 & 1 & 2 \\ 3 & -1 & -2 \\ 2 & 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 3 & -1 & -2 \\ 2 & 1 & -3 \end{pmatrix} \Rightarrow \begin{aligned} 3g - h - 2j &= 0 \\ 2g + h - 3j &= 0. \end{aligned}$$

Let $g = 1$, then
$$\begin{aligned} -h - 2j &= -3 \\ + \quad h - 3j &= -2 \\ \hline -5j &= -5 \Rightarrow j = 1. \end{aligned}$$

$h = -2 + 3j = 1$. So, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

So, $P = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.

Now $P^{-1} A P = D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 4 \end{pmatrix}$

$A = P D P^{-1}$, $A^k = P D^k P^{-1}$

$\Rightarrow A^k = P \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{pmatrix} P^{-1}$, $e^{tA} = \sum \frac{t^k A^k}{k!}$

$\Rightarrow e^{tA} = P \begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{pmatrix} P^{-1}$

Hence, the general solution of the I.V.P. is

$x(t) = e^{tA} C$

$$x(t) = P e^{tA} \overbrace{P^{-1} C}^{C} = P e^{tA} C$$

$$\Rightarrow x(t) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \; c_i \in \mathbb{R}.$$

Phase Portraits: Consider the equation $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x.$

$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$. Ch. Eqn. $\det(A - \lambda I) = 0$

$$\lambda^2 - \operatorname{tr} A \, \lambda + \det A = 0.$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0.$$

$$\Rightarrow (\lambda + 1)(\lambda - 3) = 0.$$

$$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = 3.$$

$\lambda_1 = -1$: $A v_1 = \lambda_1 v_1 = -v_1 \Rightarrow (A + I) v_1 = 0.$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{c} 2a + b = 0 \\ \cancel{4a + 2b = 0} \end{array}$$

Let $a = 1$, then $b = -2$

So $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$\lambda_2 = 3$: $A v_2 = \lambda_2 v_2 = 3 v_2 \Rightarrow (A - 3I) v_2 = 0.$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{c} -2c + d = 0 \\ \cancel{4c - 2d = 0} \end{array}$$

Let $c = 1$, then $d = 2$.

$v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. So, $P = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$.

General solution: $x(t) = e^{tA} C$

$$= P \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

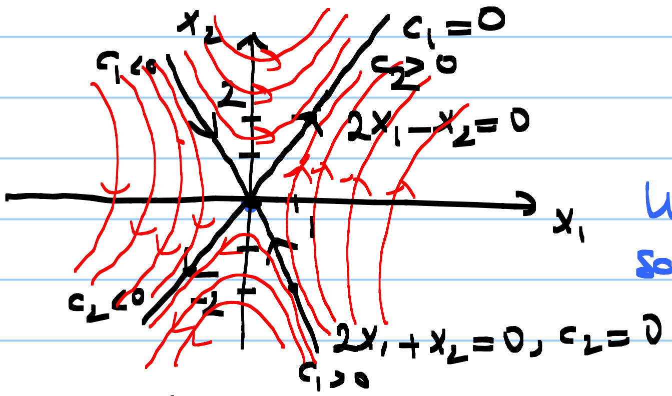
$$\Rightarrow x(t) = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Video 14

$X(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$, $x_1(t)=c_1e^{-t}+c_2e^{3t}$, $x_2(t)=-2c_1e^{-t}+2c_2e^{3t}$

$c_1, c_2 \in \mathbb{R}$.

Phase Portrait:



Unstable saddle point.

$c_1=0 \Rightarrow x_1(t)=c_2e^{3t}$, $x_2(t)=2c_2e^{3t} \Rightarrow x_2(t)=2x_1(t)$

If $c_2>0$, then $\lim_{t\to\infty} x_1(t)=\infty$ and $\lim_{t\to\infty} x_2(t)=\infty$.

Note that as $t\to-\infty$, both $x_i(t)\to 0$.

If $c_2<0$, then $x_1(t)=c_2e^{3t}$ and $x_2(t)=2c_2e^{3t}$ are negative and
$\lim_{t\to\infty} x_i(t)=-\infty$. Also, $\lim_{t\to-\infty} x_i(t)=0$.

$c_2=0$ $\quad x_1(t)=c_1e^{-t}$ and $x_2(t)=-2c_1e^{-t}$

$\Rightarrow x_2(t)=-2x_1(t)$

$c_1>0$, $\lim_{t\to\infty} x_1(t)=0=\lim_{t\to\infty} x_2(t)$, $\lim_{t\to-\infty} x_1(t)=\infty$, $\lim_{t\to-\infty} x_2(t)=-\infty$

If $c_1=c_2=0$, then $X(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ is the equilibrium solution.

This equilibrium is unstable and it is called a saddle point.

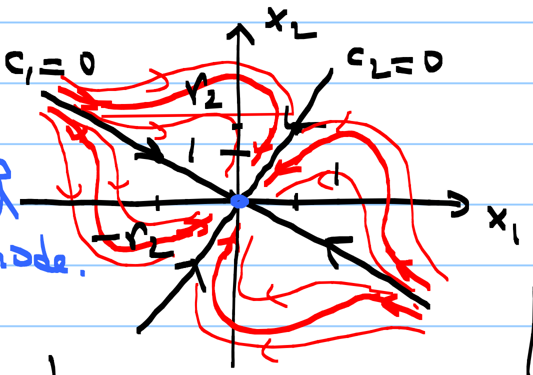
Example: $X' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} X$ $\qquad \lambda_1 = -1, \ \lambda_2 = -4$

Solution: $x^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$, $\ x^{(2)} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$

General solution: $X(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) = \begin{pmatrix} c_1 e^{-t} - \sqrt{2} c_2 e^{-4t} \\ \sqrt{2} c_1 e^{-t} + c_2 e^{-4t} \end{pmatrix}$

$$x_1 = c_1 \underbrace{e^{-t}}_{10} - \sqrt{2} c_2 \underbrace{e^{-4t}}_{10000}, \quad x_2(t) = \sqrt{2} c_1 e^{-t} + c_2 e^{-4t}.$$

$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.



The equilibrium is stable and indeed called a stable node.

$t \to \infty$
$x_1 \simeq c_1 e^{-t}$
$x_2 \simeq \sqrt{2} c_1 e^{-t}$
$c_2 = 0$

$t \to -\infty, \ c_1 = 0$

$c_1 = 0$
$x_1 = -\sqrt{2} c_2 e^{-4t}$
$x_2 = c_2 e^{-4t}$
$\Rightarrow x_1 + \sqrt{2} x_2 = 0$

$\lim_{t\to\infty} x_1(t) = 0 = \lim_{t\to\infty} x_2(t)$

$c_2 = 0 \qquad x_1 = c_1 e^{-t}, \ x_2 = \sqrt{2} c_1 e^{-t} \Rightarrow -\sqrt{2} x_1 + x_2 = 0$

$\lim_{t\to\infty} x_1(t) = 0 = \lim_{t\to\infty} x_2(t)$

§7.6. Complex Eigenvalues:

$X' = AX \qquad$ Ch. Eqn. $\det(A - rI) = 0$.

$r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ (assuming A is a real matrix)

$\xi_1 = a + ib$ eigenvector $\ \left| \ a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \ b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\} \ \xi_2 = a - ib$

If $A\xi_1 = r_1\xi_1$, where A is a real matrix, then taking conjugate of both sides we get

$$\overline{A\xi_1} = \overline{r_1\xi_1} \Rightarrow \overline{A}\,\overline{\xi_1} = \overline{r_1}\,\overline{\xi_1}. \text{ Since } \overline{A}=A$$

and thus $A\overline{\xi_1} = r_2\overline{\xi_1}$, so that $\overline{\xi_1}$ is an eigenvector associated to the eigenvalue $r_2 = \overline{r_1}$.

$$x(t) = \Xi\, e^{At} C = \begin{pmatrix} \xi_1 & \overline{\xi_1} \end{pmatrix} \begin{pmatrix} e^{r_1 t} & \\ & e^{\overline{r_1} t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$\xi_1 = a + ib$, $\overline{\xi_1} = a - ib$

$r_1 = \lambda + i\mu$, $\overline{r_1} = \lambda - i\mu$.

$$x(t) = \begin{pmatrix} e^{r_1 t}\xi_1 & e^{\overline{r_1} t}\overline{\xi_1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$c_1 = 1, c_2 = 0 \Rightarrow x^{(1)}(t) = e^{r_1 t}\xi_1$

$c_1 = 0, c_2 = 1 \Rightarrow x^{(2)}(t) = e^{\overline{r_1} t}\overline{\xi_1} = \overline{x^{(1)}}(t)$.

$$\begin{aligned} x^{(1)} &= e^{(\lambda + i\mu)t}(a+ib) \\ &= e^{\lambda t} e^{i\mu t}(a+ib) \\ &= e^{\lambda t}(\cos\mu t + i\sin\mu t)(a+ib) \\ &= e^{\lambda t}\left[(\cos\mu t\, a - \sin\mu t\, b) + i(\cos\mu t\, b + \sin\mu t\, a)\right] \end{aligned}$$

$e^{ix} = \cos x + i\sin x$

$x = \mu t$, $e^{i\mu t} = \cos\mu t + i\sin\mu t$.

$$x^{(2)} = e^{\lambda t}\left[(\quad) - i(\quad)\right]$$

Let $u(t) = \dfrac{x^{(1)} + x^{(2)}}{2} = e^{\lambda t}(\cos\mu t\, a - \sin\mu t\, b)$ and

$v(t) = \dfrac{x^{(1)} - x^{(2)}}{2i} = e^{\lambda t}(\cos\mu t\, b + \sin\mu t\, a)$.

Video 15

Hence, we can write the general solution as

$$X(t) = c_1 u(t) + c_2 v(t)$$

Example: Find the general solution of the system

$$X' = AX, \text{ where } A = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix}.$$

Solution: Ch. Eqn. $\det(A - rI) = 0$.

$$\det\begin{pmatrix} -1/2 - r & 1 \\ -1 & -1/2 - r \end{pmatrix} = 0 \Rightarrow r^2 - \operatorname{tr}A\, r + \det A = 0.$$

$$r^2 + r + 5/4 = 0.$$

$$r_{1,2} = \frac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i = \lambda \pm i\mu. \text{ So } \lambda = -1/2, \ \mu = 1$$

Eigenvectors: $\xi_1 = a + ib \quad A\xi_1 = r_1 \xi_1, \ r_1 = -1/2 + i$

$$\Rightarrow (A - r_1 I)\xi_1 = 0.$$

$$A - r_1 I = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} - \begin{pmatrix} -1/2+i & 0 \\ 0 & -1/2+i \end{pmatrix} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$$

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} -ic + d = 0. \text{ Let } c = 1, \text{ then} \\ \cancel{-c - id} = 0. \qquad d = i. \end{array}$$

$$\text{So } \xi_1 = \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a + ib$$

$$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$u(t) = e^{\lambda t}(\cos\mu t\ a - \sin\mu t\ b) = e^{-\frac{t}{2}}\left(\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$v(t) = e^{\lambda t}(\sin\mu t\ a + \cos\mu t\ b) = e^{-t/2}\left(\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$u(t) = e^{-t/2}\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}, \quad v(t) = e^{-t/2}\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}.$$

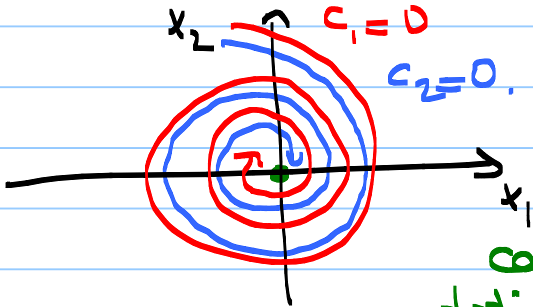
$$x(t) = c_1 u(t) + c_2 v(t) = e^{-t/2}\left(c_1\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + c_2\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\right)$$

$c_1, c_2 \in \mathbb{R}/\mathbb{C}$.

$$x(t) = \begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} \Rightarrow \begin{aligned} x_1(t) &= c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t\\ x_2(t) &= -c_1 e^{-t/2}\sin t + c_2 e^{-t/2}\cos t. \end{aligned}$$

$$\lim_{t\to\infty} x_i(t) = 0.$$

If $c_1 = 0$ then $x_1 = c_2 e^{-t/2}\sin t$, $x_2 = c_2 e^{-t/2}\cos t$.
Hence, $x_1^2 + x_2^2 = c_2^2 e^{-t}$



The origin is an equilibrium solution. In this case the solution is asymptotically stable.

§7.8. Repeated Eigenvalues:

Example: Solve the system $\underline{X}' = A\underline{X}$, where

$$A = \begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}.$$

Solution: Ch. Eqn. $\det(A - rI) = 0$, $r^2 - \operatorname{tr}A\, r + \det A = 0$.

$$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r_1 = r_2 = 2.$$

$r_1 = r_2 = 2.$ $\quad Av_1 = r_1 v_1 \Rightarrow (A - r_1 I)v_1 = 0.$

$$(A-2I)v_1 = 0 \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

$v_1 = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$ $-a-b=0$ $\Rightarrow$ let $a=1$, then $b=-1$.

$\cancel{a+b=0}$ $\quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the "only" eigenvector.

$$x^{(1)} = e^{r_1 t} v_1 = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix} \text{ a solution}$$

To write down the general solution we need a second solution, say $x^{(2)}$, which is linearly independent from $x^{(1)}$.

Note that any real matrix has a Jordan form (If not diagonalizable).

<u>How to obtain the Jordan form?</u>

$$Av_1 = r_1 v_1 \Rightarrow \boxed{(A - r_1 I)v_1 = 0. \quad v_1 : \text{eigenvector}}$$

A vector v_2 is called a generalized eigenvector of $(A - r_1 I)v_2 = v_1$. Note that in this case

$$\boxed{(A - r_1 I)^2 v_2 = (A - r_1 I)v_1 = 0.}$$

In this case, since $v_1 = (A - r_1 I)v_2$, we have

$$Av_2 = r_1 v_2 + v_1 \quad \left\{ \begin{array}{l} Av_1 = r_1 v_1 \\ Av_2 = r_1 v_2 + v_1 \end{array} \right.$$

If we let $\mathcal{B} = \{v_1, v_2\}$ and $P = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ so that $P^{-1}AP = P^{-1}A(v_1 \ \ v_2) = P^{-1}(Av_1 \ \ Av_2)$ $\quad (e_1 \ \ e_2)$

$= P^{-1}(r_1 v_1 \ \ \ r_1 v_2 + v_1)$ $\quad P^{-1}(v_1 \ \ v_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= (r_1 P^{-1} v_1 \ \ \ r_1 P^{-1} v_2 + P^{-1} v_1)$ $\quad (P^{-1}v_1 \ \ P^{-1}v_2)$

$= (r_1 e_1 \ \ \ r_1 e_2 + e_1)$ $\quad r_1 e_2 + e_1 = \begin{pmatrix} 0 \\ r_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} r_1 & 1 \\ 0 & r_1 \end{pmatrix} = J$ $\quad = \begin{pmatrix} 1 \\ r_1 \end{pmatrix}$

$X(t) = P e^{tA} P^{-1} C$

$= P e^{tA} C$

$e^{tA} = ?$ $\quad P^{-1}AP = J = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}$ $\quad r = r_1 = r_2.$

$A = PJP^{-1}, \ A^k = P J^k P^{-1} \Rightarrow e^{tA} = \sum \frac{t^k A^k}{k!}$

$= P \sum \frac{t^k J^k}{k!} P^{-1}$

$J^k = ?$

$J = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}, \ J^2 = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}\begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix} = \begin{pmatrix} r^2 & 2r \\ 0 & r^2 \end{pmatrix}$

$J^3 = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}\begin{pmatrix} r^2 & 2r \\ 0 & r^2 \end{pmatrix} = \begin{pmatrix} r^3 & 3r^2 \\ 0 & r^3 \end{pmatrix}, \ J^k = \begin{pmatrix} r^k & kr^{k-1} \\ 0 & r^k \end{pmatrix}.$

So, $e^{tJ} = \sum_{k=0}^{\infty} \frac{t^k J^k}{k!} = \begin{pmatrix} \sum \frac{t^k r^k}{k!} & \sum \frac{t^k k r^{k-1}}{k!} \\ 0 & \sum \frac{t^k r^k}{k!} \end{pmatrix}$

$\sum_{k=0}^{\infty} \frac{t^k k r^{k-1}}{k!} = t \sum_{k=1}^{\infty} \frac{t^{k-1} r^{k-1}}{(k-1)!} = t \sum_{k=0}^{\infty} \frac{t^k r^k}{k!} = t e^{rt}$

Hence, $e^{tJ} = \begin{pmatrix} e^{rt} & te^{rt} \\ 0 & e^{rt} \end{pmatrix}$.

So, $X(t) = P e^{tJ} C = (v_1 \ v_2) \begin{pmatrix} e^{rt} & te^{rt} \\ 0 & e^{rt} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

<u>Back to the Example</u> $X' = AX$, $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

$r = r_1 = r_2 = 2$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $v_2 = ?$

$(A - rI)v_2 = v_1 \Rightarrow (A - 2I)v_2 = v_1$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow$ $-c - d = 1$. let $c = 0$
$\cancel{c + d = -1}$ then $d = -1$.

So, v_2 can be chosen as $v_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

The general solution is then

$$X(t) = P e^{tJ} C = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$$X(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -e^{2t} - te^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{array}{l} x_1(t) = e^{2t}(c_1 + tc_2) \\ x_2(t) = -e^{2t}(c_1 + (1+t)c_2) \end{array}$$

Video 16

Example 1 Solve the system

$$X' = AX, \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}.$$

Solution: Ch. Eqn. $\det(A - rI) = 0$

$$0 = \det\begin{pmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -3 & 2 & 4-r \end{pmatrix} = (r-2)^3 \text{ and thus } r_1 = r_2 = r_3 = r = 2$$

Eigenvectors: $Av_1 = r v_1 = 2v_1$

$$\Rightarrow (A-2I)v_1 = 0 \Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -2 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} a - b - c &= 0 \\ b + c &= 0 \end{aligned}\right\} \quad a = 0 \text{ and } b + c = 0$$

let $b = 1$, then $c = -1$.

$v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, the only eigenvector (up to scalar multiplication)

$$\Rightarrow x^{(1)} = e^{rt} v = e^{2t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Note that we need two generalized eigenvectors.

$$(A - rI)v_2 = v_1 \quad (\Rightarrow (A - rI)^2 v_2 = 0)$$

$$(A - 2I)v_2 = v_1 \Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{r} d-e-f=0 \\ +\quad e+f=1 \\ \hline d=1. \end{array}$$

$\Rightarrow$ $d=1$ and $e+f=1$. Let $e=0$, then $f=1$.

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

For the second generalized eigenvector v_3 we have the equation

$$(A-rI)v_3 = v_2 \quad (\Rightarrow (A-rI)^3 v_3 = 0)$$

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} g \\ h \\ k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 1 & 1 \\ 2 & -1 & -1 & 0 \\ -3 & 2 & 2 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{array}{r} g-h-k=-1 \\ +\quad h+k=2 \\ \hline g=1 \end{array} \quad \Bigg| \quad h+k=2, \quad h=k=1, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$P = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$P^{-1}AP = J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

$$J = P^{-1}AP = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} P = \begin{pmatrix} 4 & -1 & -3 \\ 3 & -1 & -1 \\ -2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$J = P^{-1}AP \Rightarrow A = PJP^{-1} \Rightarrow A^k = PJ^kP^{-1}$$
$$\Rightarrow e^{tA} = Pe^{tJ}P^{-1}.$$

$$e^{tJ} = I + tJ + \frac{t^2J^2}{2!} + \frac{t^3J^3}{3!} + \cdots$$

$$J = \begin{pmatrix} r & 1 & 0 \\ 0 & r & 1 \\ 0 & 0 & r \end{pmatrix}, \quad J^2 = \begin{pmatrix} r & 1 & 0 \\ 0 & r & 1 \\ 0 & 0 & r \end{pmatrix}\begin{pmatrix} r & 1 & 0 \\ 0 & r & 1 \\ 0 & 0 & r \end{pmatrix}$$
$$= \begin{pmatrix} r^2 & 2r & 1 \\ 0 & r^2 & 2r \\ 0 & 0 & r^2 \end{pmatrix} \begin{pmatrix} r^2 & 2r & 1 \\ 0 & r^2 & 2r \\ 0 & 0 & r^2 \end{pmatrix}$$

$$\frac{t^3}{6}J^3 = \begin{pmatrix} r^3 & 3r^2 & 3r \\ 0 & r^3 & 3r^2 \\ 0 & 0 & r^3 \end{pmatrix}, \quad J^4 = \begin{pmatrix} r^4 & 4r^3 & 6r^2 \\ 0 & r^4 & 4r^3 \\ 0 & 0 & r^4 \end{pmatrix}$$

$$e^{tJ} = I + tJ + \frac{t^2J^2}{2!} + \frac{t^3J^3}{3!} + \frac{t^4J^4}{4!} + \cdots$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} rt & t & 0 \\ 0 & rt & t \\ 0 & 0 & rt \end{pmatrix} + \begin{pmatrix} \frac{r^2t^2}{2} & rt^2 & t^2/2 \\ 0 & \frac{r^2t^2}{2} & rt^2 \\ 0 & 0 & \frac{r^2t^2}{2} \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{r^3t^3}{3!} & \frac{t^3r^2}{2!} & \frac{t^3r}{2!} \\ 0 & \frac{r^3t^3}{3!} & \frac{t^3r^2}{2!} \\ 0 & 0 & r^3t^3/3! \end{pmatrix} + \cdots$$
$$= \begin{pmatrix} e^{rt} & te^{rt} & \frac{t^2}{2}e^{rt} \\ 0 & e^{rt} & te^{rt} \\ 0 & 0 & e^{rt} \end{pmatrix}$$

In our case, $e^{tJ} = \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}$.

Hence, the general solution is

$$\mathbf{X}(t) = e^{tA} C' = P e^{tJ} P^{-1} C' = P e^{tJ} C$$

$$\mathbf{X}(t) = \begin{array}{c} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \end{array} \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2} e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} v_1 & e^{2t} v_2 + te^{2t} v_1 & e^{2t} v_3 + te^{2t} v_2 + \frac{t^2}{2} e^{2t} v_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= c_1 e^{2t} v_1 + c_2 e^{2t} (v_2 + t v_1) + c_3 e^{2t} \left(v_3 + t v_2 + \frac{t^2}{2} v_1 \right)$$

$$\mathbf{X} = c_1 e^{rt} v_1 + c_2 e^{rt} (v_2 + t v_1) + c_3 e^{rt} \left(v_3 + t v_2 + \frac{t^2}{2} v_1 \right)$$

$$(A - rI) v_1 = 0, \quad (A - rI) v_2 = v_1, \quad (A - rI) v_3 = v_2.$$

Video 17

Example: Find the general solution of the system

$$\underline{X}' = A\underline{X}, \quad A = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}$$

Solution: Ch. Eqn. $0 = \det(A - rI) = (r-1)^3$.

So, $r_1 = r_2 = r_3 = r = 1$.

Eigenvectors: $(A - rI)v = 0 \Rightarrow (A - I)v = 0$.

$$\begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 4 & -3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

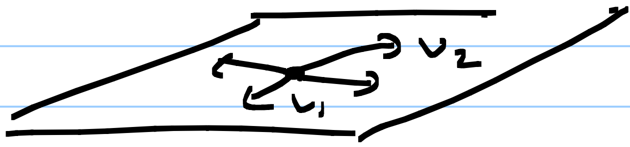
$4a - 3b - 2c = 0$. $\quad a = 1,\ b = 0 \Rightarrow 4 - 2c = 0$
$\Rightarrow c = 2$.

$v_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Now let $a = 0,\ b = 2 \Rightarrow 2c = -3b$
$\Rightarrow c = -3$.

$v_2 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$. Hence, the eigenspace for the only eigenvalue $r = 1$ has dimension 2.

So, we need one generalized eigenvector.
The Jordan form must be like

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$



$(A - rI)v_3 \in \langle v_1, v_2 \rangle$ – the eigenspace

$(A - rI)v_3 = c_1 v_1 + c_2 v_2$, for some $c_1, c_2 \in \mathbb{R}$.

$$(A - I)v_3 = c_1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} d \\ e \\ f \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 8 & -6 & -4 & 2c_2 \\ -4 & 3 & 2 & 2c_1-3c_2 \end{array}\right) \to \left(\begin{array}{ccc|c} 4 & -3 & -2 & c_1 \\ 0 & 0 & 0 & 2c_2-2c_1 \\ 0 & 0 & 0 & 3c_1-3c_2 \end{array}\right)$$

For this system to have a solution we must have $2c_2-2c_1=0$ and $3c_1-3c_2=0$. Hence, $c_1=c_2$. Let $c_1=c_2=1$. Then $4d-3e-2f=1$

$\Rightarrow d=1,\ e=1,\ f=0$. $v_3=\begin{pmatrix}1\\1\\0\end{pmatrix}$

$$(A-rI)v_3=(A-I)v_3=c_1v_1+c_2v_2=v_1+v_2=\begin{pmatrix}1\\2\\-1\end{pmatrix}.$$

Let $\mathcal{B}=\{v_1,\ v_1+v_2,\ v_3\}$

$(A-I)v_1=0,\quad (A-I)(v_1+v_2)=0,\quad (A-I)v_3=v_1+v_2$

Let $P=\begin{pmatrix}v_1 & v_1+v_2 & v_3\end{pmatrix}=\begin{pmatrix}1&1&1\\0&2&1\\2&-1&0\end{pmatrix}$, then

$$P^{-1}AP=J=\begin{pmatrix}r&0&0\\0&r&1\\0&0&r\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

$$e^{tJ}=\begin{pmatrix}e^t&0&0\\0&e^t&te^t\\0&0&e^t\end{pmatrix}$$

Hence, the general solution of the system is

$$X(t)=e^{tA}C'=Pe^{tJ}P^{-1}C'=Pe^{tJ}C$$

$$=\begin{pmatrix}1&1&1\\0&2&1\\2&-1&0\end{pmatrix}\begin{pmatrix}e^t&0&0\\0&e^t&te^t\\0&0&e^t\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}.$$

§ 7.7. Fundamental Matrices:

Let $\underline{X}' = A\underline{X}$ be an $n\times n$ homogeneous 1^{st} order linear system. Let V be the set of all solutions of this system.

Lemma: V is a vector space of dimension n.

Proof: Let $A = A(t)$ be a continuous function of matrices on an interval $I(\alpha, \beta)$ and $t_0 \in I$ any fixed real number.

To show that V is a vector space let $\underline{X}_1$ and $\underline{X}_2$ be two solutions. Then

$$\underline{X}_1' = A\underline{X}_1 \text{ and } \underline{X}_2' = A\underline{X}_2. \text{ So}$$

$$(c_1\underline{X}_1 + c_2\underline{X}_2)' = c_1\underline{X}_1' + c_2\underline{X}_2' = c_1 A\underline{X}_1 + c_2 A\underline{X}_2$$

$$= A(c_1\underline{X}_1 + c_2\underline{X}_2) \text{ and hence}$$

$c_1\underline{X}_1 + c_2\underline{X}_2$ is another solution. Hence, V is a vector space.

$\dim V = ?$

Consider the map $\phi : V \longrightarrow \mathbb{R}^n$ given by

$\phi(\underline{X}(t)) = \underline{X}(t_0)$.

Note that
$$\phi(c_1\underline{X}_1 + c_2\underline{X}_2) = (c_1\underline{X}_1 + c_2\underline{X}_2)(t_0)$$
$$= c_1\underline{X}_1(t_0) + c_2\underline{X}_2(t_0)$$
$$= c_1\,\phi(\underline{X}_1) + c_2\,\phi(\underline{X}_2)$$

and hence ϕ is a linear transformation

$\ker\phi = (0)$ so that ϕ is injective:

Let $\underline{X} \in \underline{V}$ be in the kernel of ϕ. So
$\phi(\underline{X}) = \underline{X}(t_0) = 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. Hence, $\underline{X}(t)$ must be the unique solution of the I.V.P.

$$\begin{cases} \underline{X}' = A\underline{X} \\ \underline{X}(t_0) = 0. \end{cases}$$

On the other hand, the zero function $t \longmapsto \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0$ is clearly a solution of the same I.V.P. Hence, $\underline{X}(t) = 0$, the constant zero function. So, $\ker\phi = (0)$.

$\phi : V \longrightarrow \mathbb{R}^n$ is surjective:

Let $\xi = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be any vector in $\mathbb{R}^n$. Let $\underline{X}(t)$ be the unique solution of the I.V.P.

$$\begin{cases} \underline{X}' = A\underline{X} \\ \underline{X}(t_0) = \xi. \end{cases}$$

Then clearly, $\phi(\underline{X}(t)) = \underline{X}(t_0) = \xi$, and hence ϕ is onto.

Therefore, $\phi : V \longrightarrow \mathbb{R}^n$, $\underline{X}(t) \longmapsto \underline{X}(t_0)$ is an isomorphism.

If $\mathcal{B} = \{\underline{X}_1, \ldots, \underline{X}_n\}$ is a basis for the solution space that any solution of $\underline{X}' = A\underline{X}$ is of the form

$$\underline{X}(t) = c_1 \underline{X}_1 + \cdots + c_n \underline{X}_n, \quad c_1, \ldots, c_n \in \mathbb{R}.$$

Video 18

If $\underline{X}_1, \ldots, \underline{X}_n$ are some solutions of $\underline{X}' = A\underline{X}$ then they form a basis (i.e., they are linearly independent) if and only if

$\phi(\underline{X}_1), \ldots, \phi(\underline{X}_n)$ form a linearly independent set of vectors in $\mathbb{R}^n$.

In particular, $\underline{X}_1, \ldots, \underline{X}_n$ are linearly independent solutions if and only if the matrix

$$\begin{pmatrix} \underline{X}_1(t_0) & \underline{X}_2(t_0) & \cdots & \underline{X}_n(t_0) \end{pmatrix}_{n \times n}$$ has non-zero determinant.

The determinant of the matrix of functions $\begin{pmatrix} \underline{X}_1(t) & \underline{X}_2(t) & \cdots & \underline{X}_n(t) \end{pmatrix}$ is called the Wronskian of the solutions.

$$W(t) = \det \begin{pmatrix} \underline{X}_1(t) & \underline{X}_2(t) & \cdots & \underline{X}_n(t) \end{pmatrix}_{n \times n}$$

It follows from the above consideration that $W(t)$ is either never zero on $I = (\alpha, \beta)$ or $W(t)$ is identically zero, i.e. $W(t) = 0$ for all $t \in (\alpha, \beta)$.

Definition: If $\underline{X}_1(t), \ldots, \underline{X}_n(t)$ are linearly independent solutions of the system $\underline{X}' = A\underline{X}$ ($n \times n$-system) then the matrix whose columns are $\underline{X}_i(t)$'s is called a fundamental matrix.

$$\underline{\Psi}(t) = \begin{pmatrix} \underline{X}_1(t) & \underline{X}_2(t) & \cdots & \underline{X}_n(t) \end{pmatrix}_{n \times n}.$$

Example: Recall that two solutions of the system $\underline{X}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\underline{X}$ were

$\underline{X}^{(1)} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ and $\underline{X}^{(2)} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$.

So $\underline{\Psi}(t) = \begin{pmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{pmatrix}$ is a fundamental matrix.

Since the columns of $\underline{\Psi}(t)$ are linearly independent the general solution of $\underline{X}' = A\underline{X}$ is given by

$\underline{X}(t) = \underline{\Psi}(t)C$, where C is any $n\times 1$-vector.

Remark: For the above example the Wronskian of the solutions $\underline{X}^{(1)} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ and $\underline{X}^{(2)} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ is $W(t) = \begin{vmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{vmatrix} = 4e^{2t} > 0, \forall t \in \mathbb{R}$.

The fundamental matrix $\underline{\Psi}(t)$ satisfying that $\underline{\Psi}(0) = \mathrm{Id}$ is called the fundamental matrix.

Note that $e^{0.A} = I_n + \frac{0.A}{1!} + \frac{0^2A^2}{2!} + \ldots = I_n$

and hence, the fundamental matrix is nothing but $\underline{\Phi}(t) = e^{tA}$ (of the system $\underline{X}' = A\underline{X}$).

§7.9. Non homogeneous Linear Systems:

$\underline{X}' = A\underline{X} + g(t)$ or $\underline{X}' - A\underline{X} = g(t)$.

If $\underline{X}_1$ and $\underline{X}_2$ are two solutions of the system then

$$\underline{X}_1' = A\underline{X}_1 + g \text{ and } \underline{X}_2' = A\underline{X}_2 + g$$

and thus $\underline{X}_1' - \underline{X}_2' = A(\underline{X}_1 - \underline{X}_2)$. Hence, $\underline{X}_1 - \underline{X}_2$ is a solution of the homogeneous equation.

Thus, $\underline{X}_2(t) = \underline{X}_1(t) + \underline{X}_c$, where $\underline{X}_c$ is a solution of the homogeneous system

$$\underline{X}' = A\underline{X}.$$

Hence, the general solution $\underline{X}_g(t)$ of the non homogeneous system $\underline{X}' = A\underline{X} + g$ has the form

$$\underline{X}_g(t) = \underline{X}_p(t) + \underline{X}_c(t),$$

where $\underline{X}_p(t)$ is any particular solution of $\underline{X}' = A\underline{X} + g$ and $\underline{X}_c(t)$ is the general solution of the homogeneous system $\underline{X}' - A\underline{X} = 0$. In this case, $\underline{X}_c(t)$ will be called the complementary solution.

How to obtain a particular solution $\underline{X}_p(t)$ of the system $\underline{X}' = A\underline{X} + g$.

A method to obtain a particular solution is so called Variation of Parameters:

$\underline{X}' = A\underline{X} \Rightarrow \underline{X}(t) = \Psi(t)C$, where $\Psi(t)$ is a fundamental matrix (one can choose

$\Psi(t) = e^{tA}$).

For the non-homogeneous system $X' = AX + g$ we try functions of the form

$X(t) = \Psi(t) u(t)$ to obtain a solution, where $u(t)$ is a suitable vector valued function.

To see whether such $u(t)$ exists let's plug $X(t)$ into the equation $X' = AX + g$.

$$X' = \Psi' u + \Psi u'.$$

$$\Rightarrow \Psi' u + \Psi u' = A(\Psi u) + g$$

$$\Psi' u + \Psi u' = A\Psi u + g$$

Note that $A\Psi = A(X_1\ X_2 \cdots X_n)$, where X_i's are solutions of the homogeneous equation $X' = AX$. Hence, $AX_i = X_i'$, for all $i = 1, \dots, n$.

$$\text{Hence, } A\Psi = (AX_1\ AX_2 \cdots AX_n)$$

$$= (X_1'\ X_2' \cdots X_n')$$

$$= \Psi'$$

Back to the equation

$$\cancel{\Psi' u} + \Psi u' = (A\Psi) u + g = \cancel{\Psi'} u + g$$

$$\Rightarrow \underline{\underline{\Psi u' = g}} \Rightarrow u' = \Psi^{-1} g$$

$$\Rightarrow u = u(t) = \int \Psi^{-1}(t) g(t)\, dt$$

then, $\underline{X}_p(t) = \underline{\Psi}(t)\, u(t) = \underline{\Psi}(t) \int \underline{\Psi}^{-1}(t)\, g(t)\, dt$.

<u>Remark</u>: $y' + py = q \Rightarrow e^{\int p\,dt} y' + e^{\int p\,dt} y = e^{\int p\,dt} q$

$$\frac{d}{dt}\left(e^{\int p(t)\,dt} y\right) = q(t)\, e^{\int p(t)\,dt}$$

$$y = e^{-\int p(t)\,dt} \cdot \int q(t)\, e^{\int p(t)\,dt}\, dt.$$

For systems, $\underline{X}' - A\underline{X} = g(t)$ $\quad (p = -A,\ q = g(t))$

$$\mu(t) = e^{\int p(t)\,dt} = e^{-\int A\,dt} = e^{-tA}.$$

$$\frac{d}{dt}\left(e^{-tA}\underline{X}\right) = e^{-tA} g(t) \Rightarrow e^{-tA}\underline{X} = \int e^{-tA} g(t)\,dt$$

$$\Rightarrow \underline{X} = e^{tA} \int e^{-tA} g(t)\, dt.$$

$$e^{tA} = \underline{\Phi} = \underline{\Psi} \Rightarrow \underline{X}(t) = \underline{\Psi} \int \underline{\Psi}^{-1}(t)\, g(t)\, dt.$$

<u>Example</u>: $\underline{X}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \underline{X} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$

$\left[\ \underline{X}' = A\underline{X} + g, \text{ where } A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},\ g(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}\ \right]$

$$\underline{X}_g = \underline{X}_c + \underline{X}_p$$

$\underline{X}_c$: complementary solution; i.e., general solution of the associated homogeneous equation

$$\underline{X}' = A\underline{X}.$$

$X_c = ?$ $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ Ch. Eqn. $r^2 - \mathrm{tr}A\, r + \det A = 0$

$\Rightarrow r^2 + 4r + 3 = 0$

$\Rightarrow (r+1)(r+3) = 0 \Rightarrow r_1 = -3,\ r_2 = -1.$

$\underline{r_1 = -3}$: $(A - r_1 I)v_1 = 0 \Rightarrow (A + 3I)v_1 = 0$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ $a + b = 0 \Rightarrow$ Let $a = 1$, then $b = -1$.

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. $\Rightarrow X^{(1)} = e^{r_1 t} v_1 = \begin{pmatrix} e^{-3t} \\ -e^{-3t} \end{pmatrix}$.

$\underline{r_2 = -1}$ $(A - r_2 I)v_2 = 0 \Rightarrow (A + I)v_2 = 0$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ $-c + d = 0$. Let $c = 1$, then $d = 1$

$\Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$X^{(2)} = e^{r_2 t} v_2 = e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}$.

$\Psi = \left(X^{(1)}\ X^{(2)}\right) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$, a fundamental matrix.

So, $X_c = \Psi C = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $c_i \in \mathbb{R}/\mathbb{C}$.

$X_p = ?$ Particular solution.

$X_p = \Psi(t) \int \Psi^{-1}(s)\, g(s)\, ds$

$\Psi(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}$ $\det \Psi(t) = 2e^{-4t}$

$\Psi^{-1}(t) = \frac{1}{\det \Psi(t)}\begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix} = \frac{1}{2}e^{4t}\begin{pmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}$

$$\Phi^{-1}(t) = \frac{1}{2}\begin{pmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{pmatrix}.$$

$$X_p(t) = \Phi(t)\int \frac{1}{2}\begin{pmatrix} e^{3s} & -e^{3s} \\ e^{s} & e^{s} \end{pmatrix}\begin{pmatrix} 2e^{-s} \\ 3s \end{pmatrix} ds$$

$$= \frac{\Phi(t)}{2}\int_0^t \begin{pmatrix} 2e^{2s} - 3s\,e^{3t} \\ 2 + 3s\,e^{s} \end{pmatrix} ds$$

$$\int \underbrace{s}_{u}\, \underbrace{e^{s} ds}_{dv} = s\underline{e^{s}} - \int 1 \cdot \underline{e^{s}} = se^{s} - e^{s}$$

$$= \frac{\Phi(t)}{2}\begin{pmatrix} e^{2s} - se^{3s} + \frac{e^{3s}}{3} \\ 2s + 3se^{s} - 3e^{s} \end{pmatrix}\Bigg|_0^t$$

$$= \frac{1}{2}\Phi(t)\begin{pmatrix} e^{2t} - te^{3t} + e^{3t}/3 \\ 2t + 3te^{t} - 3e^{t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}\begin{pmatrix} e^{2t} - te^{3t} + \frac{e^{3t}}{3} \\ 2t + 3te^{t} - 3e^{t} \end{pmatrix}$$

$$X_g = X_c + X_p.$$

CHAPTER 3: Second Order Linear Equations:

An n^{th} order linear differential equation has the form

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)y'(t) + a_n(t)y = g(t),$$

where $a_i(t)$ and $g(t)$ are functions of t.

For this chapter we'll consider constant coefficient equations, so that $a_i(t) = a_i$, constants.

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t), \quad a_i \in \mathbb{R}/\mathbb{C}, \; i = 1 \ldots n.$$

Example: $3y'' + 5y' - 7y = 3t + \cos 2t$.

How to solve such equations?

Idea: Convert this equation to a system of 1st order.

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = g(t).$$

Let $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $\ldots$, $x_n(t) = y^{(n-1)}(t)$.

Let $\underline{X}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$.

$x_1'(t) = y'(t) = x_2(t)$

$x_2'(t) = y''(t) = x_3(t)$

$\vdots$

$x_{n-1}'(t) = y^{(n-1)}(t) = x_n(t)$

$x_n'(t) = y^{(n)}(t) = -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + g(t)$

So $\underline{X}'(t) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n + g(t) \end{pmatrix}_{n \times 1}$

$$\underline{X}'(t) = \begin{pmatrix} 0 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & & & 0 \\ & & & & & \\ 0 & 0 & & & 0 & 1 \\ -a_n & -a_{n-1} & & & -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}$$

Hence, the equation $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = g(t)$ is equivalent to the 1^{st} order system

$$\underline{X}' = A\underline{X} + G, \text{ where } A = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \ddots & & 0 \\ & & & & \ddots & 1 \\ -a_n & \cdots & & & \cdots & -a_1 \end{pmatrix}, \; G(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ g(t) \end{pmatrix}$$

First let's consider homogeneous equations, so let $g(t) = 0$ and $G(t) = 0$.

$$\underline{X}' = A\underline{X}.$$

Ch. Eqn. of A: $\det(A - rI) = 0$.

$$\det \begin{pmatrix} -r & 1 & 0 & - & - & 0 \\ 0 & -r & 1 & \cdot & \cdot & 0 \\ \vdots & \vdots & & & -r & 1 \\ -a_n & \cdot & \cdot & \cdot & & -a_1 - r \end{pmatrix} = 0.$$

Special Case: $y'' + a_1 y' + a_2 y = 0$.

$x_1 = y$, $x_2 = y'$ $\quad \underline{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underline{X}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x_1' = y' = x_2$

$x_2' = y'' = -a_2 y - a_1 y'$ $\qquad A = \begin{pmatrix} 0 & 1 \\ -a_2 & -a_1 \end{pmatrix}.$

$\quad = -a_2 x_1 - a_1 x_2$

Ch. Eqn. $0 = \det(A - rI) = \begin{vmatrix} -r & 1 \\ -a_2 & -a_1 - r \end{vmatrix}$

$$0 = r^2 + r a_1 + a_2$$

Video 20

Note that the characteristic equation

$r^2 + a_1 r + a_2 = 0$ is nothing but the differential equation $y'' + a_1 y' + a_2 y = 0$, when $y^{(k)}$ is replaced by r^k.

Let r_1, r_2 be solutions of the ch. eqn. (eigenvalues of A) and ξ_1 and ξ_2 be corresponding eigenvectors.

Assume $r_1 \neq r_2$ so that A is diagonalizable.

Then the general solution of the system $\underline{X}' = A\underline{X}$ is

$$\underline{X}_g = \left(x^{(1)} \; x^{(2)}\right) c, \text{ where } x^{(1)} = e^{r_1 t}\xi_1 \text{ and } x^{(2)} = e^{r_2 t}\xi_2$$

$$= \left(e^{r_1 t}\xi_1 \quad e^{r_2 t}\xi_2\right)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad x_1 = y,\ x_2 = y'$$

$$\Rightarrow \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(e^{r_1 t}\xi_1 \quad e^{r_2 t}\xi_2\right)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= c_1 e^{r_1 t}\xi_1 + c_2 e^{r_2 t}\xi_2$$

Say, $\xi_1 = \begin{pmatrix} 1 \\ a \end{pmatrix}$, $\xi_2 = \begin{pmatrix} 1 \\ b \end{pmatrix}$ for some $a, b \in \mathbb{R}$. Then

$$\begin{pmatrix} y \\ y' \end{pmatrix} = c_1 \begin{pmatrix} e^{r_1 t} \\ a e^{r_1 t} \end{pmatrix} + c_2 \begin{pmatrix} e^{r_2 t} \\ b e^{r_2 t} \end{pmatrix} \Rightarrow \begin{array}{l} y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ y' = c_1 a e^{r_1 t} + c_2 b e^{r_2 t} \end{array}$$

Hence, $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where c_1, c_2 are arbitrary constants.

Example: Solve the equation $y''-3y'+2y=0$.

Solution: Ch. Eqn. $r^2-3r+2=0 \Rightarrow (r-1)(r-2)=0$

So, $r_1=1$ and $r_2=2$. Hence the general solution is $y=c_1e^{r_1t}+c_2e^{r_2t}$

$\Rightarrow y=c_1e^{t}+c_2e^{2t}$, $c_1,c_2\in\mathbb{R}/\mathbb{C}$.

Remark: The eigenvector ξ_1 and ξ_2 associated to the eigenvalues r_1 and r_2 are of the form $\xi_1=\begin{pmatrix}1\\r_1\end{pmatrix}$ and $\xi_2=\begin{pmatrix}1\\r_2\end{pmatrix}$ (provided that $r_1\neq r_2$).

$$A=\begin{pmatrix}0 & 1\\ -a_2 & -a_1\end{pmatrix} \Rightarrow r^2+a_1r+a_2=0.$$

$r_1\neq r_2$.

$y''+a_1y'+a_2y=0$

$\xi_1=?$ $(A-r_1I)\xi_1=0 \Rightarrow \begin{pmatrix}-r_1 & 1\\ -a_2 & -a_1-r_1\end{pmatrix}\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$-r_1c+d=0$. Let $c=1$ then $d=r_1$.

$\Rightarrow \xi_1=\begin{pmatrix}1\\r_1\end{pmatrix}$. Similarly, $\xi_2=\begin{pmatrix}1\\r_2\end{pmatrix}$.

So, $y=c_1e^{r_1t}+c_2e^{r_2t}$ and $y'=c_1r_1e^{r_1t}+c_2r_2e^{r_2t}$,

provided that $r_1\neq r_2$.

Ex Now consider a third order constant coefficient linear homogeneous equation, say

$$y''' + a_1 y'' + a_2 y' + a_3 y = 0.$$

So $x_1 = y$, $x_2 = y'$, $x_3 = y''$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Ch. Eqn. $0 = \det \begin{pmatrix} -r & 1 & 0 \\ 0 & -r & 1 \\ -a_3 & -a_2 & -r-a_1 \end{pmatrix}$

$$= -\left(r^3 + a_1 r^2 + a_2 r + a_3\right)$$

Assume r_1, r_2 and r_3 distinct roots of the Ch. Eqn.

If ξ_i is an associated eigenvector with r_i, then

$(A - r_i I)\xi_i = 0$. Take $i = 1$.

$$\Rightarrow \begin{pmatrix} -r_1 & 1 & 0 \\ 0 & -r_1 & 1 \\ -a_3 & -a_2 & -a_1 - r_1 \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that the first two rows are already linearly independent no matter what r_1 is. Hence, the third equation can be ignored.

The first two equations are

$-r_1 d + e = 0$	So let $d = 1$, then $e = r_1$
$-r_1 e + f = 0$	and $f = r_1^2$.

Here, $\xi_1 = \begin{pmatrix} 1 \\ r_1 \\ r_1^2 \end{pmatrix}$. Similarly, for r_i the corresponding eigenvector is $\begin{pmatrix} 1 \\ r_i \\ r_i^2 \end{pmatrix}$.

Then the solution $x^{(1)} = e^{r_1 t}\xi_1$, $x^{(2)} = e^{r_2 t}\xi_2$ and $x^{(3)} = e^{r_3 t}\xi_3$.

$$\Psi = \begin{pmatrix} e^{r_1 t} & e^{r_2 t} & e^{r_3 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & r_3 e^{r_3 t} \\ r_1^2 e^{r_1 t} & r_2^2 e^{r_2 t} & r_3^2 e^{r_3 t} \end{pmatrix}$$

Note that $\det \Psi = e^{r_1 t} e^{r_2 t} e^{r_3 t} \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix}$

$$= e^{(r_1+r_2+r_3)t} (r_1 - r_2)(r_1 - r_3)(r_2 - r_3)$$

$$= e^{-a_1 t} (r_1 - r_2)(r_1 - r_3)(r_2 - r_3)$$

and thus $\det \Psi \neq 0$ for all t, provided that the roots of the characteristic equation are distinct.

Hence, in this case $X_g = \Psi C$ is the general solution of $X' = AX$.

Moreover, the general solution of

$y''' + a_1 y'' + a_2 y' + a_3 y = 0$ is given by

$$y_g = c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t}, \quad c_1, c_2, c_3 \in \mathbb{R}/\mathbb{C}.$$

More generally, for the equation

$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$

If the roots of the Ch. Eqn. $r^n + a_1 r^{n-1} + \cdots + a_n = 0$ are distinct then the associated eigenvectors are given by

$$\xi_i = \begin{pmatrix} 1 \\ r_i \\ r_i^2 \\ \vdots \\ r_i^{n-1} \end{pmatrix} \quad i = 1, \ldots, n, \quad r_i \text{ that the}$$

general solution of the O.D.E. is given by

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}, \quad c_i \in \mathbb{R}/\mathbb{C}, \ i = 1, \ldots, n.$$

<u>Example:</u> Solve the I.V.P. $\begin{cases} y'' - y = 0 \\ \underline{\underline{y(0)}} = 1, \ \underline{\underline{y'(0)}} = -2. \end{cases}$

$$\underline{X}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$$

$$\underline{X}(0) = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Ch. Eqn. $r^2 - 1 = 0 \Rightarrow r = \pm 1$, let $r_1 = -1$, $r_2 = 1$, which are distinct. Hence the solution is given by

$$y_g(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
$$= c_1 e^{-t} + c_2 e^{t}, \quad c_1, c_2 \in \mathbb{R}.$$

For the solution of the I.V.P., we use the initial conditions:

$$\left.\begin{aligned} y(0) = 1 &\Rightarrow c_1 \cdot 1 + c_2 \cdot 1 = 1 \\ y'(0) = -2 &\Rightarrow -c_1 + c_2 = -2 \end{aligned}\right| \quad \begin{aligned} 2c_2 &= -1 \\ \Rightarrow c_2 &= -1/2 \end{aligned}$$

$$y_g' = -c_1 e^{-t} + c_2 e^{t} \qquad \Rightarrow c_1 = 3/2.$$

Hence, $y = \frac{3}{2} e^{-t} - \frac{1}{2} e^{t}$ is the unique solution.

Video 21

Summary $y^{(n)}+a_1y^{(n-1)}+\dots+a_{n-1}y'+a_ny=0$ $(*)$

Ch. Eqn. $r^n+a_1r^{n-1}+\dots+a_{n-1}r+a_n=0.$

Since Ch. Eqn. has n distinct roots, say $r_1, r_2, \dots, r_n$ then the general solution of the equation $(*)$ is given by

$$y_g=c_1y_1+c_2y_2+\dots+c_ny_n, \text{ where } c_i\in\mathbb{R}/\mathbb{C}$$

and $y_1=y_1(t)=e^{r_1t}, \dots, y_n=y_n(t)=e^{r_nt}$.

Moreover, an I.V.P. of the form

$$\begin{cases} y^{(n)}+a_1y^{(n-1)}+\dots+a_ny=0 \\ y(t_0)=b_1 \\ \vdots \\ y^{(n-1)}(t_0)=b_n \end{cases}$$

has a unique solution

The corresponding system $X'=AX$, where $X(t)=\begin{bmatrix} y(t) \\ y'(t) \\ y^{(n-1)}(t)\end{bmatrix}$ has fundamental matrix

$$\Phi=\begin{bmatrix} e^{r_1t} & \dots & e^{r_nt} \\ r_1e^{r_1t} & & r_ne^{r_nt} \\ \vdots & & \vdots \\ r_1^{n-1}e^{r_1t} & & r_n^{n-1}e^{r_nt}\end{bmatrix}, \text{ with determinant}$$

$$\det\Phi=\det\begin{bmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & & y_n^{(n-1)}\end{bmatrix}_{n\times n} \leftarrow \text{Wronskian of the solutions } y_1, y_2, \dots, y_n.$$

$$=e^{(r_1+r_2+\dots+r_n)t}\prod_{i<j}(r_i-r_j)$$

$$=e^{-a_1t}\prod_{i<j}(r_i-r_j)\neq 0 \text{ if } r_i\text{'s are all distinct.}$$

§3.3. Complex Roots of the Characteristic Equation.

$y'' + a_1 y' + a_2 y = 0$, Ch. Eqn. $r^2 + a_1 r + a_2 = 0$

Assume that the solutions are non real, say

$r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$ (assuming a_i's are real!)

$$y_1 = e^{r_1 t} \qquad y_2 = e^{r_2 t} \qquad (\mu > 0)$$

$$y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t} \cdot e^{i\mu t} = e^{\lambda t}(\cos\mu t + i\sin\mu t).$$

So, $y_1 = e^{\lambda t}\cos\mu t + i\, e^{\lambda t}\sin\mu t$. Similarly,

$$y_2 = e^{\lambda t}\cos(-\mu t) + i\, e^{\lambda t}\sin(-\mu t)$$
$$= e^{\lambda t}\cos\mu t - i e^{\lambda t}\sin\mu t.$$

Two non real solutions. To obtain real valued solutions we take the linear combinations of y_1 and y_2 as follows:

Let $u(t) = \dfrac{y_1 + y_2}{2} = e^{\lambda t}\cos\mu t$ and

$$v(t) = \frac{y_1 - y_2}{2i} = e^{\lambda t}\sin\mu t.$$

Clearly, the Wronskian is never zero:

$$W(u(t), v(t)) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t \end{vmatrix}$$

$$= \mu e^{2\lambda t} > 0.$$

Here, the general solution is given by

$$y = c_1 u + c_2 v = e^{\lambda t}(c_1 \cos\mu t + c_2 \sin\mu t), \quad c_1, c_2 \in \mathbb{R}.$$

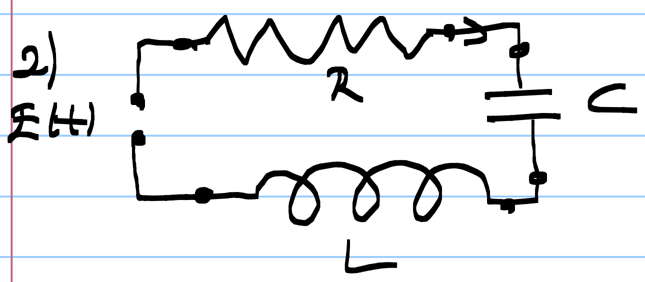
Examples: 1) $y'' + y = 0$. Find the general solution.

Ch. Eqn. $r^2 + 1 = 0 \Rightarrow r = \pm i = \lambda \pm i\mu$, $\lambda = 0$, $\mu = 1$.

$$u = e^{\lambda t}\cos\mu t = \cos t$$
$$v = e^{\lambda t}\sin\mu t = \sin t$$

Hence, the general solution is $y = c_1 \cos t + c_2 \sin t$

2)



$y(t) = Q(t)$ charge in the capacitor.

$y' = I(t)$

$$\Rightarrow \quad RI + \frac{Q(t)}{C} + LI' = E(t)$$

$$Ly'' + Ry' + \frac{1}{C}y = E(t)$$

Assume, $E(t) = 0$. Then $Ly'' + Ry' + \frac{1}{C}y = 0$.

Ch. Eqn. $Lr^2 + Rr + \frac{1}{C} = 0$

Say, $L = 1$, $R = 2$ and $C = \frac{1}{2}$. Then

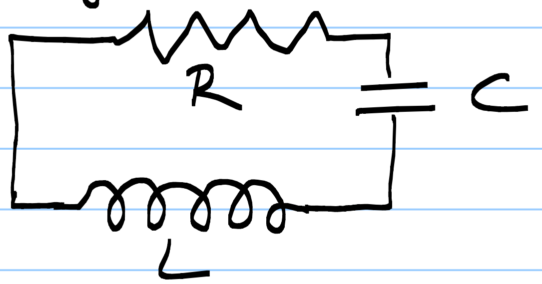
$$r^2 + 2r + 2 = 0 \Rightarrow r_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2}$$

So, $r = -1+i$ and $r_2 = -1-i$. $\lambda = -1$, $\mu = 1$.

$\Rightarrow u = e^{\lambda t}\cos\mu t = e^{-t}\cos t$
$\quad v = e^{\lambda t}\sin\mu t = e^{-t}\sin t$

$y = e^{-t}(c_1\cos t + c_2\sin t)$, $c_1, c_2 \in \mathbb{R}/\mathbb{C}$.

Charge in the capacitor C.



$y(0) = 1$
$y'(0) = 0$

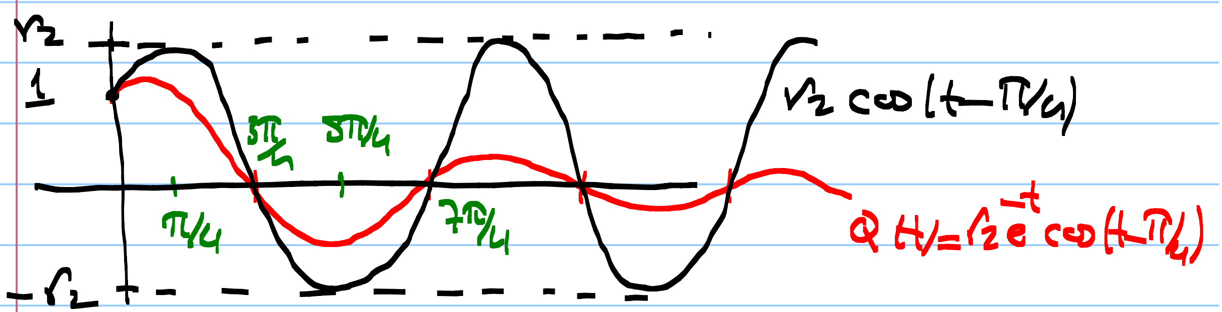
$y = e^{-t}(c_1\cos t + c_2\sin t)$
$y' = -e^{-t}(c_1\cos t + c_2\sin t) + e^{-t}(-c_1\sin t + c_2\cos t)$

$\Rightarrow 1 = y(0) = c_1$, $\quad 0 = y'(0) = -c_1 + c_2$
$\Rightarrow c_2 = c_1 = 1$.

Hence, $Q(t) = y(t) = e^{-t}(\cos t + \sin t) = \sqrt{2}e^{-t}\left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right)$

$\cos a\cos t + \sin a\sin t = \cos(t-a)$
$a = \pi/4$

$\Rightarrow Q(t) = \sqrt{2}\,e^{-t}\cos(t - \pi/4)$



Video 22

Ex: Write down a constant coefficient linear homogeneous O.D.E. whose characteristic equation is $r^4-r^2-2=0$. Also find its general solution.

Solution: $r^4-r^2-2=0 \Rightarrow y^{iv}-y''-2y=0.$

$(r^2-2)(r^2+1)=0 \Rightarrow r_1=-\sqrt{2},\ r_2=\sqrt{2},\ r_3=-i,\ r_4=i.$

$r_1=\sqrt{2} \Rightarrow y_1(t)=e^{\sqrt{2}t}$ $\qquad \lambda+i\mu \quad e^{\lambda t}\cos\mu t$

$r_2=-\sqrt{2} \Rightarrow y_2(t)=e^{-\sqrt{2}t}$ $\qquad \lambda-i\mu \quad e^{\lambda t}\sin\mu t$

$r_3=-i \Rightarrow y_3(t)=\cos t$ $\qquad (\lambda=0,\ \mu=1)$

$r_4=i \Rightarrow y_4(t)=\sin t$

Hence, the general solution is

$$y(t)=c_1y_1+c_2y_2+c_3y_3+c_4y_4$$
$$=c_1e^{\sqrt{2}t}+c_2e^{-\sqrt{2}t}+c_3\cos t+c_4\sin t,\quad c_i\in\mathbb{R}.$$

§3.4. Repeated Roots and Reductions of Order

Example: Solve the O.D.E. $y''+2y'+y=0$.

Solution: Ch. Eqn. $r^2+2r+1=0 \Rightarrow (r+1)^2=0$

Hence, $r_1=r_2=r=-1$, so we've repeated roots.

$y_1=e^{rt}=e^{-t},\ y_2=?$

Let's convert the O.D.E. $y''+2y'+y=0$ into a system:

Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Then

$x_1' = y' = x_2$

$x_2' = y'' = -y - 2y' = -x_1 - 2x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_2 \\ -x_1 - 2x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. $\underline{X}' = A\underline{X}$, where

$\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$.

<u>Ch. Eqn.</u> $r^2 - \mathrm{tr}A\, r + \det A = 0 \Rightarrow r^2 + 2r + 1 = 0$.

$\Rightarrow r_1 = r_2 = r = -1$ is the only eigenvalue.

$Av_1 = r v_1 \Rightarrow (A - rI)v_1 = 0 \Rightarrow (A + I)v_1 = 0$.

$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a + b = 0 \\ \cancel{-a - b = 0} \end{matrix}$. Let $a = 1$ then $b = -1$.

Hence, $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the only eigenvector.

Generalized eigenvector v_2: $(A - rI)v_2 = v_1$.

$\Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{matrix} c + d = 1 \\ \cancel{-c - d = -1} \end{matrix}$. Let $c = 1$, then $d = 0$.

So, $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $P = [v_1 \ v_2] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, then

$P^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. So $P^{-1}AP = J = \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.

$A = PJP^{-1} \Rightarrow e^{tA} = P e^{tJ} P^{-1}$.

$$e^{tA} = P\begin{bmatrix} e^{rt} & te^{rt} \\ 0 & e^{rt} \end{bmatrix}P^{-1} \Rightarrow \underline{X} = e^{tA}C' = Pe^{tJ}\underbrace{P^{-1}C'}$$

$$\Rightarrow \underline{X}(t) = \begin{array}{c} \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} e^{rt} & te^{rt} \\ 0 & e^{rt} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{rt} & te^{rt} + e^{rt} \\ -e^{rt} & -te^{rt} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow y(t) = c_1 e^{rt} + c_2(te^{rt} + e^{rt})$$
$$= \underline{(c_1 + c_2)e^{rt}} + c_2 \underline{\underline{te^{rt}}} \quad \text{general solution.}$$

Hence, as a second linearly independent solution we may take $y_2 = te^{rt}$.

$y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ form a basis for the solution space.

$$W = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{vmatrix} e^{rt} & te^{rt} \\ re^{rt} & e^{rt} + rte^{rt} \end{vmatrix}$$
$$= e^{2rt} > 0.$$

The general solution is $y = c_1 e^{rt} + c_2 te^{rt}$,
$$= c_1 e^{-t} + c_2 te^{-t},$$
$$c_1, c_2 \in \mathbb{R}.$$

What about roots with multiplicity more than 2?

Suppose that the characteristic equation has a root r repeated k times. One solution would be $y_1 = e^{rt}$. How can find other solution?

$$(A - rI)v = 0 \implies \begin{bmatrix} -r & 1 & - & - & 0 \\ 0 & -r & 1 & - & 0 \\ & & & & 1 \\ -a_k & & & & -r-a_1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \\ c_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{aligned} -rc_1 + c_2 &= 0 \\ -rc_2 + c_3 &= 0 \\ &\vdots \\ -rc_{k-1} + c_k &= 0 \end{aligned}$$

Let $c_1 = 1$, then $c_2 = r$, $c_3 = r^2, \dots, c_k = r^{k-1}$

$$-a_k c_1 - \quad - a_2 c_{k-1} - (r + a_1) c_k = 0$$

Hence, $v_1 = \begin{bmatrix} 1 \\ r \\ \vdots \\ r^{k-1} \end{bmatrix}$ is the only eigenvector.

So we need $k-1$ generalized eigenvectors:

$$(A - rI)v_2 = v_1, \quad (A - rI)v_3 = v_2, \dots, (A - rI)v_k = v_{k-1}.$$

Note that since rank $A - rI = k-1$ each system has 1-dim'l solution space.

$$v = v_1 \leftarrow v_2 \leftarrow v_3 \leftarrow \cdot \quad \leftarrow v_k.$$

Let $P = \begin{bmatrix} v_1 & v_2 & & v_k \end{bmatrix}$, then $P^{-1}AP = J = \begin{bmatrix} r & 1 & & 0 \\ 0 & r & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & & r \end{bmatrix}$

$$(A - rI)v_1 = 0 \implies Av_1 = rv_1 = rv_1 + 0 \cdot v_2 + \dots + 0 \cdot v_k$$

$$(A - rI)v_2 = v_1 \implies Av_2 = v_1 + rv_2 + 0 \cdot v_3 + \dots + 0 \cdot v_k$$

$$\vdots$$

$$(A - rI)v_k = v_{k-1} \implies Av_k = v_{k-1} + rv_k$$

Video 23

$$e^{tA} = P\, e^{tJ} P^{-1} = P \begin{bmatrix} e^{rt} & te^{rt} & \cdots & t^{k-1}e^{rt} \\ & \ddots & & \vdots \\ & & \ddots & \\ 0 & & & e^{rt} \end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(k)} \end{bmatrix} = e^{tA} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_k e^{rt} + c_{k-1} t e^{rt} + c_{k-2} t^2 e^{rt} + \cdots + c_1 t^{k-1} e^{rt} \\ \vdots \end{bmatrix}$$

$\Rightarrow y(t) = c_k e^{rt} + c_{k-1} t e^{rt} + \cdots + c_1 t^{k-1} e^{rt}$, the general solution in case a root is repeated k-times: $y_1 = e^{rt}, y_2 = te^{rt}, \dots, y_k = t^{k-1}e^{rt}$.

Example: Find the general solution of

$$y''' + 3y'' + 3y' + y = 0.$$

Solution: Ch. Eqn.: $r^3 + 3r^2 + 3r + 1 = 0$.

$\Rightarrow (r+1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = r = -1$.

$\Rightarrow y_1 = e^{rt} = e^{-t}, \quad y_2 = te^{-t}, \quad y_3 = t^2 e^{-t}$.

Hence, the general solution is $y = e^{-t}(c_1 + c_2 t + c_3 t^2)$, $c_i \in \mathbb{R}/\mathbb{C}$, $i = 1, 2, 3$.

Non-homogeneous linear O.D.E.'s:

We'll consider three different methods:
Reduction of order,
The method of undetermined coefficients, and
Variation of parameters.

Consider the general form of an n^{th} order linear O.D.E.:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(t).$$

Let $L[y]$ denote the linear term

$L[y] = a_n y^{(n)} + \dots + a_1 y' + a_0 y$. Then the equation is $L[y] = g(t)$.

The general solution of the corresponding homogeneous equation $L[y]=0$ is called the complementary solution, which will be denoted as y_c. A solution of $L[y]=g(t)$ is called a particular solution of $L[y]=g(t)$, denoted as again y_p.

Moreover, as in the case of systems, the general solution of $L[y]=g(t)$ will have the form

$$y_g = y_c + y_p.$$

<u>Example:</u> Consider the differential equation

$y'' + y = t$. Find its general solution.

<u>Solution:</u> $y_g = y_c + y_p$.

y_c: $y'' + y = 0$. Ch. Eqn $r^2 + 1 = 0 \Rightarrow r = \pm i$.

$\Rightarrow y_c = c_1 \cos t + c_2 \sin t$, $c_1, c_2 \in \mathbb{R}/\mathbb{C}$

$y_p = ?$ Let $y_p = t$. Since $y_p'' = 0$, we see that $y_p = t$ is a particular solution. Hence, the general solution of the equation is

$$y_g = y_c + y_p = c_1 \cos t + c_2 \sin t + t.$$

Let's find the unique solution of the I.V.P.

$$\begin{cases} y'' + y = t \\ y(0) = 0,\ y'(0) = 0. \end{cases}$$

$y(0) = 0 \Rightarrow \quad c_1 \cdot 1 + c_2 \cdot 0 + 0 = 0 \Rightarrow c_1 = 0.$

$y'(0) = 0 \Rightarrow y'(t) = -c_1 \sin t + c_2 \cos t + 1 = 0$

$\Rightarrow \quad -c_1 \cdot 0 + c_2 \cdot 1 + 1 = 0 \Rightarrow c_2 = -1.$

Hence, $y(t) = -\sin t + t$ is the unique solution of the I.V.P.

<u>Reduction of order</u>: Consider a 2nd order linear non-homogeneous equation

$$L[y] = g(t), \quad L[y] = y'' + p(t)y' + q(t)y.$$

$$y'' + p(t)y' + q(t)y = g(t).$$

Assume that $y_1 = y(t)$ is a solution of the corresponding homogeneous equation

$$L[y] = 0 \quad \text{or} \quad y'' + p(t)y' + q(t)y = 0.$$

To obtain a solution of the $L[y] = g(t)$ we try function of the form $y = v y_1(t)$, where $v = v(t)$ to be determined.

$y'' + p y' + q y = g(t).$

$y = v y_1 \Rightarrow y' = v' y_1 + v y_1'$

$y'' = v'' y_1 + v' y_1' + v' y_1' + v y_1'' = v'' y_1 + 2 v' y_1' + v y_1''.$

Plug them into the equation:

$(v'' y_1 + 2 v' y_1' + v y_1'') + p(v' y_1 + v y_1') + q v y_1 = g(t)$

$$\Rightarrow v'' y_1 + v'(2y_1' + p y_1) + v\underbrace{(y_1'' + p y_1' + q y_1)}_{=0 \text{ (by assumption)}} = g(t)$$

$\Rightarrow v'' y_1 + v'(2y_1' + p y_1) = g(t).$

Let $u = v'$, then $u' = v''$ so that above equation can be written as

$$u' y_1 + u(2y_1' + p y_1) = g(t)$$

is just a 1^{st} order O.D.E. in u.

Examples: 1) Find the general solution of

$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution: $L[y] = y'' - 3y' - 4y, \quad g(t) = 3e^{2t}.$

$y_g = y_c + y_p$

$y_c = ?$ $L[y] = 0 \Rightarrow y'' - 3y' - 4y = 0.$

Ch. Eqn $r^2 - 3r - 4 = 0 \Rightarrow (r+1)(r-4) = 0$

$\Rightarrow r_1 = -1, \quad r_2 = 4$

$y_1 = e^{r_1 t} = e^{-t} \quad y_2 = e^{r_2 t} = e^{4t}.$

$y_c = c_1 e^{-t} + c_2 e^{4t}$, $c_1, c_2 \in \mathbb{R}/\mathbb{C}$.

$y_p = ?$ To use the Reduction of Order method let $y_1 = e^{-t}$, a solution of the homogeneous equation.

Then let $y = v y_1 = e^{-t} v$. Take its derivatives:

$y = e^{-t} v$, $y' = -e^{-t} v + e^{-t} v'$ and

$$y'' = e^{-t} v - 2e^{-t} v' + e^{-t} v''.$$

Plug them into $y'' - 3y' - 4y = 3e^{2t}$.

$$(e^{-t} v - 2e^{-t} v' + e^{-t} v'') - 3(-e^{-t} v + e^{-t} v') - 4e^{-t} v = 3e^{2t}.$$

$$e^{-t} v'' + v'(-2e^{-t} - 3e^{-t}) + v\underbrace{(e^{-t} + 3e^{-t} - 4e^{-t})}_{=0} = 3e^{2t}.$$

$$e^{-t} v'' - 5e^{-t} v' = 3e^{2t}. \Rightarrow v'' - 5v' = 3e^{3t}.$$

Let $u = v'$. Then $u' = v''$. Hence, $u' - 5u = 3e^{3t}$, a 1st order linear O.D.E.

$p(t) = e^{\int p(t)dt} = e^{\int (-5)dt} = e^{-5t}$, an integrating factor.

$$\Rightarrow e^{-5t} u' - 5e^{-5t} u = 3e^{-2t}$$

$$\Rightarrow (e^{-5t} u)' = 3e^{-2t} \Rightarrow e^{-5t} u = -\frac{3}{2} e^{-2t} + c_1$$

$$\Rightarrow u = -\frac{3}{2} e^{3t} + c_1 e^{5t}.$$

$$v' = u = -\frac{3}{2} e^{3t} + c_1 e^{5t}.$$

Video 24

Hence, $v=\int u\,dt = -\frac{1}{2}e^{3t} + \frac{c_1}{5}e^{5t} + c_2.$

Now, $y = vy_1 = e^{-t}\left(-\frac{1}{2}e^{3t} + \frac{c_1}{5}e^{5t} + c_2\right)$

$= -\frac{1}{2}e^{2t} + \frac{c_1}{5}e^{4t} + c_2 e^{-t}$

$= \underbrace{-\frac{1}{2}e^{2t}}_{y_p} + \underbrace{c_1 e^{4t} + c_2 e^{-t}}_{y_c}, \quad c_1, c_2 \in \mathbb{R}$

the general solution.

2) Given that $y_1(t) = t^{-1}$ is a solution of

$2t^2 y'' + 3ty' - y = 0, \quad t>0,$ find its general solution.

<u>Solution</u>: Let $y = vy_1$, $v = ?$

$y = \frac{v}{t} \Rightarrow y' = \frac{v't - v.1}{t^2} = \frac{v'}{t} - \frac{v}{t^2}$

$y'' = \frac{v''t - v'.1}{t^2} - \frac{v't^2 - v\,2t}{t^4}$

$y'' = \frac{v''}{t} - 2\frac{v'}{t^2} + \frac{2v}{t^3}.$ Plug them into the equation:

$2t^2\left(\frac{v''}{t} - \frac{2}{t^2}v' + \frac{2v}{t^3}\right) + 3t\left(\frac{v'}{t} - \frac{v}{t^2}\right) - \frac{v}{t} = 0.$

$2t\,v'' + v'(-4+3) + v\underbrace{\left(\frac{4}{t} - \frac{3}{t} - \frac{1}{t}\right)}_{=0} = 0$

$\Rightarrow 2t v'' - v' = 0.$ Let $u = v'$, then $u' = v''$.

$\Rightarrow 2tu' - u = 0 \Rightarrow 2t\frac{du}{dt} = u \Rightarrow \frac{du}{u} = \frac{dt}{2t}$

$\ln|u| = \frac{1}{2}\ln|t| + c \Rightarrow u = t^{1/2} \cdot c_1'$

$\Rightarrow v' = u = c_1'\sqrt{t} \Rightarrow v = \frac{c_1' t^{3/2}}{3/2} + c_2$

$\Rightarrow v = c_1 t^{3/2} + c_2.$

So, $y = vy_1 = \frac{1}{t}(c_1 t^{3/2} + c_2) = c_1\sqrt{t} + c_2\frac{1}{t}$,

$c_1, c_2 \in \mathbb{R}/\mathbb{C}.$

§3.5. Nonhomogeneous Equations; Method of Undetermined Coefficients:

Method of Undetermined Coefficients:

Example: Find a particular solution y_p for

$$y'' - 3y' - 4y = 3e^{2t}.$$

Solution: Try $y_p = Ae^{2t}$, A to be determined.

$y_p = Ae^{2t}, \; y_p' = 2Ae^{2t}, \; y_p'' = 4Ae^{2t}.$

$\Rightarrow 4Ae^{2t} - 3\cdot 2Ae^{2t} - 4\cdot Ae^{2t} = 3e^{2t}$

$\Rightarrow (4A - 6A - 4A)e^{2t} = 3e^{2t}.$

$\Rightarrow -6A = 3 \Rightarrow A = -1/2.$

Hence, $y_p = -\frac{1}{2}e^{2t}$ is a particular solution

Example: Find a particular solution for

$$y'' + y = \cos t.$$

Try $y = A\cos t + B\sin t.$

$y' = -A\sin t + B\cos t$

$y'' = -A\cos t - B\sin t$

$\Rightarrow y'' + y = 0\cdot\cos t + 0\cdot\sin t = \cos t$

$0 = \cos t$, a contradiction.

Hence, we see that $y'' + y = \cos t$ does not have a particular solution of the form $y_p = A\cos t + B\sin t$.

Notation: $a_n y^{(n)} + \cdots + a_1 y' + a_0 y = L(y)$

$Dy = y',\ D^2 y = y'',\ \ldots,\ D^n y = y^{(n)}.$

Then
$$\begin{aligned} L[y] &= a_n y^{(n)} + \cdots + a_1 y' + a_0 y \\ &= a_n D^n y + \cdots + a_1 D y + a_0 y \\ &= (a_n D^n + \cdots + a_1 D + a_0) y \\ &= L[D](y). \end{aligned}$$

$L[D] \doteq a_n D^n + \cdots + a_1 D + a_0.$

Example $(D+1)(D-2) = D^2 - D - 2$

$(D^2 - D - 2)(y) = D^2 y - Dy - 2y = y'' - y' - 2y.$

$$\begin{aligned} (D+1)\underbrace{(D-2)(y)} &= (D+1)(Dy - 2y) = (D+1)(y' - 2y) \\ &= D(y' - 2y) + 1\cdot(y' - 2y) \end{aligned}$$

$$= y'' - 2y' + y' - 2y$$
$$= y'' - y' - 2y.$$

Also note that $(D+1)(D-2) = D^2 - D - 2 = (D-2)(D+1)$.

<u>Method</u>: Given a non homogeneous constant coefficient equation say $L[D](y) = g(t)$, first try to find another differential operator $L_1[D]$ so that $\underline{L_1[D]}(g(t)) = 0$.

If such $L_1[D]$ exists then apply it to the equation $L[D](y) = g(t)$:

$$L_1[D]\left(L[D](y)\right) = L_1[D]\left(g(t)\right)$$

$(L_1[D] \cdot L[D])(y) = 0$, which is just a constant coefficient homogeneous equation. Solving this we may obtain a particular solution y_p.

<u>Examples</u>: 1) Solve the equation $y'' - y = \underline{\cos t}$.

<u>Solution</u>: $L[D] = D^2 - 1$, $L[D](y) = y'' - y = \cos t$.

Find $L_1[D]$ so that $L_1[D](\cos t) = 0$.

$L_1[D] = ?$ $\quad \cos t = e^{\lambda t} \cos \mu t$, $\sin t = e^{\lambda t} \sin \mu t$

$r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$

$\lambda = 0$, $\mu = 1$

$r_1 = i$ and $r_2 = -i$. $\Rightarrow (r - r_1)(r - r_2) = (r-i)(r+i)$
$= r^2 + 1$.

So the operator that <u>kills</u> $\cos t$ is $D^2 + 1$.

Video 25

So, $L_1[D]$ can be chosen as $L_1[D] = D^2+1$.

$[\, L_1[D](\cos t) = (D^2+1)(\cos t) = (\cos t)'' + \cos t = 0 \,]$

$$L[D](y_p) = g(t) = \cos t$$

$$L_1[D]\, L[D]\,(y_p) = L_1[D]\,(\cos t)$$

$$(D^2+1)(D^2-1)(y_p) = 0. \Rightarrow (D^4-1)(y_p) = 0$$

$$\Rightarrow y_p^{iv} - y_p = 0.$$

Note that the general solution of
$L[D]y = g(t)$, $y''-y = \cos t$, has the
form

$$y_g = y_c + y_p.$$

y_c: $y''-y=0 \Rightarrow$ Ch. Eqn. $r^2-1 = 0 \Rightarrow r = \pm 1$.

So $y_c = c_1 e^{t} + c_2 e^{-t}$.

$y_p = ?$ $\quad y_p^{iv} - y_p = 0$. Ch. Eqn. $r^4-1=0$

$$\Rightarrow \underbrace{(r^2-1)}_{-1,\,1}\,\underbrace{(r^2+1)}_{-i,\,i} = 0$$

$$e^{-t},\ e^{t},\ \cos t,\ \sin t$$

$y_p = c_1 e^{-t} + c_2 e^{t} + c_3 \cos t + c_4 \sin t$. $\quad c_i = ?$

Plug this into the equation $L[D](y_p) = \underline{\underline{\cos t}}$.

Note that $c_1 e^{-t} + c_2 e^{t} = y_c$ so that $L[D](y_c) = 0$.
Thus we delete y_c from y_p so that
$y_p = A\cos t + B\sin t$

Find A and B so that $L[D](y_p) = \cos t$.

$y_p'' - y_p = \cos t$

$y_p = A\cos t + B\sin t$

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t.$

Hence $y_p'' - y_p = \cos t$ gives the equation $(-A\cos t - B\sin t) - (A\cos t + B\sin t) = \cos t$

$\Rightarrow \; -2A\cos t - 2B\sin t = \cos t + 0.\sin t$

Hence, $\left.\begin{array}{l} -2A = 1 \\ -2B = 0 \end{array}\right\} \Rightarrow A = -\frac{1}{2}, \; B = 0.$

So, $y_p = -\frac{1}{2}\cos t.$

Thus, the general solution to the equation is

$$y_g = y_c + y_p = c_1 e^{-t} + c_2 e^{t} - \frac{1}{2}\cos t.$$

Example: Solve the equation

$$y'' + 4y = e^{3t} - 2\sin 2t + \cos t.$$

Solution: $y_g = y_c + y_p$

y_c : $y'' + 4y = 0 \qquad L[D] = D^2 + 4$

Ch. Eqn. $r^2 + 4 = 0 \Rightarrow r = \mp 2i \Rightarrow \lambda = 0, \mu = 2$

$y_1 = \cos 2t$ and $y_2 = \sin 2t$ and thus

$y_c = c_1 \cos 2t + c_2 \sin 2t.$

y_p: $g(t) = e^{3t} - 2\sin 2t + \cos t$

$\boxed{e^{3t}} = e^{rt}$, $r=3$, $r-3$, $\boxed{D-3}$.

$\boxed{\sin 2t} \to \lambda + i\mu = 2i$, $\lambda = 0, \mu = 2$,

$(r-2i)(r+2i) = r^2+4$, $\boxed{D^2+4}$.

$\boxed{\cos t} \to \lambda + i\mu = i$, $\lambda = 0$, $\mu = 1$

$(r-i)(r+i) = r^2+1$ $\boxed{D^2+1}$

So let $L_1[D] = (D-3)(D^2+4)(D^2+1)$ operator of degree 5.

Hence $L_1[D](g(t)) = 0$ so that $g(t)$ is a solution of a constant coefficient equation of order 5.

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$L[D]\, y_p = g(t)$

$\Rightarrow L_1[D]\, L[D](y_p) = L_1[D](g(t)) = 0.$

$\Rightarrow (D-3)(D^2+4)(D^2+1)(D^2+4)(y_p) = 0$

Ch. Eqn. $(r-3)(r^2+4)^2(r^2+1) = 0.$

$r-3 \Rightarrow r=3 \Rightarrow y_1 = e^{rt} = \boxed{e^{3t}}$

$r^2+1 \Rightarrow r = \pm i \Rightarrow y_2 = e^{\lambda t}\cos\mu t$, $y_3 = e^{\lambda t}\sin\mu t$

$\lambda = 0, \mu = 1$ $\quad \stackrel{2}{=} \boxed{\cos t}$ $\quad = \boxed{\sin t}$

$(r^2+4)^2 \Rightarrow r = \pm 2i$, repeated twice

$\cancel{y_4 = \boxed{\cos 2t}, \; y_5 = \boxed{\sin 2t}}$

since they are in complementary solution.

$y_6 = \boxed{t\cos 2t}$, $y_7 = \boxed{t\sin 2t}$

So, y_p has the form

4 $y_p = Ae^{3t} + B\cos t + C\sin t + Dt\cos 2t + Et\sin 2t$

$y_p' = 3Ae^{3t} - B\sin t + C\cos t + D\cos 2t + E\sin 2t$
$\qquad - 2Dt\sin 2t + 2Et\cos 2t$

1. $y_p'' = 9Ae^{3t} - B\cos t - C\sin t - 2D\sin 2t + 2E\cos 2t$
$\qquad - 2D\sin 2t + 2E\cos 2t - 4Dt\cos 2t - 4Et\sin 2t$

$y_p'' + 4y_p = e^{3t} - 2\sin 2t + \cos t$

e^{3t}: $4A + 9A = 1 \Rightarrow A = 1/13$

$\cos t$: $3B = 1 \Rightarrow B = 1/3$

$\sin t$: $3C = 0 \Rightarrow C = 0.$

$\cos 2t$: $4E = 0 \Rightarrow E = 0.$

$\sin 2t$: $-4D = -2 \Rightarrow D = 1/2$

Hence, y_p can be chosen as

$$y_p = \frac{1}{13}e^{3t} + \frac{1}{3}\cos t + \frac{1}{2}t\cos 2t.$$

Remark: Note that to apply the method of undetermined coefficient to obtain a particular solution the non homogeneous part $g(t)$ must be a linear combination of the functions with types listed below:

1) $r \in \mathbb{R} \Rightarrow e^{rt}, \; t^n e^{rt}$
$(r = 0 \Rightarrow 1, \; t^n)$

2) $\lambda \pm i\mu \Rightarrow t^n e^{\lambda t}\cos \mu t, \; t^n e^{\lambda t}\sin \mu t$

Example $g = 3t^2 - 5t + 1 - 2e^{-t}\cos 5t + t^3\sin 7t$

Video 26

§3.6. Variation of Parameters:

This is another method to obtain a particular solution for a linear equation.

Consider an equation of type

$$\boxed{y'' + p(t)y' + q(t)y = g(t).}$$

Convert this to a system (2×2-system).
Let $x_1(t) = y(t)$ and $x_2(t) = y'(t)$. Then

$$x_1'(t) = y'(t) = x_2$$

$$x_2'(t) = y''(t) = -q(t)y - p(t)y' + g(t) = -qx_1 - px_2 + g$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Suppose that we have the complementary solution $\boxed{y_c = c_1 y_1(t) + c_2 y_2(t)}$. For the system one this is

$$\underline{X}_c = \left[x^{(1)} \;\; x^{(2)}\right]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \text{ where } x^{(1)} = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} \text{ and } x^{(2)} = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$$

$$= \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Psi C$$

For the particular solution we let
$\underline{X}_p = \Psi u$, where $u = u(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$.

Plug this into the equation $\underline{X}_p' = A\underline{X}_p + g$.

$$\Rightarrow (\Psi u)' = A(\Psi u) + g$$

$$\Rightarrow \Psi' u + \Psi u' = A\Psi u + g$$

$$\Rightarrow \cancel{A\Psi u} + \Psi u' = \cancel{A\Psi u} + g$$

$$\Rightarrow \Psi u' = g \Rightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

$$\boxed{\Rightarrow \begin{cases} y_1 u' + y_2 v' = 0 \\ y_1' u' + y_2' v' = g(t) \end{cases}}$$

Integrate u' and v' to obtain u and v.

Solve the above system to obtain u' and v'!

$$X_p = \Psi u = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} y_1 u + y_2 v \\ y_1' u + y_2' v \end{bmatrix}$$

$$\overset{\shortparallel}{\begin{bmatrix} y_p \\ y_p' \end{bmatrix}} \quad \Rightarrow \boxed{y_p = y_1 u + y_2 v.}$$

<u>Example:</u> $y'' + y = 2\tan t, \quad 0 < t < \pi/2.$

<u>Solution:</u> $y_g = y_c + y_p$

y_c: $y'' + y = 0 \Rightarrow$ Ch. Eqn. $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y_1 = \cos t$, $y_2 = \sin t$, $y_c = c_1 \cos t + c_2 \sin t$.

<u>y_p:</u> Use Variation of Parameters.

$y_p = u y_1 + v y_2$, where u and v satisfy

$$\begin{aligned} u' y_1 + v' y_2 &= 0 \\ u' y_1' + v' y_2' &= g(t) \end{aligned} \quad \left| \quad \begin{aligned} \cos t \; u' + \sin t \; v' &= 0 \quad &\big| \sin t \\ -\sin t \; u' + \cos t \; v' &= 2\tan t \quad &\big| \cos t \end{aligned} \right.$$

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$\hookrightarrow u' = -\frac{\sin t}{\cos t} v'$

$$0 \cdot u' + (\sin^2 t + \cos^2 t) v' = 0 \cdot \sin t + 2\tan t \cdot \cos t$$

$$v' = 2\sin t \Rightarrow v = -2\cos t$$

$$u' = ? \;, \quad u' = -\frac{\sin t}{\cos t} \cdot 2\sin t = -\frac{2\sin^2 t}{\cos t}$$

$$u' = \frac{2\cos^2 t - 2}{\cos t} \qquad (\sin^2 t = 1 - \cos^2 t)$$

$$u' = 2\cos t - 2\sec t$$

$$\Rightarrow u = 2\sin t - 2\int \sec t \, dt$$

$$= 2\sin t - 2\ln|\sec t + \tan t|$$

So, $y_p = y_1 u + y_2 v$

$$= \cos t \, (2\sin t - 2\ln|\sec t + \tan t|) + \sin t \, (-2\cos t)$$

$$= -2\cos t \ln|\sec t + \tan t|$$

Hence, $y_g = y_c + y_p = c_1 \cos t + c_2 \sin t - 2\cos t \ln|\sec t + \tan t|$.

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What about y_p for a 3rd order equation.

$$\boxed{y''' + p(t) y'' + q(t) y' + r(t) y = g(t)}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \quad \Rightarrow X' = AX + \begin{bmatrix} 0 \\ 0 \\ g(t) \end{bmatrix}$$

Finally, we'll have the equation

$\Psi u' = g$, where $\Psi = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$,

$u = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $g = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$, $X_p = \begin{bmatrix} y_p \\ y_p' \\ y_p'' \end{bmatrix} = \Psi u$.

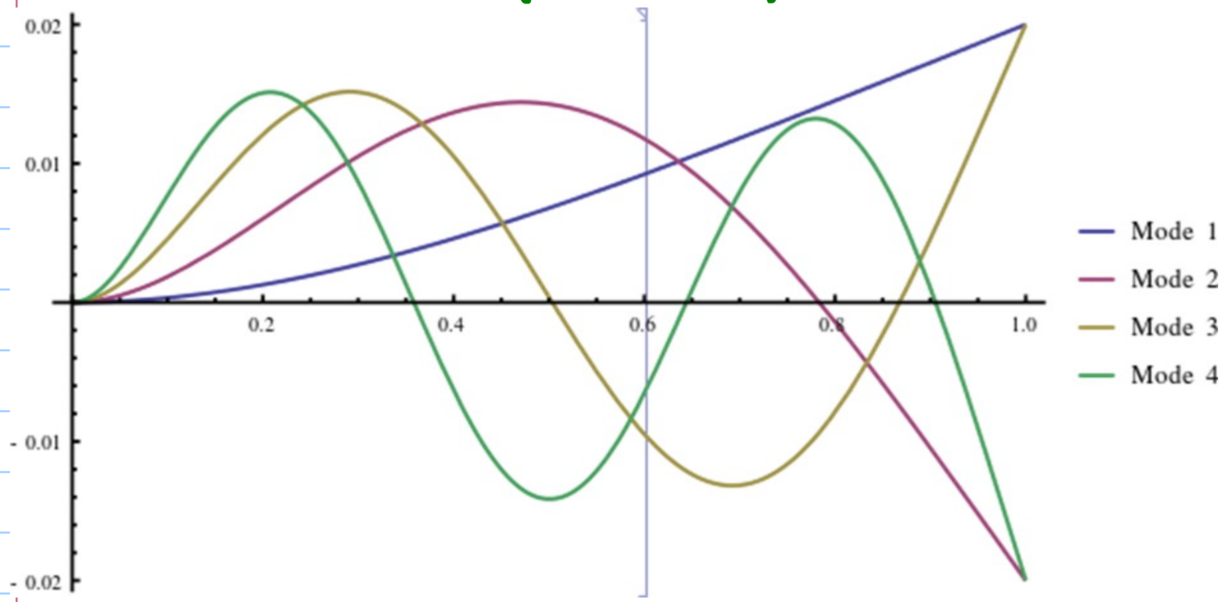
So, $\boxed{y_p = y_1 u + y_2 v + y_3 w}$, where u, v and w satisfy

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \Rightarrow \boxed{\begin{aligned} y_1 u' + y_2 v' + y_3 w' &= 0 \\ y_1' u' + y_2' v' + y_3' w' &= 0 \\ y_1'' u' + y_2'' v' + y_3'' w' &= g(t) \end{aligned}}$$

$y_g = y_c + y_p$, where $y_c = c_1 y_1 + c_2 y_2 + c_3 y_3$.

Video 27

Modes of a spring vibrating one end fixed.

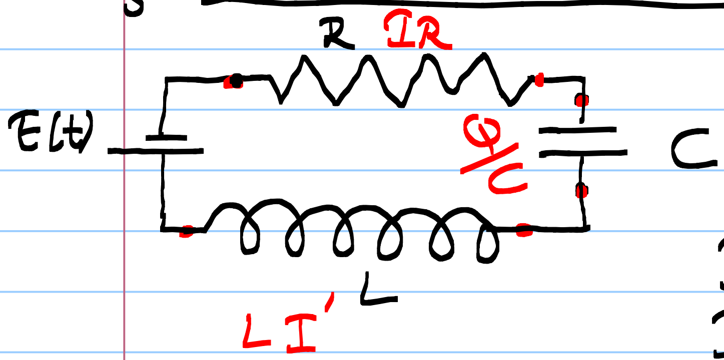


Frequencies of Modes of the Springs:

Length	22	30	65	70	105	110	275
Mode 1	39	21	2	10		2	10
Mode 2			33		39	37	
Mode 3			53			52	

Video 28

§3.7. Mechanical and Electrical Vibrations



$Q(t)$ = amount of charge in the capacitor C

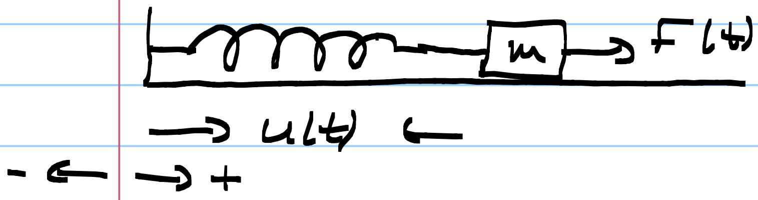
$I = I(t)$ current at time t

$I = Q'(t)$

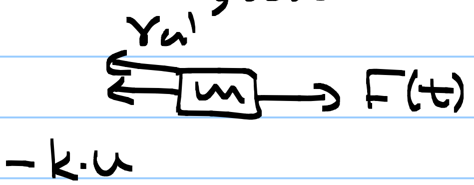
$$LI' + IR + Q/C = E(t)$$

$$Ly'' + Ry' + \frac{1}{C}y = E(t), \text{ where } y = Q(t)$$

Mechanical Vibration:



$u(t)$ = the amount of displacement from the rest position.



k: spring constant

$F(t)$: external force

$-\gamma u'$: drag force applied to the object by the table

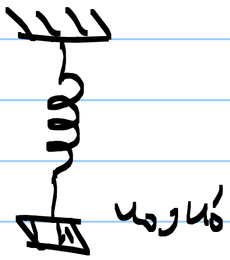
Total force applied to mass: $F(t) - ku - \gamma u'$

Newton's Law of Motion: Total force $= ma$, where $a = a(t) = u''(t)$ is the acceleration at time t.

$$mu'' = F(t) - ku - \gamma u' \Rightarrow mu'' + \gamma u' + ku = F(t)$$

Undamped Free Vibrations:

Electrical Vibrations: $R=0$.
Mechanical Vibrations: $\gamma=0$.
Free = No external force $\Rightarrow F(t)=0$.



$$mu''+ku=0, \quad k>0.$$

Ch. Eqn. $mr^2+k=0 \Rightarrow r=\pm\sqrt{-k/m}=\pm i\omega_0$

where $\omega_0=\sqrt{k/m}>0$, called the natural frequency of the system.

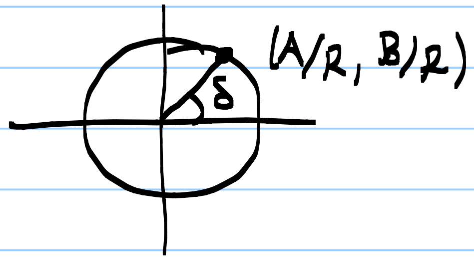
$$u(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t.$$

Assume that we are given initial conditions like $u(0)=u_0$, $u'(0)=u_0'$.

$\Rightarrow c_1=A$, $c_2=B$, $u(t)=A\cos\omega_0 t+B\sin\omega_0 t$, for some real numbers A and B.

$R=\sqrt{A^2+B^2} \Rightarrow u(t)=R\left(\frac{A}{R}\cos\omega_0 t+\frac{B}{R}\sin\omega_0 t\right)$

$(A/R)^2+(B/R)^2=1.$



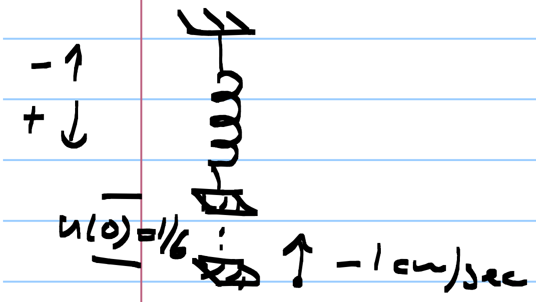
$A/R=\cos\delta$
$B/R=\sin\delta$

$$u(t)=R(\cos\omega_0 t\cos\delta+\sin\omega_0 t\sin\delta)$$
$$=R\cos(\omega_0 t-\delta)$$

δ: phase
$2\pi/\omega_0$: wave length
ω_0: natural frequency
R: amplitude of the wave.

Example: $u'' + 192u = 0$ $k = 60$ lb/ft, $m = 10/32$ lbs^2/ft

Assume $u(0) = 1/6$ ft, $u'(0) = -1$ ft/sec.



$$u = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

$$\omega_0 = \sqrt{k/m} = \sqrt{\frac{60 \times 32}{10}} = 8\sqrt{3}$$

$$u = c_1 \cos 8\sqrt{3}t + c_2 \sin 8\sqrt{3}t$$

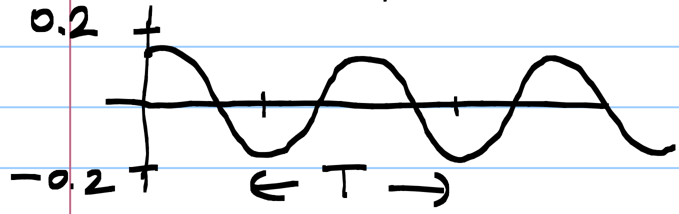
Plug the initial conditions to get $A = 1/6$, $B = -1/8\sqrt{3}$

$$\Rightarrow u = \frac{1}{6}\cos 8\sqrt{3}t - \frac{1}{8\sqrt{3}}\sin 8\sqrt{3}t$$

$$R = (A^2 + B^2)^{1/2} = \left(\frac{1}{36} + \frac{1}{192}\right)^{1/2} \approx 0.18162$$

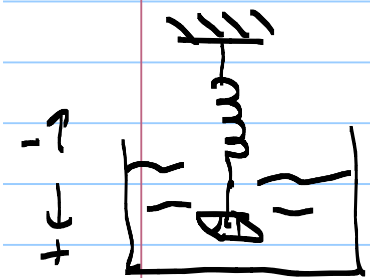
$$A = R\cos\delta, \quad B = R\sin\delta \Rightarrow \tan\delta = \frac{B}{A}.$$

$$\delta = \tan^{-1}(B/A) \cong -0.40864 \text{ rad.}$$



$T \simeq 0.453$

Damped Free Vibrations: $F(t) = 0$, $\gamma > 0$.



$$mu'' + \gamma u' + ku = 0.$$

Ch. Eqn $mr^2 + \gamma r + k = 0$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Video 29

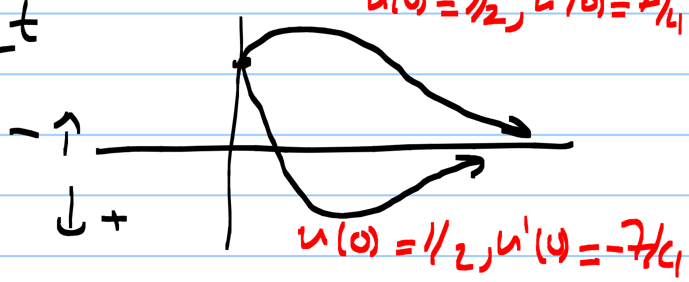
We've three cases:

Case 1: $\gamma^2 - 4mk > 0$ (over-damped)

$$r_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m} < 0, \quad r_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m} < 0.$$

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

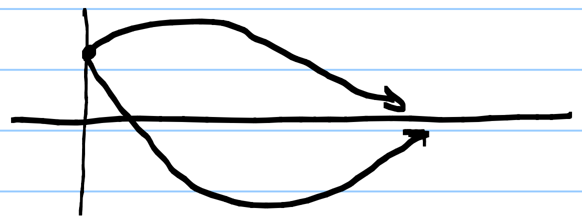
$$u(0) = u_0, \quad u'(0) = u_0'$$



Case 2 $\gamma^2 - 4mk = 0$, $r_{1,2} = \frac{-\gamma}{2m} = r$ (Critically damped)

$$u(t) = c_1 e^{rt} + c_2 t e^{rt} \qquad u(0) = u_0,\ u'(0) = u_0'$$

$$= e^{-\gamma t/2m}(A + Bt) \longrightarrow 0 \text{ as } t \to \infty.$$



Case 3: Underdamped Free Vibration

$$\gamma^2 - 4km < 0, \quad r_{1,2} = \frac{-\gamma \pm i\sqrt{4km - \gamma^2}}{2m} = \lambda \pm i\mu$$

$$\lambda = -\frac{\gamma}{2m}, \quad \mu = \frac{\sqrt{4km - \gamma^2}}{2m} = \sqrt{k/m - \frac{\gamma^2}{4m^2}} < \omega_0$$

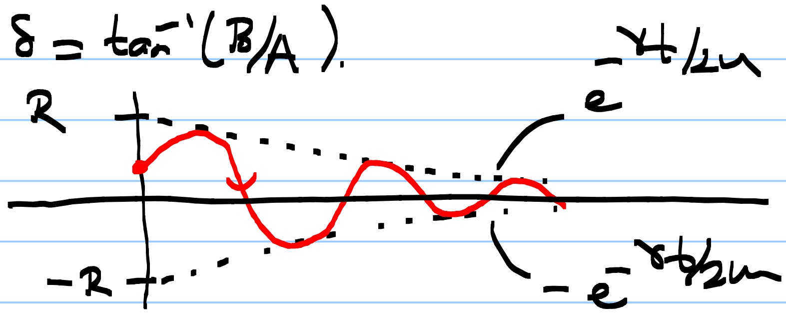
μ : quasi frequency of the system.

$u = e^{\lambda t}(c_1 \cos\mu t + c_2 \sin\mu t)$ (general solution)

$= e^{-\gamma t/2m}(A\cos\mu t + B\sin\mu t)$ $u(0)=u_0,\ u'(0)=u_0'$.

$R=(A^2+B^2)^{1/2} \Rightarrow u = R\cdot e^{-\gamma t/2m}\cos(\mu t-\delta)$

$\delta = \tan^{-1}(B/A)$.



§3.8. Forced Vibrations:

$$mu'' + \gamma u' + ku = F(t)$$

For simplicity we'll assume $F(t)$ has the form

$F(t) = F_0 \cos\omega t$, for some constants F_0 and ω.

$u_g = u_c + u_p$

u_c: $mu''+\gamma u'+ku=0 \Rightarrow u_c = c_1u_1(t)+c_2u_2(t)$,

when $u_1 = e^{-\gamma t/2m}\cos\mu t$, $u_2 = e^{-\gamma t/2m}\sin\mu t$

$u_p = ?$ $u_p = U(t)$

$\Rightarrow u_g = u_c(t) + U(t)$

Remark: If $\gamma>0$ (damped forced vibration) then $u_c(t) \to 0$ as $t \to \infty$ so that

$u(t) \approx u_p(t) = U(t)$

$u_c(t)$: transient solution
$u_p(t) = U(t)$: steady-state solution

Case 1 Forced damped vibration

$mu'' + \gamma u' + ku = F = F_0 \cos \omega t, \quad \gamma > 0.$

$$u_g = u_c + u_p = e^{-\gamma t/2m}(c_1 \cos \mu t + c_2 \sin \mu t) + u_p$$

$u_c(t) \to 0$ as $t \to +\infty$. $\quad u = u_p$.

$F = F_0 \cos \omega t, \quad u_p = ?$

Undetermined Coefficients:

$L[D]u = g(t) \qquad mu'' + \gamma u' + ku = F_0 \cos \omega t$

$L[D] = mD^2 + \gamma D + k, \quad L_1[D](g(t)) = 0.$

$L_1[D] = D^2 + \omega^2$

$L_1[D]\,L[D]\,u = L_1[D]\,g(t) = 0$

$(D^2 + \omega^2)(mD^2 + \gamma D + k)\,u = 0.$

Ch. Eqn. $(r^2 + \omega^2)(mr^2 + \gamma r + k) = 0$

$$\Rightarrow r_{1,2} = \pm i\omega, \quad r_{3,4} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

$$r_{1,2} \neq r_{3,4} \Rightarrow u_p = A\cos\omega t + B\sin\omega t$$

$$\Rightarrow u = u_c + u_p$$

$$= e^{-\gamma t/2m}(c_1 \cos\mu t + c_2 \sin\mu t)$$

$$+ A\cos\omega t + B\sin\omega t$$

$A, B = ?$

$$u_p = A\cos\omega t + B\sin\omega t$$
$$u_p' = -A\omega\sin\omega t + B\omega\cos\omega t$$
$$u_p'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$$

$$mu_p'' + \gamma u_p' + ku_p = g(t) = F_0\cos\omega t$$

$\underline{\cos\omega t}$: $\quad -mA\omega^2 + \gamma B\omega + kA = F_0$

$\sin\omega t$: $\quad -mB\omega^2 - \gamma A\omega + kB = 0$

$$\begin{aligned}(-m\omega^2 + k)A + \gamma B\omega &= F_0 \qquad / \gamma\omega \\ -\gamma\omega A + (-m\omega^2 + k)B &= 0 \qquad / (-m\omega^2 + k)\end{aligned}$$

$+$

$$\left[\gamma^2\omega^2 + (m\omega^2 - k)^2\right] B = F_0\gamma\omega$$

$$B = \frac{F_0\gamma\omega}{\gamma^2\omega^2 + (m\omega^2 - k)^2}, \quad A = \frac{(-m\omega^2 + k)F_0}{\gamma^2\omega^2 + (m\omega^2 - k)^2}$$

$$u_p(t) = \frac{F_0(-m\omega^2 + k)}{\gamma^2\omega^2 + (m\omega^2 - k)^2}\cos\omega t + \frac{F_0\gamma\omega}{\gamma^2\omega^2 + (m\omega^2 - k)^2}\sin\omega t$$

$$m\omega^2 - k = m(\omega^2 - k/m) = m(\omega^2 - \omega_0^2)$$

Note that if $\omega = \omega_0$, then

Video 30

$$u_p(t) = \frac{F_0 \gamma \omega_0}{\gamma^2 \omega_0^2} = \frac{F_0}{\gamma \omega_0} \sin \omega_0 t$$

Example (a) Solve the I.V.P.

$u'' + 2u' + 10u = \cos \omega t$, $u(0) = u_0$, $u'(0) = u_0'$.

Solution $u_g = u_c + u_p$.

u_c: $u'' + 2u' + 10u = 0 \Rightarrow$ ch. Eq. $r^2 + 2r + 10 = 0$

$$r_1 = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i \qquad q = 3 \text{ (quasi frequency)}$$

$$\Rightarrow u_c = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$$
(transient solution)

$u_c(t) \to 0$ as $t \to +\infty$.

$u_p(t)$ Steady state solution.

$u_p(t) = A \cos \omega t + B \sin \omega t \qquad A = ?, B = ?$

$u_p'(t) = -A\omega \sin \omega t + B\omega \cos \omega t$

$u_p''(t) = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$

$u_p'' + 2u_p' + 10u_p = \cos \omega t$

$\cos \omega t$: $\quad -A\omega^2 + 2B\omega + 10A = 1$

$\sin \omega t$: $\quad -B\omega^2 - 2A\omega + 10B = 0$

$\Rightarrow (10 - \omega^2)A + 2\omega B = 1 \quad / \quad 2\omega$

$+ \quad -2A\omega + (10 - \omega^2)B = 0 \quad / \quad (10 - \omega^2)$

$$\left[4\omega^2+(10-\omega^2)^2\right]B = 2\omega$$

$$B = \frac{2\omega}{4\omega^2+(10-\omega^2)^2} = \frac{2\omega}{\omega^4-16\omega^2+100}$$

$$A = \frac{(10-\omega^2)}{\omega^4-16\omega^2+100}$$

$$u_p = \frac{(10-\omega^2)}{\omega^4-16\omega^2+100}\cos\omega t + \frac{2\omega}{\omega^4-16\omega^2+100}\sin\omega t$$

$$R = \left[\frac{(10-\omega^2)^2+(2\omega)^2}{(\omega^4-16\omega^2+100)^2}\right]^{1/2} = \frac{1}{(\omega^4-16\omega^2+100)^{1/2}}$$

$$u_p = R\cos(\omega t-\delta), \quad \tan\delta = \frac{B}{A} = \frac{2\omega}{(10-\omega^2)}$$

$$u_g = u_c + u_p, \quad u_c(t) \to 0 \text{ as } t\to\infty.$$

$$u(t) \approx u_p = R\cos(\omega t-\delta)$$

$$R = \frac{1}{(\omega^4-16\omega^2+100)^{1/2}}.$$

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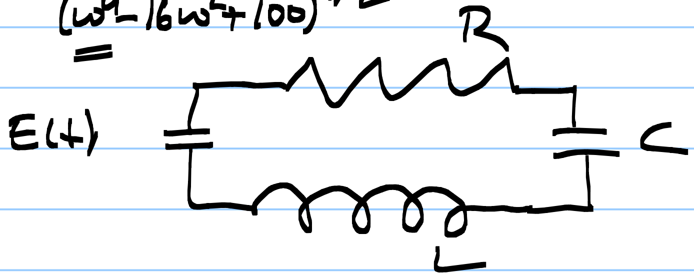


$E(t) = \cos\omega t$

$u \approx u_p = R(\cos\omega t - \delta)$

Determine ω so that $R(\omega)$ is maximal.

$$R' = 0 \Rightarrow \frac{-(4\omega^3-32\omega)}{(\omega^4-16\omega^2+100)}\cdot\frac{1}{2} = 0$$

$$4\omega^3-32\omega = 0 \Rightarrow 4\omega(\omega^2-8) = 0, \ \omega = 0$$

or $\omega = \pm 2\sqrt{2}$. $\quad \omega_{max} = 2\sqrt{2}$

$$R_{max} = R(\omega_{max}) = \frac{1}{(64 - 16.8 + 100)^{1/2}} = \frac{1}{\sqrt{36}} = \frac{1}{6}.$$

<u>Case 2 Forced Undamped Vibration</u>

$$mu'' + ku = F(t) = F_0 \cos \omega t \qquad (\gamma = 0)$$

$u_g = u_c + u_p$, $\quad u_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$

$\omega_0 = \sqrt{k/m}$ natural frequency

i) $\omega_0 \neq \omega$, $\quad L[D] = D^2 + \omega_0^2$

$L_1[D] = D^2 + \omega^2$

$\Rightarrow u_p = A \cos \omega t + B \sin \omega t.$

$$B = 0, \; A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

$$u_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

$$u_g = (c_1 \cos \omega_0 t + c_2 \sin \omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

Assume that we are given initial conditions $u(0) = 0$, $u'(0) = 0$.

This yields

$$0 = c_1 + \frac{F_0}{m(\omega_0^2 - \omega^2)} \Rightarrow c_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)}$$

$0 = u'(0) = c_2 w_0 \Rightarrow c_2 = 0$

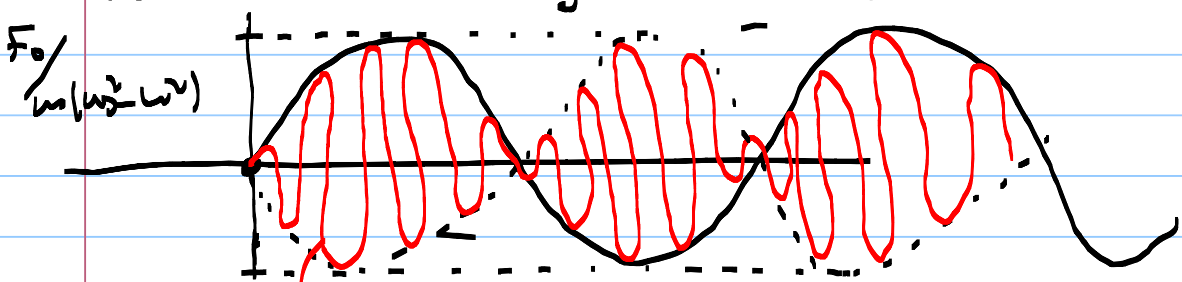
Hence the unique solution is

$$u(t) = \frac{-F_0}{m(\omega_0^2 - \omega^2)} \cos \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

$$= \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)$$

$$= \frac{F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \sin\left(\frac{\omega_0 + \omega}{2}\right)t$$

Note that if $\omega_0 \sim \omega$ then $\sin \frac{(\omega_0 - \omega)}{2} t$ has small frequency compare to the other sine term. Moreover, $\frac{F_0}{m(\omega_0^2 - \omega^2)}$ will be very big even if F_0 or m are big or small.



Case 2 $\omega_0 = \omega$. $\omega_0 = \sqrt{k/m}$

$$mu'' + ku = F_0 \cos \omega t = F_0 \cos \omega_0 t.$$

$$L_1[D] = L[D] = D^2 + \omega_0^2$$

$$r_{1,2} = r_{3,4} = \pm i \omega_0$$

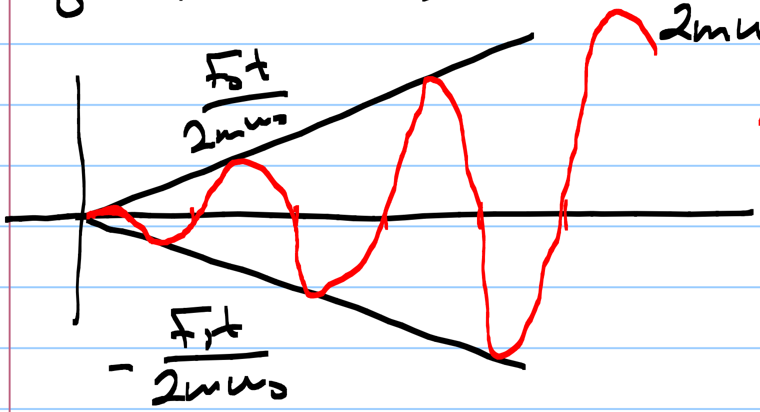
$u_g = u_c + u_p \quad , \quad u_c = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$

$u_p = ? \qquad \cancel{\cos\omega_0 t}, \ \cancel{\sin\omega_0 t}, \ t\cos\omega_0 t, \ t\sin\omega_0 t$

$u_p = At\cos\omega_0 t + Bt\sin\omega_0 t.$

$A = 0, \quad B = \dfrac{F_0}{2m\omega_0} \quad , \quad u_p = \dfrac{F_0}{2m\omega_0}\, t\sin\omega_0 t$

$$u_g = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t$$



Resonance

Video 31

CHAPTER 5: Series Solutions of Second Order Linear Equations

§ 5.1. Review of Power:

Unless it is a constant coefficient equation the solution of a linear O.D.E. is most probably not written as a combination of elementary functions such as polynomials, rational functions, exponentials, logarithms and trigonometric functions.

A power series is a formal sum of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n, \text{ where } x_0 \text{ and } a_n \in \mathbb{R}/\mathbb{C}$$

and x is an indeterminant.

Let $L = \limsup_n \left| \frac{a_{n+1}}{a_n} \right|$, then $R := \begin{cases} 1/L & \text{if } L \in \mathbb{R} \\ 0 & \text{if } L = \infty \\ \infty & \text{if } L = 0 \end{cases}$

is called the radius of convergence of the power series.

Theorem: Assume the above set up.

1) If $|x-x_0| < R$, $x \in \mathbb{R}/\mathbb{C}$, then the power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \text{ converges absolutely}$$

(i.e. $\sum_{n=0}^{\infty} |a_n (x-x_0)^n|$ is convergent)

2) If $|x-x_0| > R$ then the series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is divergent.

3) If $|x-x_0| = R$ then no conclusion.

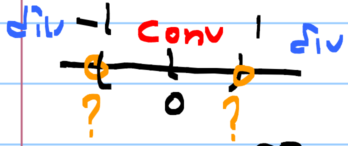
Example: Determine the radius of convergence and the interval of convergence of the following series:

a) $\sum_{n=0}^{\infty} x^n$, b) $\sum_{n=1}^{\infty} \frac{x^n}{n}$, c) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, d) $\sum_{n=0}^{\infty} n!\,x^n$, e) $\sum_{n=0}^{\infty} (-1)^n x^{2n}$.

<u>Solution:</u>

a) $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, $x_0 = 0$ and $a_n = 1, \forall n$.

$L = \limsup \left| \frac{a_{n+1}}{a_n} \right| = \limsup 1 = 1$, so $R = 1/L = 1$.



<u>End points:</u> $-1, 1$.

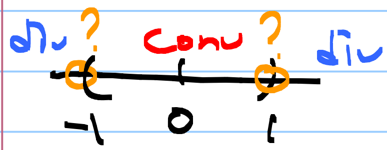
<u>$x=-1$</u> : $\sum_{n=0}^{\infty} (-1)^n$ is divergent since $\lim_n (-1)^n \neq 0$.

$x=1$: $\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$ is clearly divergent.

Hence, the interval of convergence is $(-1,1)$.

b) $\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, $x_0 = 0$, $a_n = 1/n$, $n \geq 1$.

$L = \limsup \left| \frac{a_{n+1}}{a_n} \right| = \limsup \frac{1/n+1}{1/n} = 1$, so $R = 1/L = 1$.



End points: $-1, 1$.

$x=-1$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by Alternating Series <u>Test</u>: Given a sequence (a_n) with

1) $a_n \geq 0$, 2) $a_{n+1} \leq a_n$,
3) $\lim a_n = 0$.

Then $\sum_{n=0}^{\infty} (-1)^n a_n$ is convergent.

$x=1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (it is the Harmonic Series).

Hence, the interval of convergence is $[-1, 1)$.

c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, $x_0 = 0$, $a_n = 1/n!$

So $L = \limsup \left| \frac{a_{n+1}}{a_n} \right| = \limsup \left| \frac{1/(n+1)!}{1/n!} \right| = 0.$

So, $R = \infty$. Hence, the series converges (absolutely) for all $x \in \mathbb{R}/\mathbb{C}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

d) $\sum_{n=0}^{\infty} n!\, x^n$, $x_0 = 0$, $a_n = n!$, $L = \limsup \left| \frac{a_{n+1}}{a_n} \right| = +\infty$
$\Rightarrow R = 0$.

Hence, the series is convergent only for $x = 0$.

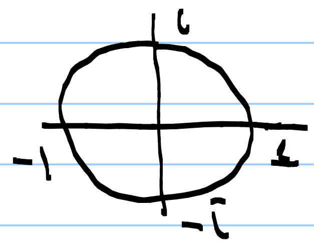
e) $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} t^n$, when $t = -x^2$.

We know that $\sum_{n=0}^{\infty} t^n$ is convergent only for $|t| < 1$.

$|t| < 1 \Leftrightarrow |-x^2| < 1 \Leftrightarrow |x| < 1$. So $R = 1$.

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}, \text{ for } |t| < 1.$$

$$\Rightarrow \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1.$$



$|x| < 1$

Suppose that $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ has radius of convergence $R>0$.

Then $f:(x_0-R, x_0+R) \longrightarrow \mathbb{R}$, $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

is a well defined function.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + \dots$$

$$f'(x) = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1} = a_1 + 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \dots + n a_n (x-x_0)^{n-1} + \dots$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x-x_0)^{n-2} = 2a_2 + 6a_3(x-x_0) + \dots + n(n-1)(x-x_0)^{n-2} + \dots$$

$$\vdots$$

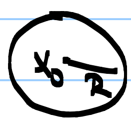
$f(x_0) = a_0$, $f'(x_0) = a_1$, $f''(x_0) = 2a_2 = 2!\,a_2$

$f'''(x_0) = 6a_3 = 3!\,a_3, \dots, f^{(n)}(x_0) = n!\,a_n$.

Hence, $a_n = \dfrac{f^{(n)}(x_0)}{n!}$, $n = 0, 1, 2, \dots$.

Suppose we have two power series with

$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n$, $\forall x$ with $|x-x_0| < R$.



$$a_n = \frac{f^{(n)}(x_0)}{n!} = b_n.$$

So, if $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$, for all x, $|x-x_0| < R$, then

$a_n = 0$, for all $n = 0, 1, 2, \dots$

§5.2 and 5.3. Series Solutions Near an Ordinary Point, Part I and Part II.

Consider a second order linear O.D.E. of the form (homogeneous)

(*) $P(x)y'' + Q(x)y' + R(x)y = 0$, where

P, Q and R are analytic functions about some point say x_0.

$P(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n$, $Q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n$, $R(x) = \sum_{n=0}^{\infty} r_n (x-x_0)^n$

We'll say that the point x_0 is an ordinary point for the equation if $P(x_0) \neq 0$.

Then (*) be written about x_0 as

$$y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0.$$

Example: (Airy's Equation) $y'' - xy = 0$, $x \in \mathbb{R}$

$y = y(x)$ $P(x) = 1$, $Q(x) = 0$, $R(x) = -x$, which are all analytic functions. Since $P(x) = 1 \neq 0$ for all x, any $x \in \mathbb{R}$ is an ordinary point for the Airy's equation

Theorem: If x_0 is an ordinary point of the equation

$P(x)y'' + Q(x)y' + R(x)y = 0$ then the equation has general solution of the form

Video 32

$y=\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1(x) + a_1 y_2(x)$, where a_0 and a_1 are arbitrary constant and $y_1(x)$ and $y_2(x)$ are linearly independent series solutions, that are analytic at x_0. Moreover, the radius of convergence of each $y_i(x)$'s is at least the radius of convergence of the functions $Q(x)/P(x)$ and $R(x)/P(x)$

<u>Example:</u> Solve the equation (Airy's Equation)

$$y'' - xy = 0.$$

<u>Solution:</u> $P(x)=1$, $Q(x)=0$, $R(x)=-x$, $\frac{Q}{P}=0$, $\frac{R}{P}=-x$ are analytic functions with radius of convergence ∞. So by theorem the general solution has the form

$y=\sum_{n=0}^{\infty} a_n x^n = a_0 y_1(x) + a_1 y_2(x)$, where $y_i(x)$ are analytic functions with radius of convergence ∞.

$$y=\sum_{n=0}^{\infty} a_n x^n, \quad y'=\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Plug them into the equation to get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x\cdot \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2\cdot 1\cdot a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_{n-1}\right] x^n = 0 = \sum_{n=0}^{\infty} 0\cdot x^n$$

$$\Rightarrow 2a_2 = 0, \quad (n+2)(n+1)a_{n+2} - a_{n-1} = 0.$$

$$\Rightarrow a_2 = 0, \quad a_{n+2} = \frac{a_{n-1}}{(n+1)(n+2)}, \quad n \geq 1.$$

$$a_3 = \frac{a_0}{6}, \quad a_4 = \frac{a_1}{12}, \quad a_5 = \frac{a_2}{\cdots} = 0.$$

$$a_6 = \frac{a_3}{5.6} = \frac{a_0}{5.6.6}, \quad a_7 = \frac{a_4}{6.7} = \frac{a_1}{6.7.12}, \ldots$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$$

$$= a_0 + a_1 x + \frac{a_0}{6}x^3 + \frac{a_1}{12}x^4 + \frac{a_0}{180}x^6 + \frac{a_1}{42 \times 12}x^7 + \cdots$$

$$= a_0 \underbrace{\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right)}_{= y_1(x)} + a_1 \underbrace{\left(x + \frac{x^4}{12} + \frac{x^7}{42 \times 12} + \cdots\right)}_{= y_2(x)}$$

Example: $y'' - xy' + 2y = 0$.

$P(x) = 1, \ Q(x) = -x, \ R(x) = 2$.

$\frac{Q}{P} = -x, \ \frac{R}{P} = 2$ analytic with $R = +\infty$.

$\Rightarrow y = a_0 y_1 + a_1 y_2$

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Plug these into the equation $y'' - xy + 2y = 0$:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum a_n x^n = 0.$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

$$2.1.a_2 x^0 + 2a_0 x^0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - n a_n + 2a_n\right]x^n = 0.$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}[\quad\quad]x^n = 0$$

$$\Rightarrow 2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0.$$

$$(n+2)(n+1)a_{n+2} + (2-n)a_n = 0, \quad n \geq 1.$$

$$\Rightarrow a_{n+2} = \frac{(n-2)a_n}{(n+2)(n+1)}, \quad n \geq 1.$$

$$a_3 = \frac{-a_1}{6}, \quad a_4 = 0, \quad a_5 = \frac{1.a_3}{5.4}, \quad a_6 = 0,$$

$$a_7 = \frac{3a_5}{7.6}$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x - a_0 x^2 - \frac{a_1}{6}x^3 - \frac{a_1}{120}x^5 - \frac{3a_1}{120 \times 42}x^7 + \cdots$$

$$= a_0\underbrace{(1 - x^2)}_{= y_1(x)} + a_1\underbrace{\left(x - \frac{x^3}{6} - \frac{x^5}{120} - \frac{x^7}{40 \times 42} + \cdots\right)}_{= y_2(x)}$$

Regular Singular Points:

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

Suppose $x_0 \in \mathbb{R}$ is a singular point, i.e., $P(x_0) = 0$.

In general obtaining the solution about a singular point is quite difficult. However, in some cases, we may obtain the solutions.

<u>Definition</u>: A singular point x_0 for the above equation is called a <u>regular singular point</u> if the following limits exist:

$$\lim_{x \to x_0} (x-x_0) \cdot \frac{Q(x)}{P(x)}, \quad \lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)}.$$

<u>Example</u>: $(1-x^2)y'' - 2xy' + 2y = 0.$

$P(x) = 1-x^2, \quad Q(x) = -2x, \quad R(x) = 2.$

$P(x) = 0 \Rightarrow 1-x^2 = 0 \Rightarrow x_0 = \pm 1$, are the only singular points of the equation.

Are they regular?

<u>$x_0 = -1$</u>: $\displaystyle \lim_{x \to x_0} (x-x_0) \frac{Q(x)}{P(x)} = \lim_{x \to -1} (x+1) \frac{(-2x)}{(1-x^2)}$

$$= \lim_{x \to -1} \frac{-2x}{1-x}$$

$$= \frac{2}{2} = 1 \checkmark$$

$$\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to -1} (x+1)^2 \frac{2}{1-x^2}$$

$$= \lim_{x \to -1} (x+1) \frac{2}{1-x}$$

$$= 0. \checkmark$$

Hence, $x_0 = -1$ is a regular singular point.

<u>$x_0 = 1$</u>, $\displaystyle \lim_{x \to x_0} (x-x_0) \frac{Q(x)}{P(x)} = \lim_{x \to 1} (x-1) \frac{-2x}{1-x^2}$

$$= \lim_{x \to 1} \frac{+2x}{1+x} = \frac{2}{2} = 1$$

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to 1} (x-1)^2 \frac{2}{1-x^2}$$

$$= \lim_{x \to 1} (x-1) \frac{-2}{1+x}$$

$$= 0$$

Again $x_0 = 1$ is a regular singular point.

Video 33

§5.4. Euler's Equations:

An equation of the form

$$x^2y'' + \alpha x y' + \beta y = 0, \text{ where } \alpha, \beta \in \mathbb{R}$$

are fixed real numbers is called an Euler's equation.

How to solve? Let's change the independent variable using $x = e^t$.

$y = y(x) = y(e^t)$

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t = x\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(x\frac{dy}{dx}\right) = \frac{d}{dx}\left(x\frac{dy}{dx}\right)\frac{dx}{dt}$$

$$= \left(1\cdot\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right)x = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}.$$

$$\frac{dy}{dt} = x\frac{dy}{dx}, \quad \frac{d^2y}{dt^2} = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}$$

$$x\frac{dy}{dx} = \frac{dy}{dt}, \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - x\frac{dy}{dx} = \frac{d^2y}{dt^2} - \frac{dy}{dt}.$$

So the Euler's equation can be written as

$$x^2\frac{d^2y}{dx^2} + \alpha x\frac{dy}{dx} + \beta y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} + \alpha\cdot\frac{dy}{dt} + \beta y = 0$$

$\Rightarrow y = y(t)$, $\dfrac{d^2y}{dt^2} + (\alpha-1)\dfrac{dy}{dt} + \beta y = 0$, which is a constant coeffient equation.

<u>Ch. Eqn.</u> $\underline{r^2 + (\alpha-1)r + \beta = 0}$.

<u>Case 1</u>: $r_1 > r_2$ two distinct real roots.

$y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$, $y_g = c_1 y_1 + c_2 y_2$

$x = e^t \Rightarrow x^{r_1} = e^{r_1 t}$ and $x^{r_2} = e^{r_2 t}$.

Hence, the $y_g = c_1 x^{r_1} + c_2 x^{r_2}$.

<u>Case 2</u>: $r_1 = r_2 = r$ double root.

$y_1 = e^{rt} = x^r$, $y_2 = t e^{rt}$, $y_2 = x^r \ln x$.

$x = e^t \Rightarrow t = \ln x$, $y_c = c_1 y_1 + c_2 y_2 = x^r (c_1 + c_2 \ln x)$

<u>Case 3</u>: $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$ non real roots,

$y_1 = e^{\lambda t}\cos\mu t$, $y_2 = e^{\lambda t}\sin\mu t$

$t = \ln x \Rightarrow e^{\lambda t} = x^\lambda$

$\Rightarrow y_1 = x^\lambda \cos\mu\ln x = x^\lambda \cos\ln x^\mu$

$y_2 = x^\lambda \sin\mu\ln x = x^\lambda \sin\ln x^\mu$.

$y_g = x^\lambda (c_1 \cos\mu\ln x + c_2 \sin\mu\ln x)$.

Example: Solve the following equations:

1) $x^2y'' + 3xy' + y = 0$. Euler equation with

$\alpha = 3$, $\beta = 1$. Ch. Eqn. $r^2 + (\alpha - 1)r + \beta = 0$

$r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$, $r_1 = r_2 = r = -1$.

$y_g = x^r (c_1 + c_2 \ln x) = x^{-1}(c_1 + c_2 \ln x)$, $x > 0$.

2) $x^2y'' + 3xy' + 2y = 0$. $\alpha = 3$, $\beta = 2$

Ch. Eqn. $r^2 + (\alpha - 1)r + \beta = 0 \Rightarrow r^2 + 2r + 2 = 0$.

$$r_{1,2} = \frac{-2 \mp \sqrt{4 - 8}}{2} = -1 \mp i, \quad \lambda = -1, \quad \mu = 1.$$

$y_g = x^{-1}(c_1 \cos \ln x + c_2 \sin \ln x)$.

3) $x^2y'' + x3y' - 3y = 0$. $\alpha = 3$, $\beta = -3$

Ch. Eqn: $r^2 + (\alpha - 1)r + \beta = 0$.

$\Rightarrow r^2 + 2r - 3 = 0 \Rightarrow (r-1)(r+3) = 0$.

$r_1 = 1$, $r_2 = -3$.

$y_g = c_1 x^{r_1} + c_2 x^{r_2} = c_1 x + c_2 x^{-3}$, $x > 0$.

§5.5 5.6. Series Solutions Near a Regular Singular Point, Part I, Part II

$P(x)y'' + Q(x)y' + R(x)y = 0$, $x_0 \in \mathbb{R}$ regular singular point.

$P(x_0) = 0$, $\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} = \alpha$, $\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \beta$.

Remark: $x^2 y'' + \alpha x y' + \beta y = 0$, $P(x) = x^2$, $Q(x) = \alpha x$, $R(x) = \beta$.

$x_0 = 0$ is the only singular point of the equation

Is it regular?

$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} = \lim_{x \to 0} x \frac{\alpha \cdot x}{x^2} = \alpha$ ✓, and

$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \frac{\beta}{x^2} = \beta$ ✓

Hence, $x_0 = 0$ is a regular singular point.

What about solutions around x_0?

In general obtaining solutions near regular singular points, in full detail, is quite complicated. Therefore, we'll just state a theorem describing solutions. We won't give any proof.

Those who are interested in the subject may look at my notes and videos of Math 424 class I gave in the spring of 2021.

Theorem: Consider the following equation

$P(x)y'' + Q(x)y' + R(x)y = 0$. Let $x_0 \in \mathbb{R}$ be a regular singular point. Let α and β be the following limits:

$$\alpha = \lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)} \quad \text{and} \quad \beta = \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}.$$

The equation $r^2 + (\alpha - 1) r + \beta = 0$ is called the indicial equation of the differential for the regular singular point x_0. Let r_1, r_2 be the roots of the indicial equation. Then we have the following (assuming $x_0 = 0$, otherwise replace x by $x - x_0$.)

1) If r_1, r_2 are real and $r_1 - r_2 > 0$ is not an integer, then

$$y_1 = |x|^{r_1} \left(1 + \sum_{n=1}^{\infty} a_n x^n\right) \text{ and } y_2 = |x|^{r_2} \left(1 + \sum_{n=1}^{\infty} b_n x^n\right)$$

for some real numbers a_n, b_n.

2) If $r_1 = r_2$ are both real then

$$y_1 = |x|^{r_1} \left(1 + \sum_{n=1}^{\infty} a_n x^n\right) \text{ and } y_2 = y_1(x) \ln|x| + |x|^{r_1} \sum_{n=1}^{\infty} b_n x^n$$

for some a_n, b_n.

3) If $r_1 - r_2 = N$ is a positive integer, then

$$y_1 = |x|^{r_1} \left(1 + \sum_{n=1}^{\infty} a_n x^n\right), \text{ and}$$

$$y_2(x) = a y_1(x) \ln|x| + |x|^{r_2} \left(1 + \sum_{n=1}^{\infty} b_n x^n\right), \text{ for some}$$

Video 34

a, a_n, b_n.

Example Solve the equation
$$2x(1+x)y'' + (3+x)y' - xy = 0$$
near the regular singular point.

Solution: $P(x) = 2x(1+x)$, $Q(x) = 3+x$, $R(x) = -x$.

$P(x) = 0 \Rightarrow 2x(1+x) = 0 \Rightarrow x = 0, x = -1$ the singular point for the equation.

Are they regular?

$x=0$, $\lim_{x \to x_0} (x-x_0) \frac{Q(x)}{P(x)} = \lim_{x \to 0} x \cdot \frac{(3+x)}{2x(1+x)} = \frac{3}{2} = \alpha$

$$\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2 \frac{(-x)}{2x(1+x)} = 0 = \beta.$$

$x=-1$ $\lim_{x \to x_0} (x-x_0) \frac{Q(x)}{P(x)} = \lim_{x \to -1} (x+1) \frac{(3+x)}{2x(1+x)} = \frac{2}{-2} = -1 = \alpha$

$$\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to -1} \frac{(x+1)^{\cancel{2}}(-x)}{2x\cancel{(1+x)}} = 0 = \beta$$

$x=0$ and $x=-1$ are regular singular points.

Solution about $x_0 = 0$:

Indicial Equation: $r^2 + (\alpha-1)r + \beta = 0$

$r^2 + (\frac{3}{2} - 1)r + 0 = 0 \Rightarrow r(r + 1/2) = 0$.
$r_1 = 0 > r_2 = -1/2$. $r_1 - r_2 = 1/2 \notin \mathbb{Z}$.

So, by the theorem we have linearly independent solutions of the form

$$y_1 = |x|^{r_1}\left(1+\sum_{n=1}^{\infty} a_n x^n\right) \text{ and } y_2 = |x|^{r_2}\left(1+\sum_{n=1}^{\infty} b_n x^n\right).$$

$$r_1 = 0 \Rightarrow y_1 = 1+\sum_{n=1}^{\infty} a_n x^n,\ y_1' = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ and}$$

$$y_1'' = \sum_{n=1}^{\infty} \underline{n(n-1)} a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Plug them into the equation $2x(1+x)y'' + (3+x)y' - xy = 0$.

$$2x(1+x)\sum_{n=1}^{\infty} n(n-1)a_n x^{n-2} + (3+x)\sum_{n=1}^{\infty} n a_n x^{n-1} - x\left(1+\sum_{n=1}^{\infty} a_n x^n\right) = 0.$$

$$\Rightarrow \sum_{n=1}^{\infty} 2n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} 2n(n-1)a_n x^n + \sum_{n=1}^{\infty} 3n a_n x^{n-1}$$
$$+ \sum_{n=1}^{\infty} n a_n x^n - x - \sum_{n=1}^{\infty} a_n x^{n+1} = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} 2(n+1)n a_{n+1} x^n + \sum_{n=1}^{\infty} 2n(n-1)a_n x^n$$
$$+ \sum_{n=0}^{\infty} 3(n+1)a_{n+1} x^n + \sum_{n=1}^{\infty} n a_n x^n - x - \sum_{n=2}^{\infty} a_{n-1} x^n = 0.$$

$$\left(\underline{4a_2 x} + \underline{\underline{3a_1}} + \underline{6a_2 x} + \underline{a_1 x} - \underline{x}\right) +$$
$$\sum_{n=2}^{\infty}\Big[2(n+1)n a_{n+1} + 2n(n-1)a_n + 3(n+1)a_{n+1} + n a_n - a_{n-1}\Big] x^n = 0.$$

x^0 : $3a_1 = 0$

x : $10a_2 + a_1 - 1 = 0$

$x^n, n \geq 2$: $\left[2n(n+1) + 3(n+1)\right] a_{n+1} + \left[2n(n-1)+n\right] a_n - a_{n-1} = 0$

$\Rightarrow a_1 = 0 \; , \; 10a_2 - 1 = 0 \Rightarrow a_2 = 1/10.$

$$a_{n+1} = \frac{a_{n-1} - (2n^2 - n)a_n}{2n^2 + 5n + 3} \; , \quad n \geq 2.$$

$$n = 2 \rightarrow a_3 = \frac{a_1 - 6a_2}{8 + 10 + 3} = -\frac{\cancel{6}^{3}}{\cancel{10}_{5}} \cdot \frac{1}{\cancel{21}_{7}} = -\frac{1}{35}$$

$$n = 3 \Rightarrow a_4 = \frac{a_2 - 15a_3}{18 + 15 + 3} = \frac{1/10 + {}^{3}\cancel{15}/\cancel{35}_{7}}{36}$$

$$= \frac{37}{70 \times 36} \; \ldots$$

So, $y_1 = 1 + \sum_{n=1}^{\infty} a_n x^n = 1 + \frac{1}{10}x^2 - \frac{1}{35}x^3 + \frac{37}{70 \times 36}x^4 + \ldots$

$$y_2 = |x|^{r_2}\left(1 + \sum_{n=1}^{\infty} b_n x^n\right) = x^{-1/2}\left(1 + \sum_{n=1}^{\infty} b_n x^n\right)$$
$$= x^{-1/2} + \sum_{n=1}^{\infty} b_n x^{n-1/2}.$$

$$y_2' = -\frac{1}{2}x^{-3/2} + \sum_{n=1}^{\infty} b_n (n - 1/2) x^{n-\frac{3}{2}}$$

$$y_2'' = \frac{3}{4}x^{-5/2} + \sum_{n=1}^{\infty} b_n (n - \tfrac{1}{2})(n - \tfrac{3}{2}) x^{n - \frac{5}{2}}.$$

Exercise: Plug these into the equation

$2x(1+x)y'' + (3+x)y' - xy = 0$ and compute b_n's.

<u>Solution about $x_0 = -1$:</u> $\alpha = -1$, $\beta = 0$.

<u>Indicial equation:</u> $r^2 + (\alpha - 1)r + \beta = 0$

$\Rightarrow r^2 - 2r = 0$, $r(r-2) = 0$
$r_1 = 2$, $r_2 = 0$.

Now by the theorem the two linearly independent solutions are given as

$$y_1 = |x - x_0|^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\right)$$

$$y_2 = a y_1(x) \ln|x - x_0| + |x - x_0|^{r_2}\left(1 + \sum_{n=1}^{\infty} b_n (x - x_0)^n\right)$$

$$y_1 = (x+1)^2\left(1 + \sum_{n=1}^{\infty} a_n (x+1)^n\right) = (x+1)^2 + \sum_{n=1}^{\infty} a_n (x+1)^{n+2}$$

$$y_1' = 2(x+1) + \sum_{n=1}^{\infty} (n+2) a_n (x+1)^{n+1}$$

$$y_1'' = 2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_n (x+1)^n$$

Plug these into the equation

$$2x(1+x)y'' + (3+x)y' - xy = 0.$$

$2(x+1-1)(1+x)y'' = 2(x+1)y'' - 2y''$
$(3+x)y' = (2 + (x+1))y' = 2y' + (x+1)y'$
$-xy = -(x+1)y + y$

So, we get $2(x+1)\left(2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_n (x+1)^n\right)$
$- 2\left(2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_n (x+1)^n\right)$

Rest is left as an exercise!

Video 35

CHAPTER 6: The Laplace Transform

This is another method to solve (basically) second order constant coefficient linear equations

$$ay'' + by' + cy = g(t),$$

where $g(t)$ is not continuous or not even a function!

§6.1. Definition of Laplace Transform:

For any function $f(t)$ its Laplace transform is defined by the following formula, whenever the integrals are convergent:

$$\mathcal{L}\{f(t)\} = F(s) \doteq \int_0^\infty e^{-st} f(t)\, dt$$

Examples: 1) $\mathcal{L}\{1\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} \cdot 1\, dt$

$f(t) = 1$

$$= \lim_{R\to\infty} \int_0^R e^{-st}\, dt$$

$$= \lim_{R\to\infty} \frac{e^{-st}}{-s} \Big|_{t=0}^{t=R}$$

$$= \lim_{R\to\infty} \left(\frac{e^{-sR} - e^{0}}{-s}\right)$$

$$= \frac{0-1}{-s} = 1/s, \text{ if } s>0.$$

So, $\mathcal{L}\{1\} = F(s) = \frac{1}{s}$, for $s>0$.

2) $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} e^{at}\,dt$

$f(t) = e^{at}$

$$\Rightarrow \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\,dt$$

$e^{at} = 1$ if $a=0$

$\frac{1}{s} = \mathcal{L}\{1\} = \mathcal{L}\{e^{at}\}$

$= \frac{1}{s-a}$

$= \frac{1}{s}$

$$= \int_0^\infty e^{-s't}\,dt, \quad s' = s-a$$

$$= F(s') = \frac{1}{s'}, \quad s' > 0$$

$$= \frac{1}{s-a}, \quad s > a.$$

Hence, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a.$

3) $\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt$

$$= \lim_{R\to\infty} \int_0^R e^{-st} \sin at\,dt$$

$$I = \int_0^R \underbrace{e^{-st}}_{u}\, \underbrace{\sin at\,dt}_{dv} = uv\Big|_0^R - \int_0^R v\,du$$

$u = e^{-st}, \; du = -se^{-st}\,dt$

$dv = \sin at\,dt$

$v = -\frac{\cos at}{a}$

$$\Rightarrow I = -\frac{e^{-st}\cos at}{a}\Big|_0^R - \int_0^R \frac{s}{a} e^{-st} \cos at\,dt$$

$$\Rightarrow I = -\frac{e^{-st}\cos at}{a}\Big|_0^R - \frac{s}{a}\underbrace{\int_0^R \underbrace{e^{-st}}_{u}\, \underbrace{\cos at\,dt}_{dv}}_{uv\big|_0^R - \int_0^R v\,du}$$

$u = e^{-st}, \; du = -se^{-st}\,dt$

$dv = \cos at\,dt \Rightarrow v = \frac{\sin at}{a}$

$$I = \frac{-e^{-st}\cos at}{a}\Big|_0^R - \frac{s}{a}\left(\frac{e^{-st}\sin at}{a}\Big|_0^R - \int_0^R -\frac{s}{a}e^{-st}\sin at\,dt\right)$$

$$I = \left(-\frac{e^{-st}\cos at}{a} - \frac{s}{a^2}e^{-st}\sin at\right)\Big|_0^R - \left(\frac{s}{a}\right)^2 \underbrace{\int_0^R e^{-st}\sin at\,dt}_{\text{``}I}$$

$$\Rightarrow \left(1+\left(\frac{s}{a}\right)^2\right) I = \left(-\frac{e^{-st}\cos at}{a} - \frac{s}{a^2}e^{-st}\sin at\right)\Big|_0^R$$

$$= \Big(-\underbrace{\frac{e^{-sR}\cos aR}{a}}_{\downarrow\, 0} - \underbrace{\frac{s}{a^2}e^{-sR}\sin aR}_{\downarrow\, 0} + \frac{1}{a}\Big)$$

and if $s>0$.

$$I = \int_0^R e^{-st}\sin at\,dt \to \frac{a^2}{s^2+a^2}\cdot\frac{1}{a} \quad \text{as } R\to\infty, \text{ provided that } s>0.$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}, \quad \text{for } s>0.$$

Similarly, $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$, for $s>0$.

__Proposition:__ Laplace transform is linear.

__Proof:__ $$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\,dt$$

$$= c_1\int_0^\infty e^{-st} f_1(t)\,dt + c_2 \int_0^\infty e^{-st} f_2(t)\,dt$$

$$= c_1\,\mathcal{L}\{f_1(t)\} + c_2\,\mathcal{L}\{f_2(t)\}.$$

∎

§6.2. Solutions of Initial Value Problems:

Theorem: Suppose that $f(t)$ is a piecewise continuous function on $[0,\infty)$. If $|f(t)| \leq Ke^{at}$ for all $t \geq M$, where a, K and M are some positive constants, then the Laplace transform of $f(t)$ exists for all $s > a$.

Proof:

$$\left| \int_0^R e^{-st} f(t)\,dt \right| \leq \int_0^R | e^{-st} f(t) |\,dt$$

$$\leq \int_0^R e^{-st} |f(t)|\,dt$$

$$\leq \int_0^M e^{-st} |f(t)|\,dt + \int_M^R e^{-st} K \cdot e^{at}\,dt$$

$$\leq \int_0^M e^{-st} |f(t)|\,dt + \int_M^R K e^{-(s-a)t}\,dt,$$

where the first integral is convergent since $f(t)$ is piecewisely continuous and the second integral is convergent provided that $s > a$.

Hence, $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$. ▪

Theorem: Suppose that f is continuous and f' is piecewise continuous on $[0,\infty)$. If $|f(t)| \leq Ke^{at}$ for all $t \geq M$, where K, a and M are some positive constants then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0).$$

Video 36

Corollary Suppose $f, f', \ldots, f^{(n-1)}$ are continuous and $f^{(n)}$ is piecewise continuous on $[0,\infty)$. If $|f^{(i)}| \le Ke^{at}$, $t \ge M$ and $i \in \{0,1,\ldots,n-1\}$, for some positive constants a, K and M, then $\mathcal{L}\{f^{(n)}(t)\}$ exists and equals

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$

for $s > a$.

Proof of the corollary By the theorem

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0).$$

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0), \text{ since } f'' = (f')'.$$
$$= s\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$
$$= s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0).$$

$$\mathcal{L}\{f'''(t)\} = s\,\mathcal{L}\{f''(t)\} - f''(0), \text{ since } f''' = (f'')'.$$
$$= s\left(s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0)\right) - f''(0)$$
$$= s^3 \mathcal{L}\{f(t)\} - s^2 f(0) - s f'(0) - f''(0)$$

$\vdots$

Similar argument finishes the proof. ∎

Proof of the theorem

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u}\, \underbrace{f'(t)\,dt}_{dv}.$$

$u = e^{-st}$
$du = -se^{-st}\,dt$
$dv = f'(t)\,dt$
$v = f(t)$

$$= uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t)\,dt$$

$$= 0 - e^{0} f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$|e^{-st}f(t)| \leq e^{-st}|f(t)| \leq e^{-st}e^{at} = e^{(a-s)t} \to 0$$

as $t \to \infty$ provided that $s > a$.

Hence, for $s > a$ we have

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0).$$

Examples 1) Solve the I.V.P.

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases}$$

Solution: $y = y(t)$, $\mathcal{L}\{y(t)\} = Y(s)$

Apply Laplace transform to the equation:

$$\mathcal{L}\{y'' - y' - 2y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0.$$

$$\Big(s^2Y(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0}\Big) - \Big(sY(s) - \underbrace{y(0)}_{=1}\Big) - 2Y(s) = 0.$$

$$s^2Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

$$\Rightarrow Y(s)(s^2 - s - 2) = s - 1$$

$$\Rightarrow Y(s) = \frac{s-1}{s^2 - s - 2}$$

We need to find $y = y(t)$ so that $\mathcal{L}\{y\} = \dfrac{s-1}{s^2-s-2}$.

$$Y(s) = \frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$\frac{A}{s+1} + \frac{B}{s-2} = \frac{A(s-2)+B(s+1)}{(s+1)(s-2)} = \frac{s-1}{(s+1)(s-2)}$$

$$\Rightarrow As + Bs + (-2A+B) = s-1$$

$$\Rightarrow \begin{array}{l} A+B=1 \\ -2A+B=-1 \\ \hline 3A = 2 \end{array} \Rightarrow A = 2/3, \; B = 1/3.$$

So, $\mathcal{L}\{y(t)\} = Y(s) = \frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$

$$\Rightarrow \mathcal{L}\{y(t)\} = \frac{2}{3}\mathcal{L}\{e^{-t}\} + \frac{1}{3}\mathcal{L}\{e^{2t}\}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\Rightarrow \mathcal{L}\{y(t)\} = \mathcal{L}\left\{\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right\}.$$

$\Rightarrow y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$, when we use the fact that "Laplace transform is injective".

Remark: Note that for a constant coefficient equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = g(t)$$

after taking the Laplace transform the coefficient of $Y(s) = \mathcal{L}\{y(t)\}$ is nothing but the characteristic polynomial with variable "s" instead of "r":

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) = \cdots.$$

2) Solve the I.V.P. $\begin{cases} y''+y=\sin 2t \\ y(0)=2, y'(0)=1. \end{cases}$

<u>Solution.</u> $Y(s)=\mathcal{L}\{y(t)\}$.

$\mathcal{L}\{y''+y\}=\mathcal{L}\{\sin 2t\}$

$\Rightarrow (s^2Y(s)-s\underset{\substack{\parallel\\2}}{y(0)}-\underset{\substack{\parallel\\1}}{y'(0)})+Y(s)=\dfrac{2}{2^2+s^2}$

$\Rightarrow (s^2+1)Y(s)-2s-1=\dfrac{2}{4+s^2}$

$\Rightarrow Y(s)=\dfrac{2s+1}{s^2+1}+\dfrac{2}{(s^2+1)(s^2+4)}$

$=2\dfrac{s}{s^2+1}+\dfrac{1}{s^2+1}+\dfrac{A}{\underset{(s^2+4)}{s^2+1}}+\dfrac{B}{\underset{(s^2+1)}{s^2+4}}$

$A+B=0 \mid 3A=2$
$4A+B=2 \mid A=\frac{2}{3}, B=-\frac{2}{3}$

$Y(s)=2\mathcal{L}\{\cos t\}+\mathcal{L}\{\sin t\}+\frac{2}{3}\mathcal{L}\{\sin t\}$
$-\frac{1}{3}\cdot\dfrac{2}{s^2+4}$

$=\mathcal{L}\{2\cos t+\frac{5}{3}\sin t\}-\frac{1}{3}\mathcal{L}\{\sin 2t\}$.

$=\mathcal{L}\{2\cos t+\frac{5}{3}\sin t-\frac{1}{3}\sin 2t\}$.

$\Rightarrow y(t)=2\cos t+\frac{5}{3}\sin t-\frac{1}{3}\sin 2t$.

$\left(y(0)=2,\ y'(0)=\frac{5}{3}-\frac{2}{3}=1\right)$

3) Solve the P.V.B. $\begin{cases} y^{IV} - y = 0 \\ y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0. \end{cases}$

Solution: $Y(s) = \mathcal{L}\{y(t)\}$.

$\mathcal{L}\{y^{IV} - y\} = 0 \Rightarrow s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0)$
$\qquad - y'''(0) - Y(s) = 0.$

$(s^4 - 1) Y(s) = s^2 \Rightarrow Y(s) = \dfrac{s^2}{s^4 - 1} = \dfrac{A}{\underset{(s^2+1)}{s^2-1}} + \dfrac{B}{\underset{(s^2-1)}{s^2+1}}$

$\left.\begin{array}{l} A+B = 1 \\ A-B = 0 \end{array}\right) \Rightarrow 2A = 1 \Rightarrow A = 1/2$
$\qquad B = 1/2$

$Y(s) = \frac{1}{2} \frac{1}{s^2-1} + \frac{1}{2} \frac{1}{s^2+1}$

$= \frac{1}{4} \left(\frac{-1}{s+1} + \frac{1}{s-1} \right) + \frac{1}{2} \frac{1}{s^2+1}$

$= \frac{1}{4} \mathcal{L}\{-e^{-t} + e^{t}\} + \frac{1}{2} \mathcal{L}\{\sin t\}$

So, $y(t) = \dfrac{e^{t} - e^{-t}}{4} + \frac{1}{2} \sin t.$

$= \frac{1}{2}(\sinh t + \sin t)$

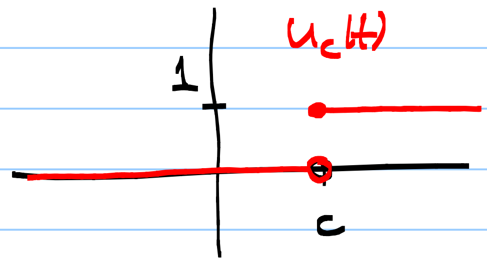
Video 37

§ 6.3. Step Functions:

Definition: Unit step function or the Heaviside function $u_c(t)$ is defined by the formula:

$$y = u_c(t), \qquad (c \in \mathbb{R} \text{ fixed})$$

$$= \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$



Theorem: If $F(s) = \mathcal{L}\{f(t)\}$ exists for all $s > a \geq 0$ and c is a positive constant then

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s).$$

Proof:

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt$$

$$= \int_c^\infty e^{-st} f(t-c)\, dt$$

$t' = t - c$

$dt' = dt$

$$= \int_0^\infty e^{-s(t'+c)} f(t')\, dt'$$

$$= \int_0^\infty e^{-s(t+c)} f(t)\, dt$$

$$= \int_0^\infty e^{-sc} e^{-st} f(t)\, dt$$

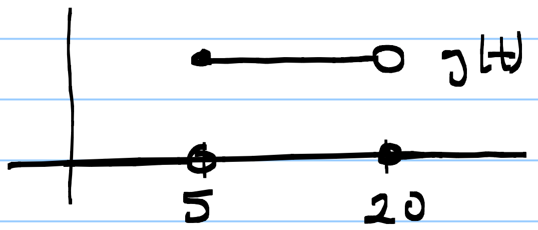
$$= e^{-sc} \int_0^\infty e^{-st} f(t)\, dt$$

$$= e^{-sc} \mathcal{L}\{f(t)\}. \quad \blacksquare$$

§ 6.4. Differential Equations with Discontinuous Forcing Functions

Example: 1) Solve the I.V.P.

$\begin{cases} 2y'' + y' + 2y = g(t) \\ y(0)=0, \ y'(0)=0 \end{cases}$, where $g(t) = \begin{cases} 1, & 5 \leq t < 20 \\ 0 & \text{otherwise} \end{cases}$



Solution $\mathcal{L}\{2y'' + y' + 2y\} = \mathcal{L}\{g(t)\}$

$2s^2 Y(s) + sY(s) + 2Y(s) = \mathcal{L}\{u_5(t) - u_{20}(t)\}$

$g(t) = \begin{cases} 1, & 5 \leq t < 20 \\ 0, & t < 5 \text{ or } t \geq 20 \end{cases}$

$= u_5(t) - u_{20}(t)$

$(2s^2 + s + 2)Y(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\}$

$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\}$

$\mathcal{L}\{u_5(t) \cdot 1\} = e^{-5s} \mathcal{L}\{1\} = e^{-5s} \frac{1}{s}$

$c = 5$

$f(t-c) = 1$

$f(t) = 1$

Similarly, $\mathcal{L}\{u_{20}(t)\} = e^{-20s} \frac{1}{s}$.

$(2s^2 + s + 2) Y(s) = \frac{1}{s} (e^{-5s} - e^{-20s})$

$Y(s) = \frac{1}{s(2s^2+s+2)} (e^{-5s} - e^{-20s})$.

$$\frac{1}{s(2s^2+s+2)} = \underset{(2s^2+s+2)}{\frac{A}{s}} + \underset{(s)}{\frac{Bs+C}{2s^2+s+2}}$$

$$A(2s^2+s+2) + (Bs+C)s = 1$$

$$2A + B = 0 \Rightarrow B = -1$$

$$A + C = 0 \Rightarrow C = -1/2$$

$$2A = 1 \Rightarrow A = 1/2$$

$$= \frac{1}{2}\frac{1}{s} - \frac{s}{2s^2+s+2} - \frac{1}{2}\frac{1}{2s^2+s+2}$$

$$\mathcal{L}\{y(t)\} = Y(s) = \left(\frac{1}{2}\frac{1}{s} - \frac{s}{2s^2+s+2} - \frac{1}{2}\frac{1}{2s^2+s+2}\right)\cdot\left(e^{-5s} - e^{-20s}\right)$$

$$Y(s) = \left(e^{-5s} - e^{-20s}\right)\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{(s+\frac{1}{4})^2 + \frac{15}{16}} - \frac{1}{4}\frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}}\right)$$

$$= \left(e^{-5s} - e^{-20s}\right)\cdot\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+1/4}{(s+\frac{1}{4})^2 + \left(\frac{\sqrt{15}}{4}\right)^2} - \frac{1}{8}\frac{1}{(s+1/4)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}\right)$$

$$= \left(e^{-5s} - e^{-20s}\right)\cdot\left(\frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\left\{\underbrace{e^{-t/4}\cos\frac{\sqrt{15}}{4}t - \frac{1}{8}e^{-t/4}\sin\frac{\sqrt{15}}{4}t}\right\}\right)$$

$$e^{-cs}\mathcal{L}\{f(t)\} = \mathcal{L}\{u_c(t) f(t-c)\}$$

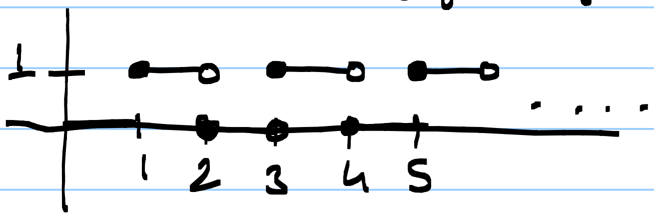
$$= \frac{1}{2}\mathcal{L}\{u_5(t) - u_{20}(t)\}$$
$$- \frac{1}{2}\mathcal{L}\Big\{u_5(t)\left(e^{-(t-5)/4}\cos\frac{\sqrt{15}}{4}(t-5) - \frac{1}{8}e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5)\right)$$
$$- u_{20}(t)\left(e^{-(t-20)/4}\cos\frac{\sqrt{15}}{4}(t-20) - \frac{1}{8}e^{-(t-20)/4}\sin\frac{\sqrt{15}}{4}(t-20)\right)\Big\}$$

Hence,

$$y(t) = \frac{u_5(t) - u_{20}(t)}{2} - \frac{u_5(t)}{2}e^{-(t-5)/4}\left(\cos\frac{\sqrt{15}}{4}(t-5) - \frac{1}{8}\sin\frac{\sqrt{15}}{4}(t-5)\right)$$
$$- \frac{u_{20}(t)}{2}e^{(t-20)/4}\left(\cos\frac{\sqrt{15}}{4}(t-20) - \frac{1}{8}\sin\frac{\sqrt{15}}{4}(t-20)\right).$$

2) Solve the I.V.P. $\begin{cases} y'' + y' + 4y = g(t) \\ y(0) = y'(0) = 0 \end{cases}$, when

$g(t)$:



$$g(t) = u_1(t) - u_2(t) + u_3(t) - u_4(t) + u_5(t) - - \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} u_n(t)$$

$$e^{-cs} \mathcal{L}\{f(t)\} = \mathcal{L}\{u_c(t) f(t-c)\}.$$

$$\mathcal{L}\{u_n(t) \cdot 1\} = e^{-ns} \mathcal{L}\{1\} = \frac{e^{-ns}}{s}$$

$f(t-n) = 1$

$f(t) = 1$

$$\mathcal{L}\{y'' + y' + 4y\} = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sum_{n=1}^{\infty} (-1)^{n-1} u_n(t)\}$$

$$(s^2 + s + 4)Y(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \mathcal{L}\{u_n(t)\}$$

$$(s^2 + s + 4)\, Y(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{-ns}}{s}$$

$$Y(s) = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-ns} \frac{1}{s(s^2+s+4)}$$

$$\frac{1}{s(s^2+s+4)} = \underset{(s^2+s+4)}{\frac{A}{s}} + \underset{(s)}{\frac{Bs+C}{s^2+s+4}} \Rightarrow \quad \begin{aligned} A + B &= 0 \Rightarrow B = -1/4 \\ A + C &= 0 \Rightarrow C = -1/4 \\ 4A &= 1 \Rightarrow A = 1/4 \end{aligned}$$

$$= \frac{1}{4} \frac{1}{s} - \frac{1}{4} \frac{s}{s^2+s+4} - \frac{1}{4} \frac{1}{s^2+s+4}$$

Video 3P

$$s^2+s+4=(s+\tfrac{1}{2})^2-\tfrac{1}{4}+4=(s+\tfrac{1}{2})^2+\tfrac{15}{16}$$

$$\frac{1}{s(s^2+s+4)}=\frac{1}{4s}-\frac{1}{4}\frac{s}{(s+\frac{1}{2})^2+\frac{15}{4}}-\frac{1}{4}\frac{1}{(s+\frac{1}{2})^2+\frac{15}{4}}$$

$$=\frac{1}{4s}-\frac{1}{4}\frac{s+1/2}{(s+1/2)^2+\frac{15}{4}}-\frac{1}{8}\frac{1}{(s+\frac{1}{2})^2+\frac{15}{4}}$$

$$=\frac{1}{4}\mathcal{L}\{1\}-\frac{1}{4}\mathcal{L}\{e^{-t/2}\cos\tfrac{\sqrt{15}}{2}t\}-\tfrac{1}{8}\mathcal{L}\{e^{-t/2}\sin\tfrac{\sqrt{15}}{2}t\}.$$

$$\mathcal{L}\{y(t)\}=Y(s)$$
$$=\sum_{n=1}^{\infty}(-1)^{n-1}e^{-ns}\Big(\qquad\qquad\Big)$$

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\Big(\frac{1}{4}\mathcal{L}\{u_n(t)\}-\frac{1}{4}\mathcal{L}\{u_n(t)e^{-(t-n)/2}\cos\tfrac{\sqrt{15}}{2}(t-n)\}$$
$$-\frac{1}{8}\mathcal{L}\{u_n(t)e^{-(t-n)/2}\sin\tfrac{\sqrt{15}}{2}(t-n)\}$$

$$e^{-cs}\mathcal{L}\{f(t)\}=\mathcal{L}\{u_c(t)f(t-c)\}$$

So, $$y(t)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{4}u_n(t)\Big(1-e^{-(t-n)/2}\cos\tfrac{\sqrt{15}}{2}(t-n)-\tfrac{1}{2}e^{-(t-n)/2}\cdot\sin\tfrac{\sqrt{15}}{2}(t-n)\Big)$$

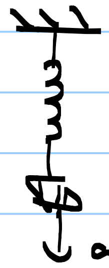
§6.5. Impulse Functions:

Unit Impulse Function, Dirac-Delta or the unit-impulse function is a generalized function (or a distribution) defined by the following properties:

$$\delta=\delta(t)=\begin{cases}0 & \text{if } t\neq 0\\ ? & \text{if } t=0\end{cases}\quad \text{and}\quad \int_{-\infty}^{+\infty}\delta(t)\,dt=1.$$

Will consider equations of the form

$$\begin{cases} ay'' + by' + cy = \delta(t-c). \\ y(0) = y'(0) = 0 \end{cases}$$

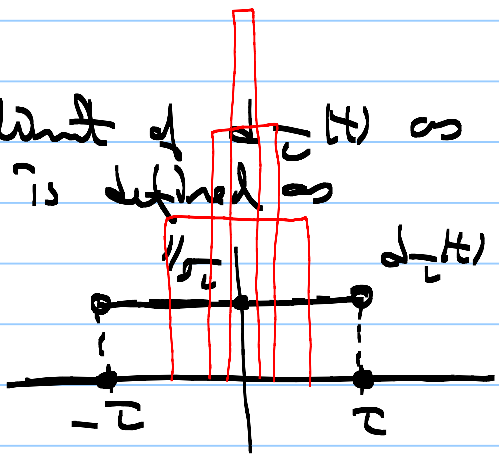


$g(t) = \delta(t-c)$

Compute $\mathcal{L}\{\delta(t-c)\}$.

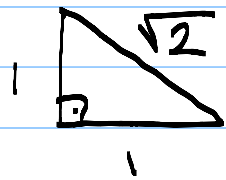
Observation: $\delta(t)$ is the limit of $d_\tau(t)$ as $\tau \to 0$, where $d_\tau(t)$ is defined as

$$d_\tau(t) = \begin{cases} 0 & \text{if } |t| > \tau \\ 1/2\tau & \text{if } |t| < \tau \end{cases}$$



$$\int_{-\infty}^{+\infty} d_\tau(t)\,dt = 2\tau \frac{1}{2\tau} = 1.$$

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t)$$



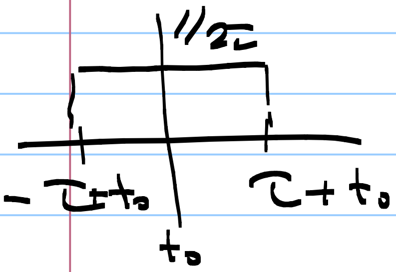
Theorem. $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$

Proof:
$$\mathcal{L}\{\delta(t-t_0)\} = \int_{-\infty}^{+\infty} e^{-st}\,\delta(t-t_0)\,dt$$

$$= \int_{-\infty}^{+\infty} e^{-st} \lim_{\tau \to 0} d_\tau(t)\,dt$$

$$= \lim_{\tau \to 0} \int_{-\infty}^{+\infty} e^{-st} d_\tau(t)\,dt$$

$$= \lim_{\tau \to 0} \int_{-\tau+t_0}^{\tau+t_0} e^{-st} \frac{1}{2\tau}\,dt$$



$\delta(t-t_0)$

$$\Rightarrow \mathcal{L}\{\delta(t-t_0)\} = \lim_{\tau\to 0} \frac{1}{2\tau} \left.\frac{e^{-st}}{-s}\right|_{-\tau+t_0}^{\tau+t_0}$$

$$= -\frac{1}{s}\lim_{\tau\to 0} \frac{1}{2\tau}\left(e^{-s\tau-st_0} - e^{s\tau-st_0}\right)$$

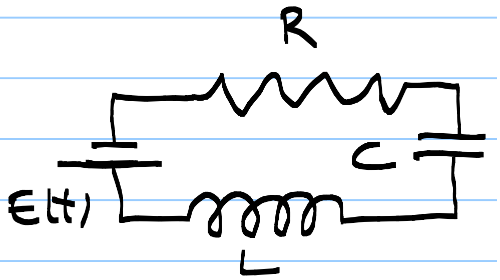
$$= -\frac{e^{-st_0}}{s}\lim_{\tau\to 0}\frac{e^{-s\tau}-e^{s\tau}}{2\tau}$$

$$= -\frac{e^{-st_0}}{s}\lim_{\tau\to 0}\frac{-se^{-s\tau}-se^{s\tau}}{2}$$

$$= -\frac{e^{-st_0}}{s}\,\frac{-s-s}{2} = e^{-st_0}$$

<u>Example.</u> Solve the I.V.P.

$$\begin{cases} y''+2y'+2y = \delta(t-5) \\ y(0)=y'(0)=0 \end{cases}$$



$E(t)=\delta(t-5)$

<u>Solution.</u> $\mathcal{L}\{y''+2y'+2y\} = \mathcal{L}\{\delta(t-5)\}.$

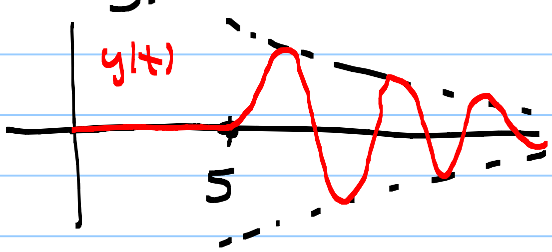
$$(s^2+2s+2)Y(s) = e^{-st_0} = e^{-5s}$$

$$Y(s) = e^{-5s}\frac{1}{s^2+2s+2} = e^{-5s}\frac{1}{(s+1)^2+1}$$

$$Y(s) = e^{-5s}\,\mathcal{L}\{e^{-t}\sin t\}.$$ $e^{-sc}\mathcal{L}\{f(t)\} = \mathcal{L}\{u_c(t)f(t-c)\}$

$$= \mathcal{L}\{u_5(t)e^{-(t-5)}\sin(t-5)\}.$$

$$y(t) = u_5(t)e^{-(t-5)}\sin(t-5)$$



Video 39

Example Solve the I.V.P.

$$\begin{cases} y'' + 2y' - 3y = 4\delta(t-2) + 3\sin 2t. \\ y(0)=0,\ y'(0)=0 \end{cases}$$

Solution: $\mathcal{L}\{y''+2y'-3y\} = \mathcal{L}\{4\underline{\underline{\delta(t-2)}}+3\sin 2t\}$

$$s^2Y(s)+2sY(s)-3Y(s) = 4e^{-2s} + 3\cdot\frac{2}{s^2+2^2}$$

$$(s^2+2s-3)Y(s) = 4e^{-2s} + \frac{6}{s^2+4}$$

$$Y(s) = \frac{4e^{-2s}}{s^2+2s-3} + \frac{6}{(s^2+4)(s^2+2s-3)} \qquad s^2+2s-3=(s-1)(s+3)$$

$$= e^{-2s}\left(\frac{A}{s-1}+\frac{B}{s+3}\right) + \left(\frac{Cs+D}{(s^2+4)}+\frac{E}{s-1}+\frac{F}{s+3}\right)$$

$$= e^{-2s}\left(\frac{1}{s-1}-\frac{1}{s+3}\right) + \quad \ldots\ldots\ldots\ldots\ldots$$

$$\frac{Cs+D}{s^2+4} + \frac{E}{s-1} + \frac{F}{s+3} = \frac{6}{\ldots\ldots}$$

$(s^2+2s-3)\quad (s^3+3s^2+4s+12)\quad (s^3-s^2+4s-4)$

$s^3:\ C+E+F=0$	$C=-E-F$	
$s^2:\ D+2C+3E-F=0$	$D+E-3F=0$	$D=-E+3F$
$s:\ 2D-3C+4E+4F=0$	$2D+7E+7F=0$	$5E+13F=0$
$1:\ -3D+12E-4F=6$	$-3D+12E-4F=0$	$15E-13F=6$

$$+\quad 20E=6 \Rightarrow E=\frac{3}{10}$$

$$F=-\frac{5E}{13}=\frac{-15}{130},\quad C=-\frac{3}{10}+\frac{15}{130}=\frac{-24}{130}$$

$$D=-E+3F=-\frac{3}{10}-\frac{45}{130}=\frac{-84}{130}$$

$$Y(s) = e^{-2s}\left(\frac{1}{s-1} - \frac{1}{s+3}\right) + \frac{Cs+D}{s^2+4} + \frac{E}{s-1} + \frac{F}{s+3}$$

$$\mathcal{L}\{y(t)\} = e^{-2s}\mathcal{L}\{e^{t} - e^{-3t}\} + C\cdot\mathcal{L}\{\cos 2t\} + \frac{D}{2}\mathcal{L}\{\sin 2t\}$$
$$+ E\,\mathcal{L}\{e^{t}\} + F\,\mathcal{L}\{e^{-3t}\}.$$

$$e^{-cs}\mathcal{L}\{f(t)\} = \mathcal{L}\{u_c(t) f(t-c)\}.$$

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{u_2(t)\,(e^{t-2} - e^{-3(t-2)})\} + \cdots\cdots$$

$$\Rightarrow y(t) = u_2(t)\,(e^{t-2} - e^{-3(t-2)}) + C\cos 2t + \frac{D}{2}\sin 2t$$
$$+ E e^{t} + F e^{-3t}.$$

§6.6. The Convolution Integral:

Definition: For two functions $f(t)$ and $g(t)$ their convolution, denoted as $f*g$, is defined by the formula

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau,$$ provided that the integral is well defined.

Remark: (a_n), (b_n) two sequences.

$f(x) = a_0 + a_1x + a_2x^2 + \cdots$, $g(x) = b_0 + b_1x + b_2x^2 + \cdots$

$h(x) = f(x)\,g(x) = c_0 + c_1x + c_2x^2 + \cdots$

$c_0 = a_0b_0$, $c_1 = a_0b_1 + a_1b_0$ $\cdots$

$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_kb_{n-k} + \cdots + a_nb_0$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} a(k)\, b(n-k) \text{ “}=\text{” } \int_0^n a(k)\, b(n-k)\, dk$$

Theorem: If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \geq 0$, then $H(s) = F(s)G(s)$ is equal to $\mathcal{L}\{h(t)\}$, where

$$h(t) = (f * g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau.$$

First some properties:

1) $f * g = g * f$

2) $f * (g_1 + g_2) = f * g_1 + f * g_2$

3) $f * (g * h) = (f * g) * h$

4) $f * 0 = 0 = 0 * f$.

Proof of the properties:

1) $$(f*g)(t) = \int_{\tau=0}^{\tau=t} f(t-\tau)\, g(\tau)\, d\tau = \int_t^0 f(t')\, g(t-t')\, (-dt')$$

$t - \tau = t' \Rightarrow -d\tau = dt'$

$$\Rightarrow (f*g)(t) = \int_0^t g(t-t')\, f(t')\, dt' = (g*f)(t)$$

2) $f*(g_1+g_2)(t) = \int_0^t f(t-\tau)(g_1(\tau)+g_2(\tau))\,d\tau$

$$= \int_0^t f(t-\tau) g_1(\tau)\,d\tau + \int_0^t f(t-\tau) g_2(\tau)\,d\tau$$

$$= (f*g_1)(t) + (f*g_2)(t)$$

$\Rightarrow f*(g_1+g_2) = f*g_1 + f*g_2.$

4) $(f*0)(t) = \int_0^t f(t-\tau)\overset{0}{g(\tau)}\,d\tau = 0.$

3) $(f*g)*h = f*(g*h)$ (Left as an exercise!)

<u>Proof of the theorem:</u>

$F(s) = \int_0^\infty e^{-s\xi} f(\xi)\,d\xi$, $G(s) = \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$

$$F(s)\,G(s) = \left(\int_0^\infty e^{-s\xi} f(\xi)\,d\xi\right)\left(\int_0^\infty e^{-s\tau} g(\tau)\,d\tau\right)$$

$$= \int_0^\infty e^{-s\tau} g(\tau)\left[\int_0^\infty e^{-s\xi} f(\xi)\,d\xi\right] d\tau$$

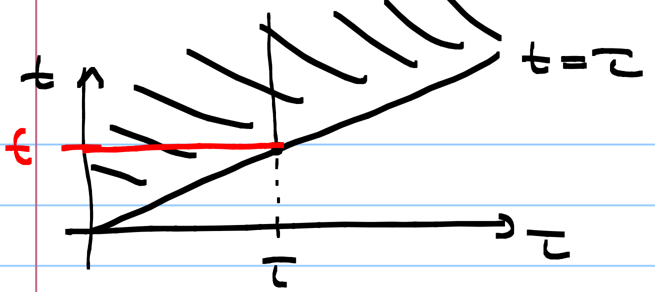
Let $t = \xi + \tau$, for fixed τ. Then $dt = d\xi$.

$$= \int_0^\infty e^{-s\tau} g(\tau)\left[\int_\tau^\infty e^{-s(t-\tau)} f(t-\tau)\,dt\right] d\tau$$

$$= \int_0^\infty g(\tau)\left[\int_\tau^\infty e^{-s\tau}\, e^{-s(t-\tau)} f(t-\tau)\,dt\right] d\tau$$

$$= \int_0^\infty g(\tau)\left[\int_\tau^\infty e^{-st} f(t-\tau)\,dt\right] d\tau$$

Video 40



$$= \int_0^\infty \left(\int_0^t e^{-st} g(\tau) f(t-\tau) \, d\tau \right) dt$$

$$= \int_0^\infty e^{-st} \left(\int_0^t \underline{\underline{g(\tau)}}\, \underline{\underline{f(t-\tau)}} \, d\tau \right) dt$$

$$= \int_0^\infty e^{-st} \, (f*g)(t) \, dt$$

$$= \mathcal{L}\{(f*g)(t)\}.$$

<u>Notation:</u> If $F(s) = \mathcal{L}\{f(t)\}$ then we'll write $f(t) = \mathcal{L}^{-1}\{F(s)\}$, the inverse Laplace transform of $F(s)$.

<u>Example:</u> 1) $\mathcal{L}^{-1}\left\{ \dfrac{a}{s(s^2+a^2)} \right\} = \mathcal{L}^{-1}\left\{ \dfrac{1}{s} \cdot \dfrac{a}{s^2+a^2} \right\}$

$f(t) = 1$

$g(t) = \sin at$

$$= \mathcal{L}^{-1}\{ \mathcal{L}\{1\} \cdot \mathcal{L}\{\sin at\}\}$$

$$= \mathcal{L}^{-1}\{ \mathcal{L}\{ 1 * \sin at \}\}$$

$$= 1 * \sin at$$

$$= \int_0^t f(t-\tau) g(\tau) \, d\tau$$

$$= \int_0^t 1 \cdot \sin a\tau \, d\tau$$

$$= -\frac{\cos a\tau}{a} \Big|_0^t = \frac{1 - \cos at}{a}.$$

2) Find the unique solution of the I.V.P.

$$\begin{cases} y'' + 4y = g(t) \\ y(0) = 3, \; y'(0) = 1 \end{cases}$$

<u>Solution:</u> $Y(s) = \mathcal{L}\{y(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}$$

$$\Rightarrow \left(s^2 Y(s) - s\underbrace{y(0)}_{3} - \underbrace{y'(0)}_{1}\right) + 4Y(s) = G(s)$$

$$(s^2+4)\,Y(s) = 3s + 1 + G(s)$$

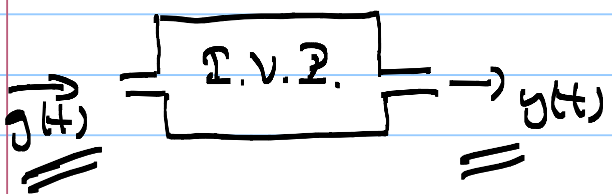
$$Y(s) = 3\,\frac{s}{s^2+4} + \frac{1}{s^2+4} + G(s)\,\frac{1}{s^2+4}$$

$$= 3\,\mathcal{L}\{\cos 2t\} + \frac{1}{2}\mathcal{L}\{\sin 2t\} + \mathcal{L}\left\{\mathcal{L}\{g(t)\}.\frac{\mathcal{L}\{\sin 2t\}}{2}\right\}$$

$$= 3\mathcal{L}\{\cos 2t\} + \mathcal{L}\{\tfrac{1}{2}\sin 2t\} + \frac{1}{2}\mathcal{L}\{\mathcal{L}\{g * \sin 2t\}\}$$

So, $y(t) = 3\cos 2t + \frac{1}{2}\sin 2t + \frac{1}{2}(g(t) * \sin 2t)$.

$$y(t) = 3\cos 2t + \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t g(t-\tau)\sin 2\tau \, d\tau.$$



<u>Theorem</u> $\mathcal{L}\{t.f(t)\} = -F'(s)$, when $F(s) = \mathcal{L}\{f(t)\}$.

<u>Proof:</u> $F(s) = \mathcal{L}\{f(t)\}$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

$$\frac{dF}{ds} = F'(s) = \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)\,dt\right)$$
$$= \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right) dt$$
$$= \int_0^\infty -t\,e^{-st} f(t)\,dt$$
$$= -\int_0^\infty e^{-st}\left(t f(t)\right) dt$$
$$= -\mathcal{L}\{t f(t)\}.$$

Hence, $\mathcal{L}\{t f(t)\} = -F'(s)$, where $F(s) = \mathcal{L}\{f(t)\}$.

<u>Example:</u> $\mathcal{L}\{1\} = \frac{1}{s}$, $s > 0$.

$$\mathcal{L}\{t\} = \mathcal{L}\{t \cdot 1\} = -\frac{d}{ds}\left(\mathcal{L}\{1\}\right)$$
$$= -\frac{d}{ds}\left(1/s\right)$$
$$= \frac{1}{s^2}.$$

$$\mathcal{L}\{t^2\} = \mathcal{L}\{t \cdot t\} = -\frac{d}{ds}\mathcal{L}\{t\}$$
$$= -\frac{d}{ds}\left(1/s^2\right)$$
$$= \frac{2}{s^3}$$

Similarly, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

<u>System Response</u>: Consider the I.V.P.

$$\begin{cases} ay'' + by' + cy = g(t) \\ y(0) = y_0, \; y'(0) = y_0'. \end{cases}$$

Let $Y(s) = \mathcal{L}\{y(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$.
Take the Laplace transform of the equation to get

$$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = G(s)$$

$$(as^2 + bs + c)\,Y(s) = (ay_0' + by_0) + ay_0 s + G(s)$$

$$Y(s) = \frac{(ay_0' + by_0) + ay_0 s}{as^2 + bs + c} + G(s)\,\frac{1}{as^2+bs+c}$$

Consider the solution of the I.V.P.

$$(*) \begin{cases} ay'' + by' + cy = \delta(t) \\ y(0) = 0, \; y'(0) = 0 \end{cases}$$

Taking Laplace transform of both sides we get

$$(as^2 + bs + c)\,Y(s) = \mathcal{L}\{\delta(t)\} = e^{-st_0} = 1 \quad (t_0 = 0).$$

$$Y(s) = \frac{1}{as^2+bs+c}.$$

Let $\phi(t)$ denote the unique solution of $(*)$, which we'll call the system response (the response of the system to the unit impulse function).

$$\mathcal{L}\{\phi(t)\} = Y(s) = \frac{1}{as^2+bs+c}$$

Back to the original problem:

$$Y(s) = \frac{(ay_0' + by_0) + ay_0 s}{as^2 + bs + c} + G(s)\,\frac{1}{as^2+bs+c}$$

$$= (As+B)\,\mathcal{L}\{\phi(t)\} + \mathcal{L}\{g(t)\}\cdot\mathcal{L}\{\phi(t)\}$$

$$= (As+B)\,\mathcal{L}\{\phi(t)\} + \mathcal{L}\{g*\phi\}$$

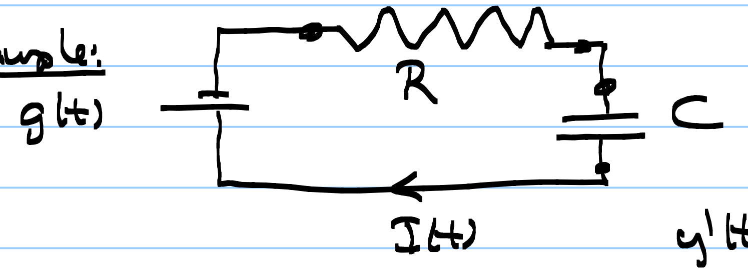
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{(As+B)\,\mathcal{L}\{\phi(t)\}\} + (g*\phi)(t)$$

Special Case $y_0 = y_0' = 0 \Rightarrow A = B = 0$ & $\phi(t)$

$$y(t) = (g*\phi)(t) \qquad \begin{cases} ay'' + by' + cy = g(t) \\ y(0) = 0,\ y'(0) = 0 \end{cases}$$

So one can determine the solution of the system for any external force function $g(t)$ by just taking the convolution of $g(t)$ with the response of the system to the unit impulse function, which is $\phi(t)$.

Example:



$y(t) = Q(t)$ the charge in the capacitor C.

$y'(t) = I(t)$ current

$$RI + \frac{Q}{C} = E(t) = g(t)$$

Assume that $Q(0) = 0 \Rightarrow y(0) = 0$.

$$\Rightarrow Ry' + \frac{1}{C}y = g(t)$$

$$\Rightarrow \mathcal{L}\{Ry' + \tfrac{1}{C}y\} = \mathcal{L}\{g(t)\}$$

$$R\left(sY(s)-\overset{\,0}{\cancel{y(0)}}\right)+\frac{1}{C}Y(s)=G(s)$$

$$Y(s)\left(Rs+\frac{1}{C}\right)=G(s)$$

$$Y(s)=G(s)\cdot\frac{1}{Rs+1/C}$$

Consider the equation $Ry'+\frac{1}{C}y=\delta(t)$ to get

$$\Phi(s)\left(Rs+\frac{1}{C}\right)=\mathcal{L}\{\delta(t)\}=1,\ \text{where } \Phi(s)=\mathcal{L}\{\phi(t)\}$$

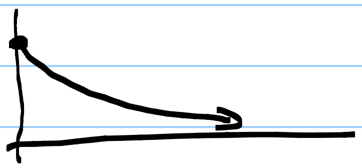
and $\phi(t)$ is the solution of $\begin{cases} Ry'+\frac{1}{C}y=\delta(t) \\ y(0)=0 \end{cases}$.

$$\Phi(s)=\frac{1}{Rs+1/C}=\frac{C}{1+RCs}=\frac{1}{R}\,\frac{1}{s+1/RC}=\frac{1}{R}\mathcal{L}\{e^{-\frac{t}{RC}}\}.$$

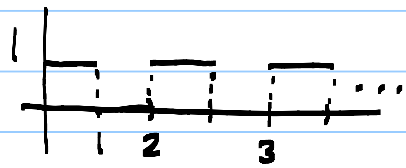
$$\Rightarrow Y(s)=G(s)\cdot\Phi(s)=\mathcal{L}\{g(t)\}\cdot\mathcal{L}\{\phi(t)\}$$

$\Rightarrow y(t)=g*\phi$, where $\phi(t)=\frac{1}{R}e^{-t/RC}$, the system response.

$1/R$



Now let $g(t)=\sum_{n=0}^{\infty}(-1)^n u_n(t)$



Then the solution is $y(t)=g*\phi$

$$y(t)=\left(\sum_{n=0}^{\infty}(-1)^n u_n(t)\ *\ \phi\right)$$

$$=\sum_{n=0}^{\infty}(-1)^n u_n(t)*\phi$$

$$(u_c*\phi)(t)=\int_0^t u_c(\tau)f(t-\tau)\,d\tau$$

$$\Rightarrow (u_c * \phi)(t) = \begin{cases} 0 & \text{if } t < c \\ \int_c^t f(t-\tau)\,d\tau, & \text{if } t \geq c. \end{cases}$$

$$= \left(\begin{cases} 0 & t<c \\ 1 & t \geq c \end{cases}\right) \cdot \int_c^t f(t-\tau)\,d\tau$$

$$= u_c(t) \cdot \int_c^t f(t-\tau)\,d\tau$$

$$= u_c(t) \left.\frac{F(t-\tau)}{-1}\right|_{\tau=c}^{\tau=t}$$

$$= u_c(t)\left(F(t-c) - F(0)\right),$$ where $F(t)$

is an antiderivative for $f(t)$. Without loss of generality we may assume that $F(0)=0$

So, $\underline{\underline{(f * u_c)(t) = u_c(t)\,F(t-c)}}$, where $F' = f$ and $F(0)=0$.

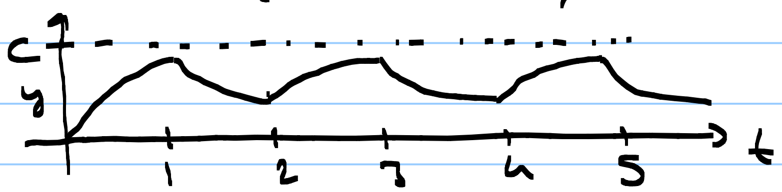
So, the solution of the original problem is the

$$y(t) = \sum_{n=0}^{\infty} (-1)^n u_n(t) * \phi(t) \qquad , \quad \phi(t) = \frac{1}{R} e^{-t/RC}$$

$$= \sum_{n=0}^{\infty} (-1)^n u_n(t) \cdot \Phi(t-n) \qquad \Phi(t) = -C \cdot e^{-t/RC} + C$$

$$= C\left(1 - e^{-t/RC}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} u_n(t)\, C\left(1 - e^{-(t-n)/RC}\right)$$



More examples

1) (p. 348, 7) Solve the I.V.P. $\begin{cases} y''+y=\delta(t-2\pi)\cos t \\ y(0)=y'(0)=1. \end{cases}$

Solution: $\delta(t-c)f(t)=\delta(t-c)f(c)$

So, $\delta(t-2\pi)\cos t=\delta(t-2\pi)\cos 2\pi=\delta(t-2\pi)$.

$$\begin{cases} y''+y=\delta(t-2\pi) \\ y(0)=y'(0)=1. \end{cases}$$ $\leftarrow$ not continuous (not even a function)

$$\mathcal{L}\{y''+y\}=\mathcal{L}\{\delta(t-2\pi)\}$$

$$\Rightarrow \left(s^2Y(s)-s\underbrace{y(0)}_{=1}-\underbrace{y'(0)}_{=1}\right)+Y(s)=e^{-2\pi s}$$

$$\Rightarrow (s^2+1)Y(s)=s+1+e^{-2\pi s}$$

$$\Rightarrow Y(s)=\frac{s}{s^2+1}+\frac{1}{s^2+1}+e^{-2\pi s}\frac{1}{s^2+1}$$

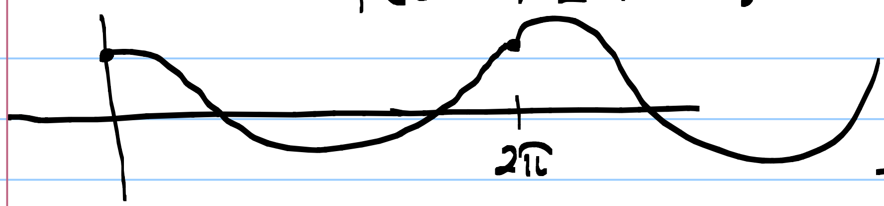
$$=\mathcal{L}\{\cos t\}+\mathcal{L}\{\sin t\}+e^{-2\pi s}\mathcal{L}\{\sin t\}.$$

$$e^{-cs}\mathcal{L}\{f(t)\}=\mathcal{L}\{u_c(t)f(t-c)\}.$$

$$\Rightarrow Y(s)=\mathcal{L}\{\cos t\}+\mathcal{L}\{\sin t\}+\mathcal{L}\{u_{2\pi}(t)\underbrace{\sin(t-2\pi)}_{=\sin t}\}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\cos t+\sin t+u_{2\pi}(t)\sin t.$$

$$=\begin{cases} \cos t+\sin t, & t\le 2\pi \\ \cos t+2\sin t, & t>2\pi \end{cases}$$



Solution is a continuous function.

2) (p. 355, 13) Solve the I.V.P. $\begin{cases} y'' + \omega^2 y = g(t) \\ y(0) = 1, \; y'(0) = 1. \end{cases}$

Solution: $\mathcal{L}\{y'' + \omega^2 y\} = \mathcal{L}\{g(t)\}$

$$(s^2 Y(s) - s\underset{1}{\underset{\shortparallel}{y(0)}} - \underset{1}{\underset{\shortparallel}{y'(0)}}) + \omega^2 Y(s) = G(s).$$

$$\Rightarrow (s^2+\omega^2) Y(s) = s + 1 + G(s)$$

$$Y(s) = \frac{1}{s^2+\omega^2}\left(s+1+G(s)\right)$$

$$= \frac{s}{s^2+\omega^2} + \frac{1}{s^2+\omega^2} + \frac{1}{s^2+\omega^2} G(s)$$

$$= \mathcal{L}\{\cos\omega t\} + \frac{1}{\omega}\mathcal{L}\{\sin\omega t\} + \frac{1}{\omega}\mathcal{L}\{\sin \omega t\} \cdot \mathcal{L}\{g(t)\}$$

$$= \mathcal{L}\{\cos\omega t\} + \frac{1}{\omega}\mathcal{L}\{\sin\omega t\} + \frac{1}{\omega}\mathcal{L}\{g(t) * \sin\omega t\}$$

$$\Rightarrow y(t) = \cos\omega t + \frac{1}{\omega}\sin\omega t + \frac{1}{\omega} g(t) * \sin\omega t.$$

3) (p. 340, 9) Solve the I.V.P. $\begin{cases} y'' + y = g(t) \\ y(0) = 0, \; y'(0) = 2 \end{cases}$,

when $g(t) = \begin{cases} t/2, & 0 \le t < 6 \\ 3, & t \ge 6 \end{cases}$

Solution: $g(t) = \dfrac{t}{2} + \begin{cases} 0, & 0 \le t < 6 \\ 3 - \dfrac{t}{2}, & t \ge 6 \end{cases}$

$$= \frac{t}{2} + \left(3 - \frac{t}{2}\right)\begin{cases} 0 & \cdots \\ 1 & \cdots \end{cases}$$

$$= t/2 + (3 - t/2)\, u_6(t)$$

$$y'' + y = \frac{t}{2} + (3 - t/2)\, u_6(t)$$

$$\Rightarrow \mathcal{L}\{y'' + y\} = \mathcal{L}\{t/2\} + \mathcal{L}\{(3 - t/2)\, u_6(t)\}$$

$$(s^2 Y(s) - s\overset{0}{y(0)} - \overset{2}{y'(0)}) + Y(s) = \frac{1}{2}\frac{1}{s^2} + e^{-6s}\mathcal{L}\{t\}$$

$$e^{-cs}\mathcal{L}\{f(t)\} = \mathcal{L}\{u_c(t) f(t-c)\} \qquad = \frac{1}{2}\frac{1}{s^2} + e^{-6s}\frac{1}{s^2}$$

$$c = 6, \quad f(t-c) = 3 - t/2$$

$$f(t-6) = 3 - t/2$$

$$f(t) = 3 - \frac{t+6}{2} = t.$$

$$\Rightarrow (s^2+1) Y(s) = 2 + \frac{1}{2}\frac{1}{s^2} + e^{-6s}\frac{1}{s^2}$$

$$\Rightarrow Y(s) = \frac{2}{s^2+1} + \frac{1}{2}\frac{1}{s^2}\cdot\frac{1}{s^2+1} + e^{-6s}\frac{1}{s^2}\cdot\frac{1}{s^2+1}$$

$$= 2\mathcal{L}\{\sin t\} + \frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right) + e^{-6s}\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$$

$$= 2\mathcal{L}\{\sin t\} + \frac{1}{2}\mathcal{L}\{t - \sin t\} + e^{-6s}\mathcal{L}\{t - \sin t\}$$

$$= \ldots\ldots \qquad + \mathcal{L}\{u_6(t)(t - 6 + \sin(t-6))\}$$

$$\Rightarrow y(t) = 2\sin t + \frac{1}{2}(t - \sin t) + u_6(t)(t - 6 + \sin(t-6))$$

$$= \frac{3}{2}\sin t + \frac{1}{2}t + u_6(t)(t - 6 + \sin(t-6)).$$

Video 42

CHAPTER 10: Partial Differential Equations and Fourier Series:

We'll study so called Heat equation and Wave equation:

$$\alpha^2 u_{xx} = u_t \quad , \qquad \alpha^2 u_{xx} = u_{tt}$$

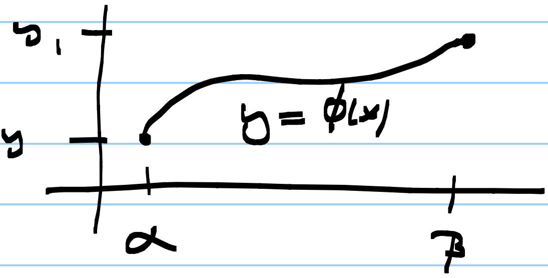
Separation of Variable: $u(x,t) = X(x)\,T(t)$

$\Rightarrow$ O.D.E.'s in for $X(x)$ and $T(t)$.

§10.1. Two Point Boundary Value Problem

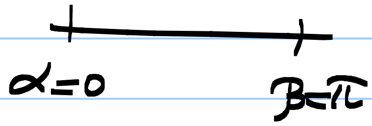
A second order linear two point boundary value problem has the form

$$\begin{cases} y'' + p(x)y' + q(x)y = g(x) \\ y(\alpha) = y_0 , \quad y(\beta) = y_1 \end{cases}$$



Unlike I.V.P.'s a two point boundary value problem may have no solution, unique solution or even infinitely many solutions, even if p, q and g are continuous on the interval $[\alpha, \beta]$.

Examples 1) $\begin{cases} y'' + 2y = 0 \\ y(0) = 1, \quad y(\pi) = 0 \end{cases}$



Solutions: $y'' + 2y = 0$ Ch. Eqn. $r^2 + 2 = 0 \Rightarrow r_{1,2} = \pm\sqrt{2}\,i$

$y_g = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

Plug the boundary conditions to determine c_i's.

$y(0)=1 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = 1 \Rightarrow c_1 = 1.$

$y(\pi)=0 \Rightarrow c_1 \cdot \cos\sqrt{2}\pi + c_2 \sin\sqrt{2}\pi = 0$

$$c_2 = -\frac{\cos\sqrt{2}\pi}{\sin\sqrt{2}\pi} = -\cot\sqrt{2}\pi$$

$y = \cos\sqrt{2}x - \cot\sqrt{2}\pi \sin\sqrt{2}x.$

Hence, in this case we obtained a unique solution

2) $\begin{cases} y'' + y = 0 \\ y(0) = 1, \ y(\pi) = a \end{cases}$

Solution Ch. Eqn. $r^2 + 1 = 0 \Rightarrow r = \mp i$

$y = c_1 \cos x + c_2 \sin x.$

$y(0)=1 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = 1 \Rightarrow c_1 = 1$

$y(\pi) = a \Rightarrow -c_1 + c_2 \cdot 0 = a \Rightarrow c_1 = -a.$

If $a \neq -1$, then there is no solution.
If $a = -1$, then $c_1 = 1$ and $c_2 \in \mathbb{R}$ so that we have infinitely many solutions

$$y = \cos x + c_2 \sin x, \quad c_2 \in \mathbb{R}.$$

3) $y'' + y = 0, \ y(0) = 0, \ y(\pi) = 0.$

Solution Ch. Eqn. $r^2 + 1 = 0 \Rightarrow r = \pm i$

$y = c_1 \cos x + c_2 \sin x.$

$y(0) = 0 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

$y(\pi) = 0 \Rightarrow c_1(-1) + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0.$

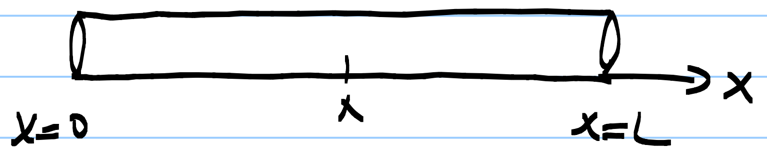
Hence, $y = c_2 \sin x, \ c_2 \in \mathbb{R}$, infinitely many solutions.

§10.5. Separation of Variables:

Heat Equation in a Rod:

Consider the following boundary value problem

$$(*)\quad \begin{cases} \alpha^2 u_{xx} = u_t \\ u(x,0) = f(x), \quad 0 \le x \le L \quad \text{Initial Temp. Distribution} \\ u(0,t) = 0, \quad t > 0 \\ u(L,t) = 0; \quad t > 0 \end{cases}$$



$u = u(x,t)$: temperature at position x at time t.

Homogeneous boundary conditions. That's why the problem is called homogeneous two point B.V.P. for Heat Equation.

How to solve? We make use of so called the method of separation of variables.

Idea: Look for solutions of the form
$u = u(x,t) = X(x)T(t)$, the product of a function of x and a function of t.

$u(x,t) = X(x)T(t)$

$u_x(x,t) = X'(x)T(t)$, $u_{xx}(x,t) = X''(x)T(t)$

$u_t(x,t) = X(x)T'(t)$

Plug these into the equation to get

$$\alpha^2 u_{xx} = u_t \Rightarrow \alpha^2 X'' T = X T'$$

$$\frac{X''}{X} = \frac{T'}{T}\frac{1}{\alpha^2} \qquad X = X(x),\ T = T(t)$$

Note that the L.H.S. is a function of x only and the R.H.S. is a function of t only. This implies that both terms are constant functions.

Say $\frac{X''}{X} = \frac{T'}{T}\frac{1}{\alpha^2} = -\lambda$, where λ is a constant.

So we get two equations:

$\frac{X''}{X} = -\lambda$ and $\frac{T'}{T}\frac{1}{\alpha^2} = -\lambda.$

$\frac{X''}{X} = -\lambda \Rightarrow \boxed{X'' + \lambda X = 0}$ a constant coefficient homogeneous equation.

$T' + \lambda\alpha^2 T = 0$, also a constant coefficient 1st order equation.

What about boundary conditions?

$u(x,t) = X(x)T(t)$

$u(0,t) = 0 \Rightarrow X(0)T(t) = 0$
$u(L,t) = 0 \Rightarrow X(L)T(t) = 0$

Note that if $X(0) \neq 0$ or $X(L) \neq 0$ then the function $T(t) \equiv 0$ and thus $u(x,t) \equiv 0$, for all x, t.

Hence, to obtain non zero solutions we must let $\boxed{X(0) = 0}$ and $\boxed{X(L) = 0.}$

Thus for $X = X(x)$ we have a two point boundary value problem:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0, \ X(L) = 0. \end{cases}$$

Let's solve the above two point B.V.P.:

<u>Ch. Eqn.</u> $r^2 + \lambda = 0. \Rightarrow r^2 = -\lambda$

<u>Case 1</u> $\lambda < 0. \Rightarrow r^2 = -\lambda > 0$
$\Rightarrow r_{1,2} = \pm\sqrt{-\lambda}$

$X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ $\quad r_1 = -r_2 = \sqrt{-\lambda}$

Use the boundary value:

$X(0) = 0 \Rightarrow c_1 \cdot 1 + c_2 \cdot 1 = 0 \Rightarrow c_1 = -c_2.$
$X(L) = 0 \Rightarrow c_1 e^{r_1 L} + c_2 e^{r_2 L} = 0$
$\Rightarrow c_1 e^{r_1 L} - c_1 e^{-r_1 L} = 0 \Rightarrow c_1 e^{r_1 L}\left(1 - \frac{1}{e^{2r_1 L}}\right) = 0$

$2r_1 L \neq 0 \Rightarrow e^{2r_1 L} \neq 1 \Rightarrow \left(1 - \frac{1}{e^{2r_1 L}}\right) \neq 0$

$\Rightarrow c_1 e^{r_1 L} = 0 \Rightarrow c_1 = 0. \Rightarrow c_2 = -c_1 = 0.$

Hence, $X(x) \equiv 0$ the only solution

<u>Case 2</u> $\lambda = 0$ $\quad r^2 = -\lambda = 0 \Rightarrow r_{1,2} = 0.$

$y_1 = e^{r_1 x} = 1, \ y_2 = x \cdot y_1 = x \cdot 1 = x.$

$y_0 = c_1 \cdot 1 + c_2 \cdot x = c_1 + c_2 x.$

Video 43

Boundary Conditions:

$X(0)=0 \Rightarrow c_1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0.$
$X(L)=0 \Rightarrow c_1 + c_2 \cdot L = 0 \Rightarrow c_2 \cdot L = 0 \quad (L>0)$
$\Rightarrow c_2 = 0.$

$\Rightarrow X = X(x) \equiv 0$ is the only solution

Case 3: $\lambda > 0$. $\quad r^2 = -\lambda < 0.$
$r_{1,2} = \pm\sqrt{\lambda}\, i$

Let $p = \sqrt{\lambda} > 0$. $r_1 = pi$, $r_2 = -pi$

$X(x) = c_1 \cos px + c_2 \sin px$

Apply the boundary conditions:

$X(0) = 0 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0.$

$X(x) = c_2 \sin px.$

$X(L) = 0 \Rightarrow c_2 \sin pL = 0.$

So, we see that to avoid getting the trivial solution $\sin pL = 0$. This implies
$pL = n\pi$, $n \in \mathbb{Z}$.

So $p_n = \frac{n\pi}{L}$ and $X(x) = c_2 \sin \frac{n\pi}{L} x.$

$X_n(x) = \sin \frac{n\pi x}{L}$, $\quad n = 1, 2, 3, \ldots$

$u(x,t) = X(x)T(t)$ $\quad \frac{X_n''}{X_n} = \frac{1}{\alpha^2}\frac{T_n'}{T_n} = -\lambda_n = -p_n^2$
$= -\left(\frac{n\pi}{L}\right)^2.$

For $T_n(t)$ we have $T_n' + \alpha^2 p_n^2 T_n = 0$.

$$\frac{dT_n}{T_n} = -\alpha^2 p_n^2\, dt \Rightarrow \int \frac{dT_n}{T_n} = -\int \alpha^2 p_n^2\, dt$$

$$\ln|T_n| = -\alpha^2 p_n^2 t + c'$$

$$\Rightarrow T_n(t) = c\, e^{-\alpha^2 p_n^2 t}$$

$$u_n(x,t) = X_n(x)\, T_n(t) = c_n \sin\frac{n\pi x}{L}\, e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$$

$$\Rightarrow u_n(x,t) = c_n \sin\frac{n\pi x}{L}\, e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t}, \quad n = 1, 2, \ldots$$

Our Heat Problem were

$$\begin{cases} \alpha^2 u_{xx} - u_t = 0 \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

The first three equations are homogeneous and thus any linear combination of the solutions of the first three equations is also a solution.

In particular, any sum of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$
$$= \sum_{n=1}^{\infty} c_n\, e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \sin\frac{n\pi}{L} x$$

is a solution of the first three equations (Heat equation + Boundary Conditions), provided that the infinite sum is meaningful.

What about the initial condition (initial temp. distribution)?

$u(x,0) = f(x)$, for all $x \in [0, L]$.

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \underbrace{e^{-\left(\frac{\alpha n\pi}{L}\right)^2 \cdot 0}}_{=1} \sin\frac{n\pi x}{L}.$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}.}$$

To obtain a unique solution for the Heat Problem (*) c_n's should be uniquely determined from the above equation.

§10.2. Fourier Series:

(Joseph Fourier (1807 1811))

A Fourier series is a series of functions of the form

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right),$$ where

a_0, a_m, b_m are constants.

Note that $\cos x$ and $\sin x$ are periodic with period 2π. Thus $\cos ax$ and $\sin ax$ are periodic with period $2\pi/a$.

Hence, $\cos\frac{m\pi x}{L}$ and $\sin\frac{m\pi x}{L}$ are periodic with period $\frac{2\pi}{(m\pi/L)} = \frac{2L}{m}$. Hence, the series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L} \quad \text{is}$$

periodic with period $2L$.

Note that the collection of all functions with period $2L$ is a vector space, say V.

$$V = \{ f:\mathbb{R} \to \mathbb{R} \mid f(x+2L) = f(x),\ \forall x \in \mathbb{R} \}.$$

Let $f, g \in V$ and $c_1, c_2 \in \mathbb{R}$, then

$$\begin{aligned}(c_1 f + c_2 g)(x+2L) &= c_1 f(x+2L) + c_2 g(x+2L) \\ &= c_1 f(x) + c_2 g(x) \\ &= (c_1 f + c_2 g)(x)\end{aligned}$$

$\Rightarrow c_1 f + c_2 g \in V.$

Hence V is a subspace of the vector space of all real valued functions from $\mathbb{R}$ to $\mathbb{R}$.

__An Inner Product on V:__

$\mathbb{R}^n$, $\quad v = (v_1, \dots, v_n)$, $\quad u = (u_1, \dots, u_n)$

$$v \cdot u = v_1 u_1 + v_2 u_2 + \dots + v_n u_n = \sum_{k=1}^{n} v_k u_k.$$

$v_k = v(k)$, $\quad u_k = u(k)$

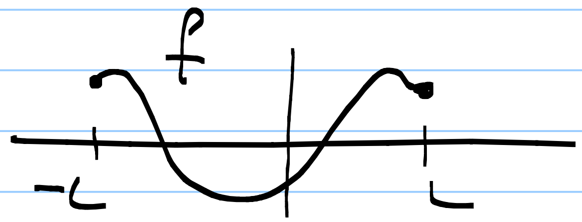
$v, u : \{1, 2, \dots, n\} \to \mathbb{R}$

$$u \cdot v = \sum_{k=1}^{\infty} v(k) u(k)$$

$f, g : \mathbb{R} \to \mathbb{R}$, $f \cdot g = \sum_{\underline{\underline{x \in \mathbb{R}}}} f(x) g(x)$ would not work!

Instead use Integral:

$$f \cdot g \doteq \int_{-L}^{L} f(x) g(x) \, dx$$



The inner product on V is defined as

$$f \cdot g \doteq \int_{-L}^{L} f(x) g(x) \, dx.$$

Is this really an inner product?
We must check the following conditions:

1) $(c_1 f_1 + c_2 f_2) \cdot g = c_1 f_1 \cdot g + c_2 f_2 \cdot g$

2) $f \cdot g = g \cdot f$

3) $f \cdot f = \|f\|^2 \geq 0$ and $f \cdot f = 0$ if and only if $f = 0$.

Proof: 1) $(c_1 f_1 + c_2 f_2) \cdot g = \int_{-L}^{L} (c_1 f_1 + c_2 f_2)(x) g(x) \, dx$

$$= c_1 \int_{-L}^{L} f_1(x) g(x) \, dx + c_2 \int_{-L}^{L} f_2(x) \cdot g(x) \, dx$$

$$= c_1 (f_1 \cdot g) + c_2 (f_2 \cdot g).$$

2) $f \cdot g = \int_{-L}^{L} f(x) g(x) \, dx = \int_{-L}^{L} g(x) f(x) \, dx = g \cdot f.$

3) $f \cdot f = \int_{-L}^{L} f(x) \cdot f(x)\, dx = \int_{-L}^{L} f^2(x)\, dx \geq 0.$

Assume now $f \cdot f = 0. \Rightarrow \int_{-L}^{L} f^2(x)\, dx = 0$

<u>must show:</u> $f(x) = 0$ for all $x \in \mathbb{R}$.

For simplicity assume that $f(x)$ is continuous.

Let $h(x) = \int_{-L}^{x} f^2(t)\, dt$. Since f is continuous

h is differentiable and $h'(x) = f^2(x)$, by the fundamental theorem of calculus.

On the other hand,

$$0 = \int_{-L}^{L} f^2(t)\, dt = \underbrace{\int_{-L}^{x} f^2(t)\, dt}_{\geq 0} + \underbrace{\int_{x}^{L} f^2(t)\, dt}_{\geq 0}$$

$$\Rightarrow h(x) = \int_{-L}^{x} f^2(t)\, dt = 0, \text{ for all } x.$$

$h'(x) = 0$, for all x. Hence $f^2(x) = h'(x) = 0$, for all x. Thus, $f(x) = 0$, for all $x \in \mathbb{R}$.

Therefore, $(V, \cdot)$ is an inner product space.

Section 4.4

What about an orthogonal basis for this inner product space?

<u>Theorem</u> The set $\left\{ \sin \frac{m\pi x}{L}, \cos \frac{m\pi x}{L} \right\}_{m=0}^{\infty}$ is an orthogonal set.

<u>Proof</u>: There are several cases:

1) $\cos \frac{m\pi x}{L} \cdot \sin \frac{n\pi x}{L} = \int_{-L}^{L} \cos \underbrace{\frac{m\pi x}{L}}_{a} \sin \underbrace{\frac{n\pi x}{L}}_{b} \, dx$

$2 \sin a \cos b = \sin(a+b) + \sin(a-b)$

$$= \frac{1}{2} \int_{-L}^{L} \left(\sin \frac{(m+n)\pi x}{L} + \sin \frac{(m-n)\pi x}{L} \right) dx$$

$= 0$, because both functions in the integral are odd function and the integral is over an interval symmetric about the origin.

2) $\sin \frac{m\pi x}{L} \cdot \sin \frac{n\pi x}{L} = \int_{-L}^{L} \sin \frac{m\pi x}{L} \cdot \sin \frac{n\pi x}{L} \, dx$

$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$

$$= \frac{1}{2} \int_{-L}^{L} \left[\cos \frac{(m-n)\pi x}{L} - \cos \frac{(m+n)\pi x}{L} \right] dx$$

<u>Subcases</u>: $m = n \geq 1$

$$\sin \frac{n\pi x}{L} \cdot \sin \frac{n\pi x}{L} = \frac{1}{2} \int_{-L}^{L} \left(1 - \cos \frac{2n\pi x}{L} \right) dx$$

$$= \frac{1}{2}\left(x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right)\Bigg|_{-L}^{L}$$

$$= \frac{1}{2}\left(2L - \frac{L}{2n\pi}(0-0)\right)$$

$$= L.$$

Subcase $m \neq n$.

$$\sin\frac{n\pi x}{L} \cdot \sin\frac{m\pi x}{L} = \frac{L}{(m-n)\pi}\sin\frac{(m-n)\pi x}{L} - \frac{L}{(m+n)\pi}\sin\frac{(m+n)\pi x}{L}\Bigg|_{-L}^{L}$$

$$= 0.$$

Similarly, $\cos\frac{n\pi x}{L} \cdot \cos\frac{m\pi x}{L} = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \geq 1. \end{cases}$

$$1 \cdot 1 = \int_{-L}^{L} 1 \cdot 1 \, dx = 2L$$

Hence, the set $\left\{1, \cos\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}\right\}_{m=1}^{\infty}$ is orthogonal.

<u>The Euler–Fourier Formula:</u>

Let $f(x)$ be a function defined by the infinite sum

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}.$$

Let's try to compute the coefficients:

$$1 = \cos\frac{0\cdot\pi x}{L}$$

$$f\cdot 1 = \frac{a_0}{2}\cdot 1 + \sum_{m=1}^{\infty}\underbrace{\left(\cos\frac{m\pi x}{L}\cdot 1\right)}_{=0} + \underbrace{\left(\sin\frac{m\pi x}{L}\cdot 1\right)}_{=0}$$

$$f\cdot 1 = \frac{a_0}{2}\, 1\cdot 1 = \frac{a_0}{2}\cdot 2L \Rightarrow a_0 = \frac{f\cdot 1}{L}$$

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.}$$

$$f\cdot\cos\frac{m\pi x}{L} = \cos\frac{m\pi x}{L}\cdot\left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right)$$

$m \geq 1$

$$= \frac{a_0}{2}\underbrace{\left(\cos\frac{m\pi x}{L}\cdot 1\right)}_{=0} + \sum_{k=1}^{\infty} a_k\underbrace{\left(\cos\frac{k\pi x}{L}\cdot\cos\frac{m\pi x}{L}\right)}_{=\delta_{km}\cdot L}$$

$$+ \sum_{k=1}^{\infty} b_k\underbrace{\left(\sin\frac{k\pi x}{2}\cdot\cos\frac{m\pi x}{L}\right)}_{=0}$$

Hence, $f\cdot\cos\frac{m\pi x}{L} = a_m\cdot L \Rightarrow a_m = \frac{1}{L}\, f\cdot\cos\frac{m\pi x}{L}$

$$\Rightarrow \boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx.}$$

Similarly, we obtain $\boxed{b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx}$

<u>Definition:</u> The Fourier series of a function $f(x)$ is defined to be the series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L},$$ where

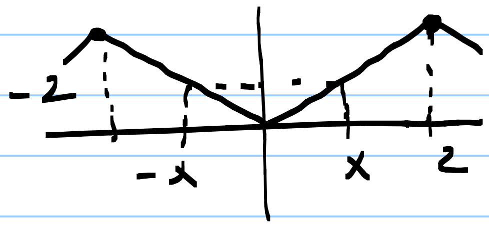
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx, \quad m = 0, 1, 2, \ldots, \text{ and}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx, \quad m = 1, 2, \ldots$$

Example: Find the Fourier series of the function

$$f(x) = \begin{cases} -x, & -2 \le x < 0 \\ x, & 0 \le x < 2 \end{cases}$$

, which is periodic with period $2L = 4$.



$L = 2$

$\left\{1, \cos\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}\right\}_{m=1}^{\infty}$

$= \left\{1, \cos\frac{m\pi x}{2}, \sin\frac{m\pi x}{2}\right\}_{m=1}^{\infty}$

$f(-x) = f(x) \Rightarrow f$ is even function

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\cdot 4 = 2.$$

$$m \ge 1, \quad a_m = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{m\pi x}{2}}_{\text{even}}\,dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{m\pi x}{2}\,dx$$

$$= \frac{2}{2}\int_{0}^{2} \underbrace{x}_{u}\,\underbrace{\cos\frac{m\pi x}{2}\,dx}_{dv}$$

$u = x \Rightarrow du = dx$

$dv = \cos\frac{m\pi x}{2}$

$v = \frac{2}{m\pi}\sin\frac{m\pi x}{2}$

$$= uv\Big|_0^2 - \int_0^2 v\,du$$

$$= x\sin\frac{m\pi x}{2}\Big|_0^2 - \int_0^2 \frac{2}{m\pi}\sin\frac{m\pi x}{2}\,dx$$

$$=\left(2\overbrace{\sin m\pi}^{\overset{0}{\shortparallel}}-0\right)+\left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}\Big|_0^2$$

$$=\left(\frac{2}{m\pi}\right)^2\left(\cos m\pi-1\right)$$

$$=\begin{cases}0, & m \text{ even}\\ \dfrac{-8}{m^2\pi^2} & m \text{ odd}\end{cases}$$

$$b_n=\frac{1}{L}\int_{-L}^{L}\overbrace{\underset{\text{even}}{f(x)}\,\underset{\text{odd}}{\sin\frac{m\pi x}{2}}}^{\text{odd}}\,dx=0.$$

Hence the Fourier series of $f(x)$ is

$$\frac{a_0}{2}+\sum_{m=1}^{\infty}a_m\cos\frac{m\pi x}{2}+b_m\sin\frac{m\pi x}{2}$$

$$=\frac{2}{2}+\sum_{\substack{m=1\\ m \text{ odd}}}^{\infty}\frac{-8}{m^2\pi^2}\cos\frac{m\pi x}{2}$$

$m=2k-1$

$$=1+\sum_{k=1}^{\infty}\frac{-8}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{2}$$

Video 45

§10.3 Fourier Convergence Theorem

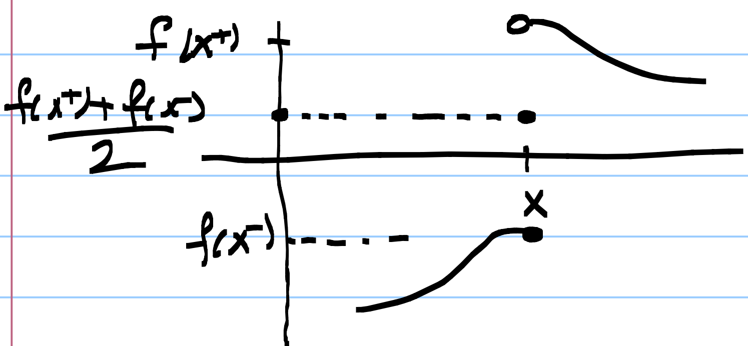
Theorem: Suppose that f and f' are piecewise continuous on the interval $-L \le x \le L$. Further suppose that f is defined outside the interval $[-L, L]$ so that it is periodic with period $2L$ ($f(x+2L) = f(x)$, for all $x \in \mathbb{R}$). Then f has a Fourier series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right), \text{ where}$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx, \quad b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx.$$

Moreover, the series converges to $f(x)$ at all points where f is continuous and to $(f(x^+)+f(x^-))/2$ at all points where f is discontinuous.

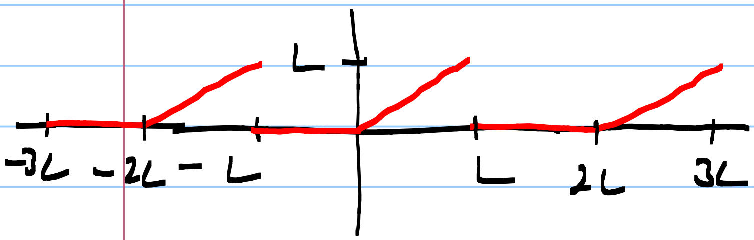
$$f(x^+) \doteq \lim_{x\to x^+} f(x) \quad \text{and} \quad f(x^-) \doteq \lim_{x\to x^-} f(x)$$



Example: $f(x) = \begin{cases} 0 & -L < x \le 0 \\ x & 0 < x < L \end{cases}$, $f(x+2L) = f(x)\ \forall x$

Compute its Fourier series.

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx$$



$$a_m = \frac{1}{L}\int_0^L \underline{x}\,\underbrace{\cos\frac{m\pi x}{L}}\,dx = \frac{1}{L}\left(x\frac{L}{m\pi}\sin\frac{m\pi x}{L}\Big|_0^L - \int_0^L 1\cdot\frac{L}{m\pi}\sin\frac{m\pi x}{L}\,dx\right)$$

$$= \frac{1}{L}\left(0 + \frac{L^2}{m^2\pi^2}\cos\frac{m\pi x}{L}\Big|_0^L\right)$$

$$= \frac{1}{L}\frac{L^2}{m^2\pi^2}\left(\cos m\pi - 1\right)$$

$$= \frac{L}{m^2\pi^2}\left((-1)^m - 1\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\frac{x^2}{2}\Big|_{-L}^{L} = 0.$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = \frac{1}{L}\int_0^L x\sin\frac{m\pi x}{L}\,dx$$

$$= \frac{1}{L}\left(x\cdot\frac{-L}{m\pi}\cos\frac{m\pi x}{L}\Big|_0^L - \int_0^L 1\cdot\frac{-L}{m\pi}\cos\frac{m\pi x}{L}\,dx\right)$$

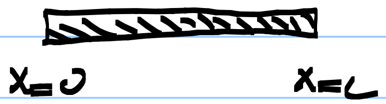
$$= \frac{1}{L}\left(\frac{-L^2}{m\pi}\cos m\pi + \underbrace{\left(\frac{L}{m\pi}\right)^2\sin\frac{m\pi x}{L}\Big|_0^L}_{=0}\right)$$

$$= \frac{-L}{m\pi}(-1)^m = \frac{(-1)^{m+1}L}{m\pi}$$

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}$$

§10.4 Even and Odd Functions

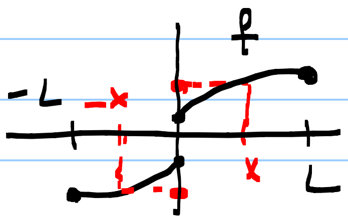
Let $f: \mathbb{R} \to \mathbb{R}$. We say that f is an even function if $f(-x) = f(x)$, for all $x \in \mathbb{R}$.
Similarly, f is called an odd function if $f(-x) = -f(x)$, for all $x \in \mathbb{R}$.



$f: [0,L] \to \mathbb{R}$

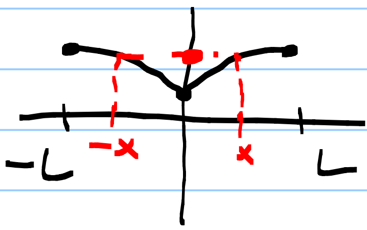
Let $f: [0,L] \to \mathbb{R}$ be a function. The odd extension $f_o : [-L,L] \to \mathbb{R}$ is defined as follows:

$$f_o(-x) = -f(x), \quad x \in [0,L].$$



Similarly, we define the even extension f_e of f to $[-L,L]$ as follows:

$$f_e(-x) = f(x), \quad \text{if } x \in [0,L].$$



Using these extensions we may obtain two different Fourier series of $f: [0,L] \to \mathbb{R}$.

<u>Odd extension:</u> f_o

$$a_m = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f_o(x)}_{\text{odd}} \underbrace{\cos\frac{m\pi x}{L}}_{\text{even}}}_{\text{odd}} dx = 0, \quad \text{for all } m.$$

$$b_m = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f_o(x)}_{\text{odd}}\ \underbrace{\sin\frac{m\pi x}{L}}_{\text{odd}}}_{\text{even}}\, dx = \frac{1}{L}\cdot 2\cdot\int_0^L f_o(x)\sin\frac{m\pi x}{L}\,dx$$

$$b_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx.$$

Hence, the Fourier series of $f_o(x)$ is as

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L} = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{L},$$

where $b_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx.$

$$\boxed{f(x) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{L}, \text{ called the sine series of } f(x).}$$

<u>Even extension</u> f_e

$$a_m = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f_e(x)\cos\frac{m\pi x}{L}}_{\text{even}}}_{\text{even}}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

and

$$b_m = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f_e(x)}_{\text{even}}\ \underbrace{\sin\frac{m\pi x}{L}}_{\text{odd}}}_{\text{odd}}\,dx = 0.$$

So, we obtain

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L}\,dx,}$$

where $a_m = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx.$

This is called the cosine series of $f(x)$.

Back to the B.V.P.

$$\begin{cases} \alpha^2 u_{xx} = u_t \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

We've obtain a solution of the form

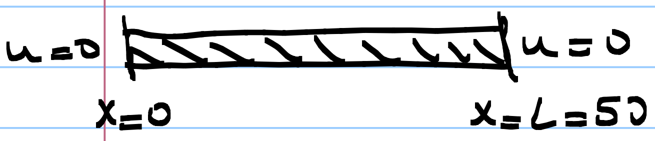
$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}$$

so that $f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$.

This is just the sine series for $f(x)$. Then

$$c_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Example: $L = 50$, $f(x) = 20$, $0 < x < 50$



$u = u(x,t)$ the temp. at x at time t

$$u = u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L},$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$

So $b_n = \frac{2}{50}\int_0^{50} 20 \cdot \sin\frac{n\pi x}{50}\, dx$

$= \frac{4}{5}\left(\frac{-50}{n\pi}\right)\cos\frac{n\pi x}{50}\Big|_0^{50}$

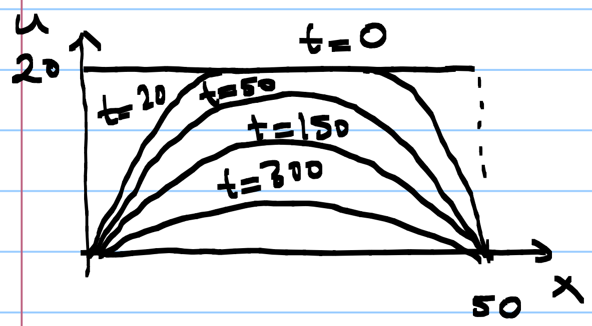
$= \frac{-40}{n\pi}\left(\cos n\pi - 1\right)$

$= \frac{-40}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ \frac{80}{n\pi} & \text{if } n \text{ odd} \end{cases}$

So, $u(x,t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} e^{-\left(\frac{n\pi\alpha}{50}\right)^2 t}\, \frac{80}{n\pi} \sin\frac{n\pi x}{50}$

$n = 2k-1$

$= \sum_{k=1}^{\infty} e^{-\left(\frac{(2k-1)\pi\alpha}{50}\right)^2 t}\, \frac{80}{(2k-1)\pi} \sin\frac{n\pi x}{50}.$



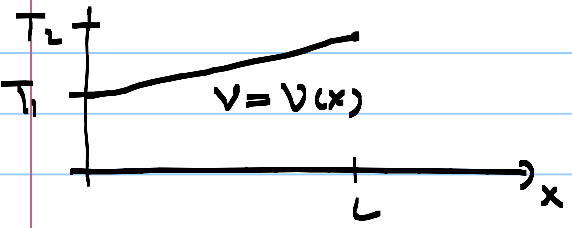
Video 46

§10.6. Other Heat Conduction Problems

Non-homogeneous Boundary Conditions

$$\begin{cases} \alpha^2 u_{xx} = u_t & 0<x<L,\ t>0 \\ u(0,t) = T_1 \\ u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases}$$

First we look for a "particular solution" for this non-homogeneous B.V.P.. As time passes we expect the temp. distribution $u(x,t)$ converges to the so called steady state solution, which is the linear solution $v = v(x)$.



$$v(x) = (T_2 - T_1)\frac{x}{L} + T_1$$

$$v(0) = T_1, \quad v(L) = T_2.$$

Note that $v_x = (T_2 - T_1)/L$, $v_{xx} = 0$ and $v_t = 0$.

So it solves the heat equation $\alpha^2 u_{xx} = u_t$.
Moreover, it satisfies the B.V.'s $v(0) = T_1$ and $v(L) = T_2$

Let $w(x,t) = u(x,t) - v(x)$. Then $w_{xx} = u_{xx}$ and $w_t = u_t$ so that $w(x,t)$ satisfies the heat equation $\alpha^2 u_{xx} = u_t$.

Moreover, $w(0,t) = u(0,t) - v(0) = T_1 - T_1 = 0$ and $w(L,t) = u(L,t) - v(L) = T_2 - T_2 = 0$.

Finally, $w(x,0) = u(x,0) - v(x) = f(x) - v(x)$.

Hence, $w(x,t)$ is the solution of the B.V.P.

$$\begin{cases} \alpha^2 w_{xx} = w_t \\ w(0,t) = 0 \\ w(L,t) = 0 \\ w(x,0) = f(x) - v(x) \end{cases}$$

Hence, $w(x,t) = \sum_{n=1}^{\infty} b_n \, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin \frac{n\pi x}{L}$,

where $b_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \cdot (f(x) - v(x)) \, dx$.

So, the solution $u(x,t) = w(x,t) + v(x)$.

Example: Find the solution of the B.V.P.

$$\begin{cases} u_{xx} = u_t & 0 < x < 30, \ t > 0 \\ u(0,t) = 20, & t > 0 \\ u(30,t) = 50, & t > 0 \\ u(x,0) = 60 - 2x, & 0 < x < 30. \end{cases}$$

Solution: $\alpha = 1$, $L = 30$, $T_1 = 20$, $T_2 = 50$

$f(x) = 60 - 2x$.

Steady state solution $v(x) = \frac{T_2 - T_1}{L} x + T_1 \Rightarrow$

$v(x) = \frac{50 - 20}{30} x + 20 = x + 20$.

$w(x,t) = \sum_{n=1}^{\infty} b_n \, e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \sin \frac{n\pi x}{L}$, where

$$b_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \cdot (f(x) - v(x)) \, dx$$

$$= \frac{2}{30} \int_0^{30} \sin \frac{n\pi x}{30} (60 - 2x - x - 20) \, dx$$

$$= \frac{1}{15}\int_0^{30} \overbrace{\sin\frac{n\pi x}{30}\,\overbrace{(40-3x)}^{u}\,dx}^{dv}$$

$$= \frac{1}{15}\left((40-3x)\frac{-30}{n\pi}\cos\frac{n\pi x}{30}\Big|_0^{30} - \int_0^{30}(-3)\frac{-30}{n\pi}\cos\frac{n\pi x}{30}\,dx\right)$$

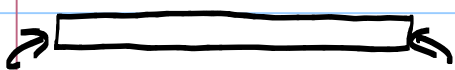
$$= \frac{1}{15}\left(\frac{1500}{n\pi}\cos n\pi + \frac{1200}{n\pi} - \underbrace{\frac{90}{n\pi}\,\frac{30}{n\pi}\sin\frac{n\pi x}{30}\Big|_0^{30}}_{=0}\right)$$

$$= \frac{80}{n\pi} + \frac{100}{n\pi}(-1)^n$$

So, $b_n = \frac{1}{n\pi}\left(80+100(-1)^n\right)$.

$$u(x,t) = w(x,t) + v(x)$$
$$= 20+x+\sum_{n=1}^{\infty} b_n\, e^{-\left(\frac{n\pi}{30}\right)^2 t}\sin\frac{n\pi x}{30}.$$

Bar with Isolated Ends:



no heat conduction at the ends

$$\begin{cases} \alpha^2 u_{xx} = u_t \\ u_x(0,t)=0, \\ u_x(L,t)=0, \\ u(x,0)=f(x) \end{cases}$$

Solution: Use separation of variables.

$u(x,t) = X(x)T(t) \Rightarrow u_x = X'T,\ u_{xx} = X''T,\ u_t = XT'.$

$\Rightarrow \alpha^2 X''T = XT' \Rightarrow \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda,\ \lambda$ constant

$\Rightarrow X'' + \lambda X = 0$ and $T' + \lambda \alpha^2 T = 0.$

$u_x(0,t) = 0 \Rightarrow X'(0)\,T(t) = 0$
$u_x(L,t) = 0 \Rightarrow X'(L)\,T(t) = 0$

To get nonzero solutions we must have $X'(0) = X'(L) = 0.$

Hence, for $X = X(x)$ we have the following two point B.V.P.

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0,\ X'(L) = 0. \end{cases}$$

Ch. Eqn. $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{-\lambda}.$

Case 1 $\lambda < 0 \Rightarrow r_1 = \sqrt{-\lambda}$ and $r_2 = -\sqrt{-\lambda} = -r_1$

$X(x) = c_1 e^{r_1 x} + c_2 e^{-r_1 x}, \quad X'(x) = c_1 r_1 e^{r_1 x} - c_2 r_1 e^{-r_1 x}.$

$X'(0) = 0 \Rightarrow c_1 r_1 - c_2 r_1 = 0 \Rightarrow c_1 = c_2.$
$X(x) = c_1 (e^{r_1 x} + e^{-r_1 x})$

$X'(L) = 0 \Rightarrow c_1 (\underbrace{e^{r_1 L} + e^{-r_1 L}}_{>0}) = 0 \Rightarrow c_1 = 0 \Rightarrow c_2 = 0.$

$\Rightarrow X(x) = 0$ the trivial solution

Case 2 $\lambda = 0 \Rightarrow r^2 = -\lambda = 0 \Rightarrow r_{1,2} = 0.$

$X_1(x) = e^{r_1 x} = 1, \quad X_2(x) = x \cdot X_1(x) = x.$

$\Rightarrow X(x) = c_1 \cdot 1 + c_2 \cdot x, \quad X'(x) = c_2$

$X'(0) = 0 \Rightarrow c_2 = 0, \quad X'(L) = 0 \Rightarrow c_2 = 0.$
Hence, $X(x) = C$ a constant function.

Video 47

Case 3 $\lambda > 0$, $r^2 = -\lambda \Rightarrow r = \pm i\mu$, where $\mu^2 = \lambda$.

$X(x) = c_1 \cos \mu x + c_2 \sin \mu x$.
$X'(x) = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x$.

$X'(0) = 0 \Rightarrow c_2 \mu \cdot 1 = 0 \Rightarrow c_2 = 0$

$X'(L) = 0 \Rightarrow -c_1 \mu \sin \mu L = 0$.

Hence, to obtain non zero solutions $\sin \mu L = 0$. Hence $\mu L = n\pi$, for some $n \in \mathbb{Z}$.

$\Rightarrow \mu_n = \frac{n\pi}{L}$, $n \in \mathbb{Z}$. , $\lambda_n = \mu_n^2 = \left(\frac{n\pi}{L}\right)^2$

$X_n = \cos \mu_n x = \cos \frac{n\pi}{L} x$, $n = 1, 2, 3, \ldots$

$X_0 = 1$

What about $T_n(t)$?

$T' + \alpha^2 \lambda T = 0$

For $\lambda = 0 \Rightarrow T' = 0 \Rightarrow T(t) = C$.

So, $T_0 = 1$, and hence $u_0(x,t) = X_0(x) T_0(t) = 1$.

For $\lambda > 0 \Rightarrow \lambda = \mu^2 = \left(\frac{n\pi}{L}\right)^2$

$\Rightarrow \frac{dT}{T} = -\alpha^2 \left(\frac{n\pi}{L}\right)^2 dt \Rightarrow T_n(t) = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t}$.

So, $u_n(x,t) = T_n(t) X_n(x) = e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos \frac{n\pi x}{L}$.

Hence, the solution $u(x,t)$ is as follows

$$u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\frac{n\pi x}{L}.$$

$c_n = ?$ Use the Initial Temp. Distr. $u(x,0) = f(x)$

$$\Rightarrow f(x) = u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cdot \cos\frac{n\pi x}{L},$$ where

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

Example: Solve the B.V.P.

$$\begin{cases} u_{xx} = u_t \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) = x \end{cases} \qquad (\alpha = 1) \quad L = 25\text{ cm}.$$

Solution: $c_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{25}\int_0^{25} x\,dx = 25$

$$c_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{25}\int_0^{25} x\cos\frac{n\pi x}{25}\,dx$$

$$= \frac{2}{25}\left(\underbrace{x \cdot \frac{25}{n\pi}\sin\frac{n\pi x}{25}\Big|_0^{25}}_{=0} - \int_0^{25} 1 \cdot \frac{25}{n\pi}\sin\frac{n\pi x}{25}\,dx\right)$$

$$= \frac{2}{n\pi}\cdot\frac{25}{n\pi}\cos\frac{n\pi x}{25}\Big|_0^{25}$$

$$= \frac{50}{(n\pi)^2}(\cos n\pi - 1) = \frac{50}{(n\pi)^2}((-1)^n - 1) = \begin{cases} 0, & n \text{ even} \\ \frac{-100}{(n\pi)^2}, & n \text{ odd} \end{cases}$$

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{25}\right)^2 t} c_n \cdot \cos\frac{n\pi x}{25}$$

Example: Solve the B.V.P.

$$\begin{cases} (1+t^2)u_{xx} = u_t \\ u(0,t)=0, \quad u(L,t)=0 \\ u(x,0)=20 \end{cases} \qquad (L=50 \text{ cm})$$

Solution: Use Separation of Variables.

$u(x,t) = X(x)\,T(t)$.

$u_x = X'T, \quad u_{xx} = X''T, \quad u_t = XT'$.

$$\Rightarrow (1+t^2)X''T = XT' \Rightarrow \frac{X''}{X} = \frac{1}{1+t^2}\frac{T'}{T} = -\lambda.$$

$$\Rightarrow \begin{cases} X''+\lambda X=0 \\ X(0)=0 \\ X(L)=0 \end{cases} \qquad \text{and} \quad T' + \lambda(1+t^2)T = 0$$

$\Downarrow$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n=1,2,3.$

$X_n(x) = \sin\frac{n\pi x}{L}$

$T_n' + \lambda_n(1+t^2)T_n = 0.$

$\Rightarrow \frac{dT_n}{T_n} = -(1+t^2)\,dt$

$\ln(T_n) = -\left(t+\frac{t^3}{3}\right)$

$T_n(t) = e^{-(t+t^3/3)}$

$$u_n(x,t) = X_n(x)\,T_n(t) = e^{-(t+t^3/3)}\sin\frac{n\pi x}{L}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-(t+t^3/3)}\sin\frac{n\pi x}{L} \qquad b_n = ?$$

$$u(x,0) = f(x) \Rightarrow \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = 20.$$

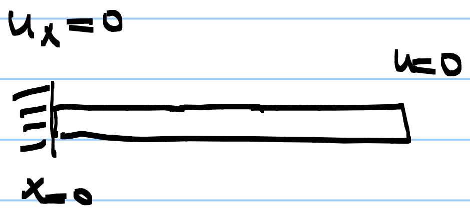
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{50}\cdot 20\cdot\left(\frac{-L}{n\pi}\right)\cos\frac{n\pi x}{L}\Big|_0^L$$

Video 4 P

$$b_n = \frac{4}{5}\,\frac{-50}{n\pi}\,(\cos n\pi - 1)$$

$$= \frac{-40}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0, & n \text{ even} \\ \frac{80}{n\pi}, & n \text{ odd} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n\, e^{-(t+t^3/3)} \sin \frac{n\pi x}{L}$$



Ex: Solve the B.V.P.

$$\begin{cases} \alpha^2 u_{xx} = u_t \\ u_x(0,t) = 0,\ u(L,0) = 0 \\ u(x,0) = f(x) \end{cases}$$

Solution: Separation of Variables $u = XT$

$u_x = X'T$, $u_{xx} = X''T$, $u_t = XT'$

$$\Rightarrow \alpha^2 X''T = XT' \Rightarrow \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$$

$$\Rightarrow X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda\alpha^2 T = 0$$

$u_x(0,t) = 0 \Rightarrow X'(0)\,T(t) = 0 \Rightarrow X'(0) = 0$

$u(L,t) = 0 \Rightarrow X(L)\,T(t) = 0 \Rightarrow X(L) = 0.$

So for $X = X(x)$ we have the following two point B.V.P.

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 \\ X(L) = 0 \end{cases}$$

Ch. Eqn. $r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda$

Case 1 $\lambda < 0 \Rightarrow r^2 = -\lambda > 0 \Rightarrow r_1 = -r_2 = \sqrt{-\lambda}$

$X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{r_1 x} + c_2 e^{-r_1 x}$.

$X'(x) = r_1 (c_1 e^{r_1 x} - c_2 e^{-r_1 x})$.

$X'(0) = 0 \Rightarrow r_1 (c_1 - c_2) = 0 \Rightarrow c_1 = c_2$.

$X(x) = c_1 (e^{r_1 x} + e^{-r_1 x})$.

$X(L) = 0 \Rightarrow c_1 (\underset{>0}{e^{r_1 L}} + \underset{>0}{e^{-r_1 L}}) = 0 \Rightarrow c_1 = 0 \Rightarrow c_2 = 0$.

$\Rightarrow X(x) = 0$.

<u>Case 2</u> $\lambda = 0$, $r_1 = r_2 = 0$. $X(x) = c_1 + c_2 x$

$X'(x) = c_2 \Rightarrow X'(0) = 0 \Rightarrow c_2 = 0$

$X(x) = c_1 \Rightarrow X(L) = 0 \Rightarrow c_1 = 0$.

$\Rightarrow X(x) = 0$.

<u>Case 3</u> $\lambda > 0$. $r^2 = -\lambda < 0 \Rightarrow r_{1,2} = \pm i\mu$, when

$\mu = \sqrt{\lambda} > 0$.

$\Rightarrow X(x) = c_1 \cos \mu x + c_2 \sin \mu x$.

$X'(x) = \mu(-c_1 \sin \mu x + c_2 \cos \mu x)$

$0 = X'(0) = \mu c_2 \Rightarrow c_2 = 0$.

$\Rightarrow X(x) = c_1 \cos \mu x$.

$X(L) = 0 \Rightarrow c_1 \cos \mu L = 0$. So obtain non zero solutions we must choose μL so that $\cos \mu L = 0$.

So, $\mu L = (2n-1)\pi/2$, $n = 1, 2, \ldots$

$\Rightarrow X_n(x) = \cos \frac{(2n-1)\pi}{2} x \qquad n = 1, 2, \ldots$

<u>What about $T(t)$?</u> $\lambda_n = \mu_n^2 = \left(\frac{2n-1}{2}\pi\right)^2$

$T' + \lambda \alpha^2 T = 0$

$\Rightarrow \frac{dT}{T} = -\lambda \alpha^2 dt$

$T_n = e^{-\lambda \alpha^2 t} = e^{-\left(\frac{(2n-1)\pi\alpha}{2}\right)^2 t} \qquad n = 1, 2, \ldots$

$u(x,t) = \sum_{n=1}^{\infty} a_n T_n(t) X_n(x)$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\left(\frac{(2n-1)\pi a}{2}\right)^2 t} \cos\frac{(2n-1)\pi}{2} x$$

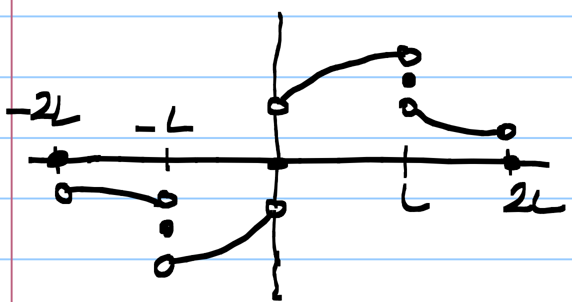
To determine a_n's use the initial condition

$$u(x,0) = f(x) \Rightarrow f(x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n-1}{2}\right)\pi x.$$

p. 622 (Problems 38, 39 and 40) More Specialized Fourier Series

Suppose $f:[0,L] \to \mathbb{R}$ is given.

38) Extend f to $(L, 2L]$ in an arbitrary way and then to $(-2L, 0)$ as an odd function.



Now it is an odd function periodic with period $4L$.

So the corresponding basis $\left\{1, \cos\frac{m\pi x}{2L}, \sin\frac{m\pi x}{2L}\right\}_{m=1}^{\infty}$

$$f(x) = \frac{a_0}{2} + \sum a_m \cos\frac{m\pi x}{2L} + b_m \sin\frac{m\pi x}{2L}$$

$$a_0 = \frac{1}{2L}\int_{-2L}^{2L} f(x)\,dx \quad \text{odd} \quad = 0.$$

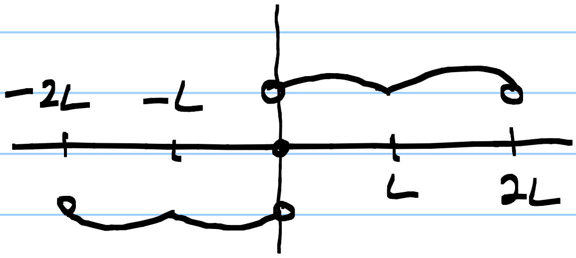
$$a_m = \frac{1}{2L}\int_{-2L}^{2L} \underbrace{f(x)}_{\text{odd}} \cos \underbrace{\frac{m\pi x}{2L}}_{\text{even}}\, dx = 0.$$

$$b_m = \frac{1}{2L}\int_{-2L}^{2L} f(x) \sin\frac{m\pi x}{2L}\, dx.$$

Hence, we obtain $f(x) = \sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{2L}$

40) Similarly, obtain series for $f(x)$ using cosine function instead of sine.

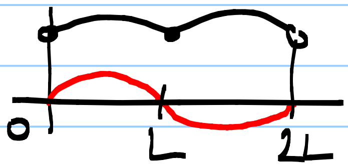
39) Given $f:[0,L]\to\mathbb{R}$ extend f to $(L,2L)$ so that it is symmetric about $x=L$ so that it satisfies $f(2L-x)=f(x)$ for $0\le x<L$. Next extend f to $(-2L,0)$ again as an odd function.



Now since f is periodic with period $4L$ and since it is an odd function as above we have

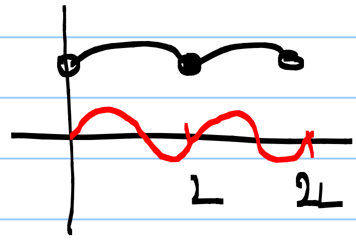
$$f(x)=\sum_{m=1}^{\infty} b_m \sin\frac{m\pi x}{2L}, \text{ where}$$

$$b_m=\frac{1}{2L}\int_{-2L}^{2L}\underbrace{f(x)\sin\frac{m\pi x}{2L}}_{\text{odd}\cdot\text{odd}=\text{even}}=\frac{1}{L}\int_0^{2L} f(x)\sin\frac{m\pi x}{2L}$$



If $m=2n$ is even, then

$$b_m=\frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx=0.$$



If $m=2n-1$ is odd then

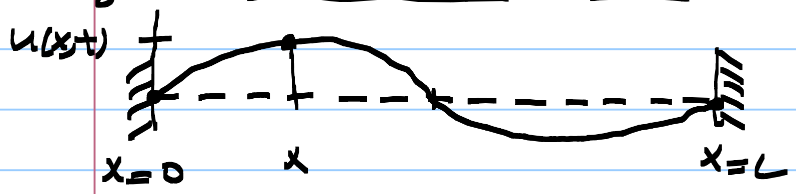
$$b_m=\frac{1}{L}\int_0^{2L} f(x)\sin\frac{m\pi x}{2L}\,dx=$$

$$=\frac{2}{L}\int_0^{L} f(x)\sin\frac{(2n-1)\pi x}{2L}$$

So, $$f(x)=\sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)\pi x}{2L}$$

Video 49

§10.7. The Wave Equation:



$u(x,t)$ = the amount of displacement of the string at x at time t.

It is known that $u(x,t)$ satisfies so called the wave equation

$$a^2 u_{xx} = u_{tt}.$$

A B.V.P. for the wave equation has the form

$$\begin{cases} a^2 u_{xx} = u_{tt} & \text{(Wave Equation)} \\ \left.\begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array}\right) & \text{Homogeneous boundary conditions} \\ u(x,0) = f(x) \quad 0 \le x \le L & \text{Initial position.} \\ u_t(x,0) = g(x) \quad 0 \le x \le L & \text{Initial velocity.} \end{cases}$$

We'll use again the separation of Variables to solve the problem.

A) First assume $g(x) = 0$.

$u(x,t) = X(x)T(t)$, $u_x = X'T$, $u_{xx} = X''T$, $u_t = XT'$, $u_{tt} = XT''$.

$\Rightarrow a^2 u_{xx} = u_{tt} \Rightarrow a^2 X''T = XT'' \Rightarrow \dfrac{X''}{X} = \dfrac{1}{a^2}\dfrac{T''}{T} = -\lambda.$

$\Rightarrow X'' + \lambda X = 0$ and $T'' + a^2\lambda T = 0.$

B.C. $u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0.$

$u(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0.$

So, for $X(x)$ we have again the same two point B.V.P.

$$\left\{\begin{array}{l} X'' + \lambda X = 0 \\ X(0) = 0 \\ X(L) = 0 \end{array}\right\} \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n = 1, 2, \ldots$$

$$X_n(x) = \sin \frac{n\pi x}{L}.$$

<u>What about $T_n(t)$?</u>

$$T_n'' + \lambda_n a^2 T_n = 0$$

$$0 = g(x) = u_t(x, 0) = X(x) T'(0) \Rightarrow T'(0) = 0.$$

Ch. Eqn $\quad r^2 + \lambda_n a^2 = 0 \Rightarrow r^2 = -\lambda_n a^2 = -\left(\frac{n\pi}{L}\right)^2 a^2$

$$\Rightarrow r_{1,2} = \pm i \left(\frac{n\pi a}{L}\right)$$

$$\Rightarrow T_n(t) = k_1 \cos \frac{n\pi a}{L} t + k_2 \sin \frac{n\pi a}{L} t.$$

$$T_n'(t) = -a\frac{k_1 n\pi}{L} \sin \frac{n\pi a}{L} t + a\frac{k_2 n\pi}{L} \cos \frac{n\pi a}{L} t$$

$$0 = T_n'(0) = a k_2 \cdot \frac{n\pi}{L} \cdot 1 \Rightarrow k_2 = 0$$

$$T_n(t) = \quad \cos \frac{n\pi a}{L} t.$$

$$u_n(x,t) = \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t, \ n = 1, 2, \ldots$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t$$

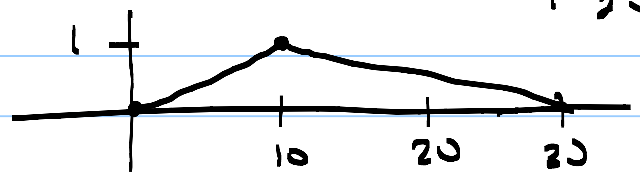
To determine c_n's we use the initial position condition, namely,

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}.$$

Here, $c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx.$

Example: Solve the B.V.P. for wave equation

$$\begin{cases} 4u_{xx} = u_{tt} & a^2 = 4,\ a = 2 \\ u(0,t) = u(L,t) = 0 & L = 30 \\ u(x,0) = f(x) & \text{, where } f(x) = \begin{cases} x/10 & 0 \le x \le 10 \\ \frac{30-x}{20}, & 10 < x \le 30 \end{cases} \\ u_t(x,0) = 0 \end{cases}$$



$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \cos \frac{n a \pi t}{L}$$
$$= \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{30} \cos \frac{n\pi t}{15}, \text{ where}$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$
$$= \frac{2}{30} \int_0^{30} f(x) \sin \frac{n\pi x}{30}\, dx$$
$$= \frac{9}{n^2\pi^2} \sin \frac{n\pi}{3}, \quad n = 1, 2, \ldots$$

B) $f(x) \equiv 0$ and let $u_t(x,0) = g(x)$.

$X_n(x) = \sin \frac{n\pi x}{L}$, $T_n(t) = k_1 \cos \frac{n a\pi t}{L} + k_2 \sin \frac{n a \pi t}{L}$, when $T_n(0)=0$. So $k_1 \cdot 1 + k_2 \cdot 0 = 0$. $\Rightarrow k_1 = 0$.

Hence, $T_n(t) = \sin \frac{n\pi a t}{L}$.

$$u_n(x,t) = \sin \frac{n\pi x}{L} \sin \frac{n\pi a t}{L}.$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \sin \frac{n\pi a t}{L}$$

To determine c_n's we use initial velocity condition: $u_t(x,0) = g(x)$.

$$u_t(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \frac{n\pi a}{L} \cos \frac{n\pi a t}{L}$$

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} c_n \frac{n\pi a}{L} \sin \frac{n\pi x}{L}.$$

So, $$c_n \frac{n\pi a}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx$$

$$\Rightarrow c_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx.$$

c) General Case

$$\begin{cases} a^2 u_{xx} = u_{tt} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(t) \end{cases}$$

Note that $u = v + w$, where v and w are the solutions of the B.V.P's

$$\begin{cases} a^2 v_{xx} = v_{tt} \\ v(0,t) = v(L,t) = 0 \\ v(x,0) = f(x) \\ v_t(x,0) = 0 \end{cases} \quad \text{and} \quad \begin{cases} a^2 w_{xx} = w_{tt} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = 0 \\ w_t(x,0) = g(x) \end{cases}.$$

Video 50

§10.8. Laplace's Equation:

In two dimensions: $u_{xx}+u_{yy}=0$

In three dimensions: $u_{xx}+u_{yy}+u_{zz}=0$.

In polar coordinates it becomes:

$u_{xx}+u_{yy}=0 \qquad x=r\cos\theta,\ y=r\sin\theta$

$$u_r=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=u_x\cos\theta+u_y\sin\theta$$

$$u_{rr}=(u_{xx}\cos\theta+u_{yx}\sin\theta)\frac{\partial x}{\partial r}+(u_{xy}\cos\theta+u_{yy}\sin\theta)\frac{\partial y}{\partial r}$$

$$=u_{xx}\cos^2\theta+2u_{xy}\cos\theta\sin\theta+u_{yy}\sin^2\theta$$

$$u_\theta=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}=u_x\cdot(-r\sin\theta)+u_y(r\cos\theta)$$

$$u_{\theta\theta}=\big(u_{xx}(-r\sin\theta)+u_{yx}(r\cos\theta)\big)\frac{\partial x}{\partial\theta} \quad =-r\sin\theta$$

$$+\big(u_{xy}(-r\sin\theta)+u_{yy}(r\cos\theta)\big)\frac{\partial y}{\partial\theta} \quad =r\cos\theta$$

$$+u_x(-r\cos\theta)+u_y(-r\sin\theta)$$

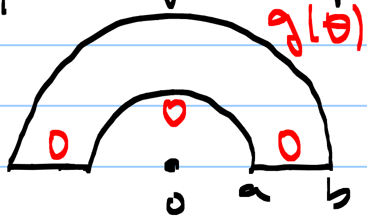
$$=r^2\sin^2\theta\, u_{xx}+r^2\cos^2\theta\, u_{yy}-2r^2\sin\theta\cos\theta\, u_{xy}$$

$$+\underbrace{u_x(-r\cos\theta)+u_y(-r\sin\theta)}_{=-ru_r}$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=u_{xx}+u_{yy}=0.$$

Example: Solve the B.V.P. for Laplace equation

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \\ u(r,0)=0 \qquad a<r\le b \\ u(r,\pi)=0 \\ u(a,\theta)=0 \qquad 0\le\theta\le\pi \\ u(b,\theta)=g(\theta) \end{cases}$$



Solution: Separation of Variables:

$u(r,\theta) = R(r)\,\Theta(\theta)$

$u_r = R'\Theta$, $u_{rr} = R''\Theta$, $u_\theta = R\Theta'$, $u_{\theta\theta} = R\Theta''$

$$\Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$r^2R''\Theta + rR'\Theta = -R\Theta''$$

$$\Rightarrow \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$\Rightarrow \Theta'' + \lambda\Theta = 0 \quad \text{and} \quad r^2R'' + rR' - \lambda R = 0$$
(Euler's Eqn.)

$u(r,0) = 0 \Rightarrow R(r)\,\Theta(0) = 0 \Rightarrow \Theta(0) = 0$

$u(r,\pi) = 0 \Rightarrow R(r)\,\Theta(\pi) = 0 \Rightarrow \Theta(\pi) = 0.$

So for $\Theta(\theta)$ we have the 2-point B.V.P.

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(0) = 0 \\ \Theta(\pi) = 0 \end{cases} \qquad L = \pi$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 = n^2, \quad n = 1, 2, \ldots$$

$$\Theta_n = \sin\frac{n\pi\theta}{L} = \sin n\theta.$$

For $R(r)$ we have

$$r^2R_n'' + rR_n' - \lambda_n R_n = 0, \quad \lambda_n = n^2.$$

Euler's equation has solutions of the form

$$R_n(r) = r^s, \text{ for some } s.$$

Then $R_n'(r) = s\,r^{s-1}$, $R_n''(r) = s(s-1)r^{s-2}$

$$\Rightarrow s(s-1)r^s + s\,r^s - n^2 r^s = 0.$$

$r^s\left(s(s-1)+s-n^2\right)=0$, for all $r>0$.

$\Rightarrow s^2-s+s-n^2=0 \Rightarrow s_{1,2}=\pm n.$

$\Rightarrow r^n$ or r^{-n}

So, $R_n(r)=c_1 r^n + c_2 r^{-n}$.

c_1, c_2?

$u(a,\theta)=0 \Rightarrow \left(c_1 a^n + c_2 a^{-n}\right)\Theta(\theta)=0$

$\Rightarrow c_1 a^n + c_2 a^{-n}=0 \Rightarrow c_2=-c_1 r^{2a}$

$\Rightarrow R_n(r)=c_1\left(r^n - r^{2a} r^{-n}\right)$ or we may take

$R_n(r)=r^{n-a}-r^{-(n-a)} \qquad \left(c_1=r^{-a}\right)$

$u_n(r,\theta)=R_n(r)\,\Theta_n(\theta)$
$=\left(r^{n-a}-r^{-(n-a)}\right)\sin n\theta$

$$u(r,\theta)=\sum_{n=1}^{\infty} c_n\left(r^{n-a}-r^{-(n-a)}\right)\sin n\theta.$$

$c_n=?$

$$g(\theta)=u(b,\theta)=\sum_{n=1}^{\infty} c_n\left(b^{n-a}-b^{-(n-a)}\right)\sin n\theta$$

Hence by the Fourier Theory

$$c_n\left(b^{n-a}-b^{-(n-a)}\right)=\frac{2}{L}\int_0^L g(\theta)\sin\frac{n\pi\theta}{L}\,d\theta$$
$$=\frac{2}{\pi}\int_0^{\pi} g(\theta)\sin n\theta\,d\theta.$$

So, $c_n = \dfrac{2}{\pi\left(b^{n-a} - b^{-(n-a)}\right)} \displaystyle\int_0^{\pi} g(\theta) \sin n\theta \, d\theta.$