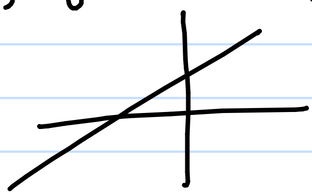
Chapter 1: Fundamental Concepts

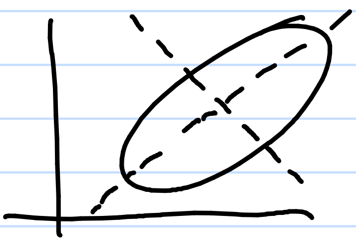
§1. Algebraic Curves in the Complex Projective Plane $\mathbb{P}^2\mathbb{C} = \mathbb{CP}^2$.

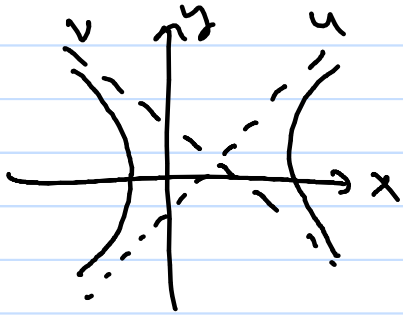
Let $f(x,y) \in \mathbb{R}[x,y]$ be a polynomial and consider its graph in $\mathbb{R}^2$ given by the equation $f(x,y)=0$. It will be called a real algebraic curve of degree d, where d is the degree of the polynomial $f(x,y)$.

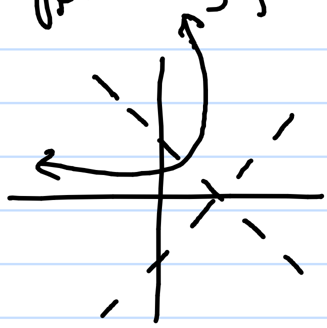
$d=1$: $f(x,y) = ax+by+c=0$ a line in $\mathbb{R}^2$

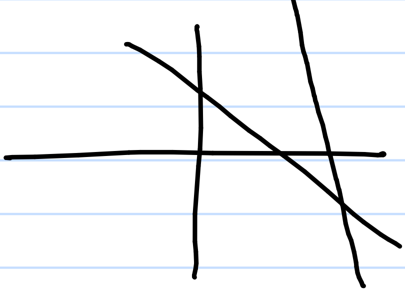


$d=2$, $f(x,y) = ax^2+bxy+cy^2+dx+ey+f=0$, a quadratic in $\mathbb{R}^2$. It is either an ellipse, hyperbola, parabola or a pair of lines.









The difficulty of classification becomes huge as the degree gets bigger. One reason is that the field $\mathbb{R}$ is not algebraically closed. For example, finding the number of intersections of the graph with a straight line is a difficult problem: Assume that the line L is parametrized as $x=\alpha t$, $y=\beta t$, for some constants α, β (w.l.o.g. we assume that L passes through the origin).

Let write $f(x,y) = f_n(x,y) + \dots + f_0$, where $f_i(x,y)$ is homogeneous of degree i. Then

$$f(x,y) = f(\alpha t, \beta t) = f_n(\alpha t, \beta t) + \dots + f_1(\alpha t, \beta t) + f_0(\alpha t, \beta t)$$
$$= t^n f_n(\alpha,\beta) + \dots + t f_1(\alpha,\beta) + f_0$$

So, to find the intersection points one needs to solve the equation

$$f_n(\alpha,\beta)t^n + \cdots + f_1(\alpha,\beta)t + f_0 = 0$$ for t, which is not a simple matter at all.

It may have no solutions, one solution or any number of solutions between 0 and n.

However, in case of complex numbers, considering $f(x,y) \in \mathbb{C}[x,y]$ as long as $f_n(\alpha,\beta) \neq 0$ the above equation has exactly n solutions, counting with multiplicity, and hence the line L intersects the curve in exactly n points in $\mathbb{C}^2$ (again counted with multiplicity).

Next question is what if $f_n(\alpha,\beta) = 0$? Indeed, assume that $f_n(\alpha,\beta) = \cdots = f_{m+1}(\alpha,\beta) = 0$, but $f_m(\alpha,\beta) = 0$. In this case there will be m intersection points in $\mathbb{C}^2$. The remaining $n-m$ intersection points will appear at infinity! Namely, let $t = 1/s$, then the the equation becomes:

$$f_n(\alpha,\beta) + f_{n-1}(\alpha,\beta)s + \cdots + f_0(\alpha,\beta)s^n = 0.$$ Moreover, in this case $s=0$ will be a zero with multiplicity $n-m$ (because $f_n(\alpha,\beta) = \cdots = f_{m+1}(\alpha,\beta) = 0$). Also note that $s=0$ corresponds $\infty = \lim_{s \to 0} 1/s = \lim_{s\to 0} t = \lim_{t\to\infty} t$.

By the above arguments, it is useful to add $\mathbb{C}^2$ the points at infinity, which we'll call the line at infinity. The total space will be called the complex projective plane and denoted as $\mathbb{P}^2\mathbb{C}$ or $\mathbb{CP}^2$.

$\mathbb{P}^2\mathbb{C} = \mathbb{CP}^2$ is defined as follows:

$$\mathbb{CP}^2 = \mathbb{C}^3 \setminus \{(0,0,0)\}/\sim, \quad (x,y,z) \sim (\lambda x, \lambda y, \lambda z),\ \lambda \in \mathbb{C}\setminus\{0\}.$$

the equivalence class of (x,y,z) will be denoted as $[x,y,z]$.

The topology on $\mathbb{CP}^2$ is the quotient topology. It turns out that $\mathbb{CP}^2$ is a second countable compact Hausdorff space. Moreover, it is two dimensional complex manifold.

Let $U_x = \{[x,y,z] \in \mathbb{CP}^2 / x \neq 0\}$, $U_y = \{[x,y,z] \in \mathbb{CP}^2 / y \neq 0\}$

and $U_z = \{[x,y,z] \in \mathbb{CP}^2 \mid z \neq 0\}$.

Note that $\mathbb{CP}^2 = U_x \cup U_y \cup U_z$ and each U_x is homeomorphic to $\mathbb{C}^2$:

$\mathbb{C}^2 \longrightarrow U_z$, $(x,y) \longmapsto [x,y,1]$ with inverse map

$[x,y,z] \longmapsto (x/z, y/z)$.

The line at infinity L_∞ is defined as $L_\infty = \mathbb{CP}^2 \setminus U_z$.

Indeed, $L_\infty = \{[x,y,0] \mid (x,y) \in \mathbb{C}^2\} = \mathbb{C}^2 \setminus \{(0,0)\} / \sim$, which is the complex projective line.

So, $L_\infty = \mathbb{CP}^1 \cong \mathbb{C} \dot{\cup} \mathbb{C} / z \sim 1/z \simeq S^2$.

What about functions? Consider $f(x,y) \in \mathbb{C}[x,y]$.

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, leftrightarrow] & U_z = \{[x,y,z] \mid z \neq 0\} \\[-20pt]
(x,y) \arrow[r, mapsto] \arrow[dr, "f"'] & {[x,y,1]} \arrow[d, "{F(x,y,1)}"] \\
& \mathbb{C}
\end{tikzcd}$$

If $\deg f(x,y) = n$, let

$F(x,y,z) \doteq z^n f(x/z, y/z)$. Then clearly, $F(x,y,z)$ satisfies $F(x,y,1) = f(x,y)$.

Hence, algebraic curves in $\mathbb{P}^2$ (or $\mathbb{CP}^2$, $\mathbb{P}^2\mathbb{C}$) is defined to be the zero set of a homogeneous polynomial $F(x,y,z) \in \mathbb{C}[x,y,z]$.

If $F = F_1^{m_1} F_2^{m_2} \cdots F_\ell^{m_\ell}$, where each F_i is irreducible then

If $C = \{F = 0\}$ and $C_i = \{F_i = 0\}$, then we write

$C = m_1 C_1 + m_2 C_2 + \cdots + m_\ell C_\ell$. Each C_i is called an irreducible component of C.

§2. Riemann Surfaces:

Definition 2.3. A Riemann surface is a connected Hausdorff topological space C with an open covering $\{U_\alpha\}$ of C and a family of mappings $z_\alpha : U_\alpha \to \mathbb{C}$ such that

a) Each $z_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism U_α onto the open subset $z_\alpha(U_\alpha)$ of $\mathbb{C}$;
b) If $U_\alpha \cap U_\beta \neq \emptyset$, then the function

$$z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \longrightarrow z_\beta(U_\alpha \cap U_\beta) \text{ is holomorphic.}$$

We call such a (U_α, z_α) a local holomorphic coordinate, and $\{(U_\alpha, z_\alpha)\}$ a holomorphic coordinate covering.

Example 1. $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Let $U_0 = \mathbb{CP}^1 \setminus \{\infty\} = \mathbb{C}$ and

$U_1 = \mathbb{CP}^1 \setminus \{0\}$. Moreover, $z_0 : U_0 \to \mathbb{C}$, $z \mapsto z$ and

$z_1 : U_1 \to \mathbb{C}$, $z \mapsto \begin{cases} 0 & \text{if } z = \infty \\ 1/z & \text{if } z \neq \infty. \end{cases}$

Then $z_1 \circ z_0^{-1} : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$, $z \mapsto 1/z$, which is clearly holomorphic.

Note that we may then write $\mathbb{CP}^1 = \mathbb{C} \,\dot\cup\, \mathbb{C} / z \sim 1/z,\ z \neq 0$

Now let $S = S^2 = \{(x,y,z) \in \mathbb{R}^2 \mid x^2+y^2+z^2 = 1\}$, $U_0 = S \setminus P$, $U_1 = S \setminus N$, where $N = (0,0,1)$ and $P = (0,0,-1)$. Also, let

$\Phi_0 : U_0 \to \mathbb{C}$, $(x,y,z) \mapsto (x - iy)/(1+z)$, and

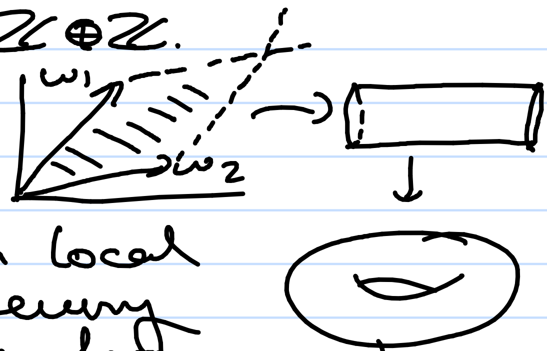
$\Phi_1 : U_1 \to \mathbb{C}$, $(x,y,z) \mapsto (x+iy)/(1-z)$.

Then $\Phi_0 \circ \Phi_1^{-1} : \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$, $z \mapsto 1/z$ is again a biholomorphism. So, this gives an alternative construction of $\mathbb{CP}^1$.

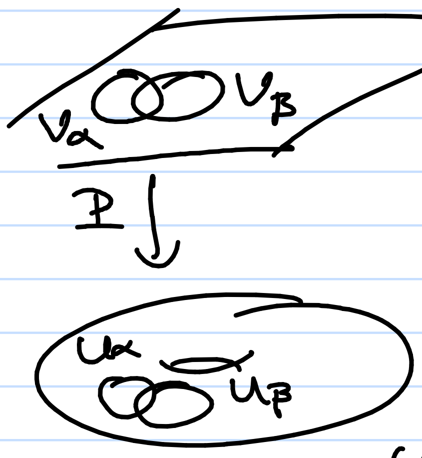
Example 2: The complex torus $C=\mathbb{C}/\Lambda$, where Λ is a discrete subgroup of $\mathbb{C}$ generated by two elements, say ω_1 and ω_2 so that they are linearly independent over real numbers.

So, $\Lambda=\{n_1\omega_1+n_2\omega_2 \mid n_1,n_2\in\mathbb{Z}^2\}\simeq\mathbb{Z}\oplus\mathbb{Z}$.

C is called a complex torus:



The quotient map $\mathbb{C}\to\mathbb{C}/\Lambda$ is a local homeomorphism and thus around every point of C we can find some homeomorphism $\varphi_\alpha: U_\alpha\to V_\alpha\subseteq\mathbb{C}$.



$\varphi_\alpha = P|_{V_\alpha}\to U_\alpha$, $z\mapsto z+\Lambda$

If $U_\alpha\cap U_\beta\neq\emptyset$ then there is some $v\in\Lambda$ so that

$\varphi_\alpha(z)=\varphi_\beta(z+v)$ and hence

$$\varphi_\alpha^{-1}\circ\varphi_\beta^{-1}: \varphi_\alpha(V_\alpha\cap V_\beta)\longrightarrow\varphi_\beta(V_\alpha\cap V_\beta)$$
$$z\longmapsto z+v.$$

Exercise: C is a compact Riemann surface.

Now let $w=f(z)$ ($w=u+iv$, $z=x+iy$) is a holomorphic map which maps an open set U of $\mathbb{C}$ into an open set V of $\mathbb{C}$.

$$u=u(x,y),\quad v=v(x,y).$$

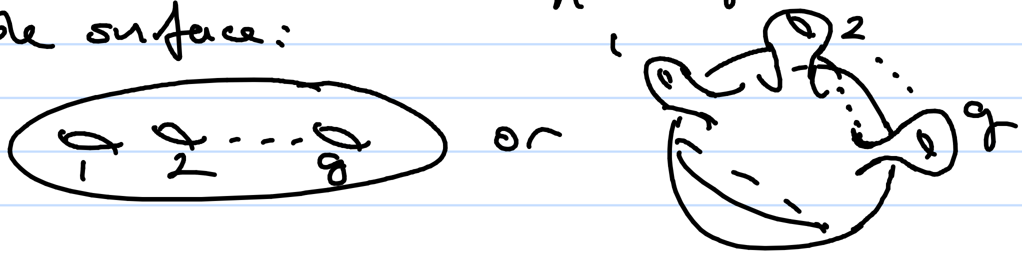
Since f is holomorphic u and v satisfies the Cauchy-Riemann equations:

$$\begin{cases} u_x=v_y \\ u_y=-v_x.\end{cases}\quad \text{Thus } \det\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}=u_xv_y-v_xu_y=(u_x)^2+(u_y)^2>0$$

(assuming f is biholomorphic).

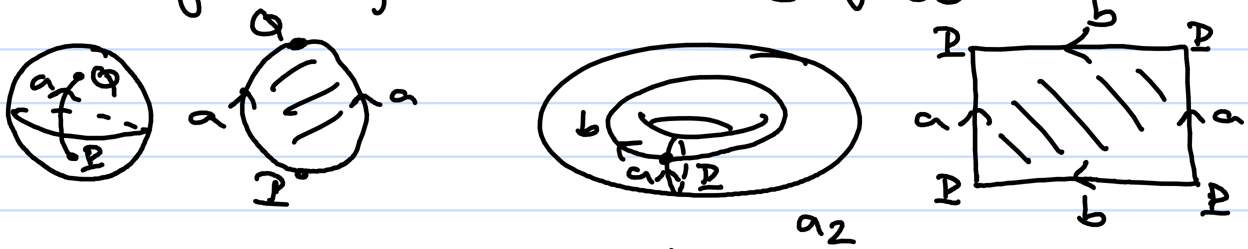
This shows that any Riemann surface is a real differentiable surface. Moreover, it is orientable since the Jacobians of $\varphi_\alpha^{-1} \circ \varphi_\beta$ is always positive.

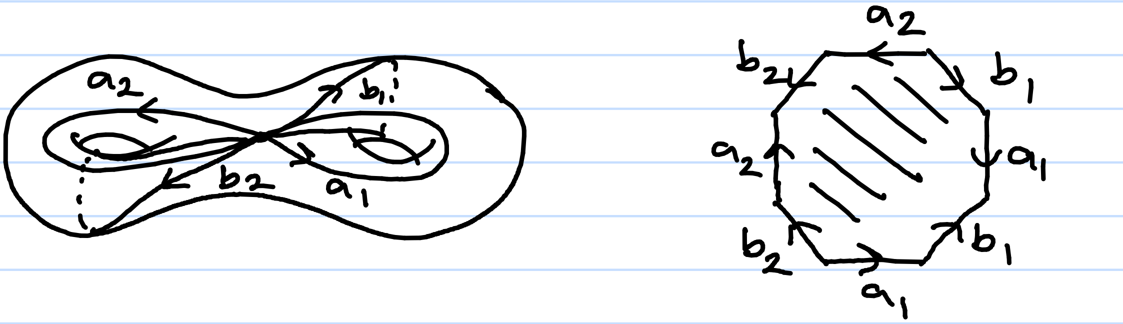
If the Riemann surface is also compact then by topological classification it should be diffeomorphic to some genus g orientable surface:

Σ_g: 

Clearly, any genus 0 Riemann surface is a sphere and any genus 1 Riemann surface is a torus, topologically.

Plane diagrams for orientable surfaces.





The sequences aa^{-1}, $aba^{-1}b^{-1}$, $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}$ are called canonical representations of the surfaces. In general for the surface of genus g the canonical representation is the sequence of edges of a $4g$-gon, namely $a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$.

Definition: The Euler characteristic of a two-dimensional, orientable, compact surface of genus g is defined to be $\chi = 2-2g$. Euler characteristic is a very important topological invariant.

§3. Holomorphic and Meromorphic Functions:

Suppose C is a Riemann surface and $\{(U_i, z_i)\}$ is its holomorphic coordinate covering. A meromorphic (resp. holomorphic) function f on C is definition a family of mappings $f_i : U_i \to \Sigma$ satisfying the following conditions:

a) $f_i = f_j$ on $U_i \cap U_j$ for $U_i \cap U_j \neq \emptyset$,

b) For all i, $f_i \circ z_i^{-1}$ are meomorphic (resp. holomorphic) functions.

Remark 3.2. The set of meromorphic functions on C form a field, called the field of meromorphic functions, denoted as $K(C)$. On the other hand, the set of holomorphic functions on C, denoted as $\mathcal{O}(C)$, form an algebra over $\mathbb{C}$.

Theorem 3.3.: The only holomorphic functions on a compact Riemann surface C are the constant functions.

Proof: Let $f: C \to \mathbb{C}$ be a holomorphic function. Since C is compact the continuous function $|f|: C \to \mathbb{C}$ has a maximum at a point say $p \in C$. If p is contained in the local coordinate (U_α, z_α) then $|f \circ z_\alpha^{-1}|: V_\alpha \to \mathbb{C}$ has a maximum on $\alpha = z_\alpha(U_\alpha) \subseteq \mathbb{C}$ at the point $z_\alpha(p)$. This implies that $f \circ z_\alpha^{-1}$ is constant on V_α. Hence, f is constant of U_α. If $f_\alpha = f|_{U_\alpha}$ and $U_\alpha \cap U_\beta \neq \emptyset$ then, since $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$ we see that f_β is constant on the open set $U_\alpha \cap U_\beta$. This implies that f_β is constant on U_β, because U_β is connected (we may assume that each local coordinate U_β is connected).

Finally a connectedness argument finishes the proof:

Let $q \in C$ be any other point and let $\gamma: [a,b] \to C$ be continuous path with $\gamma(a) = q$ and $\gamma(b) = p$. Cover $\gamma([a,b])$ with finitely many U_α's say $U_{\alpha_1}, \dots, U_{\alpha_n}$ so that $q \in U_{\alpha_1}$, $p \in U_{\alpha_n}$ and $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$ for any $i = 1, \dots, n-1$. (This can be done since $[a,b]$ is compact.) Now, f is constant on U_{α_n} since $p \in U_{\alpha_n} \cap U_\alpha \neq \emptyset$. The f is constant on $U_{\alpha_{n-1}}$ and so on. Hence, f is constant on each U_{α_i} and in particular $f(q) = f(p)$. This finishes the proof. ∎

Definition 3.4. Suppose C is a compact Riemann surface, with $f \in K(C)$, $p \in C$. Select a local coordinate z in a neighborhood of the point p such that $z(p) = 0$. Then in a neighborhood of p we have $f = z^\nu h(z)$, where $h(z)$ is a holomorphic function, $h(0)\neq 0$, and $\nu\in\mathbb{Z}$. This integer ν is well defined and called the multiplicity of f at the point p, denoted $\nu_p(f)$. If $\nu_p(f)>0$, p is called a zero of f, and if $\nu_p(f)$ is negative then p is called a pole of f and $|\nu_p(f)|$ is called the order of the pole p.

Theorem 3.5.: The meromorphic function field $K(S)$ of the Riemann sphere is isomorphic to the rational function field $\mathbb{C}(z)$.

Proof: $\mathbb{CP}^1 = S = \mathbb{C}\,\dot{\cup}\,\mathbb{C}/_{z\sim 1/z,\ z\neq 0}$.

Let $f\in K(S)$. Then on the first copy of $\mathbb{C}$ it has an expansion of the form

$$f(z)=\sum_{i=1}^{N} P_i(z)+q(z), \text{ where each } P_i(z)=\frac{b_{i\nu_i}}{(z-a_i)^{\nu_i}}+\cdots+\frac{b_{i1}}{z-a_i}$$

and $q(z)$ is holomorphic. Since f is meromorphic on the other copy of $\mathbb{C}$, $f(1/z)$ must be meromorphic and thus $q(z)=\sum_{n=0}^{\infty} c_n z^n$ must be a polynomial. This finishes the proof. ∎

Now let's study the case of complex torus $C=\mathbb{C}/\Lambda$, where $\Lambda=\{m_1\omega_1+m_2\omega_2 \mid m_1,m_2\in\mathbb{Z}\}$ is a lattice in $\mathbb{C}$.

Theorem 3.6. $K(\mathbb{C}/\Lambda)$ is isomorphic to the field of doubly periodic functions with period $\{\omega_1,\omega_2\}$.

Proof Let $\pi:\mathbb{C}\to\mathbb{C}/\Lambda$ be the natural projection which is a holomorphic function. Now, if $f\in K(\mathbb{C}/\Lambda)$ then $f\circ\pi:\mathbb{C}\to\mathbb{C}$ is clearly a doubly periodic meromorphic function. Moreover, any doubly periodic meromorphic function $g:\mathbb{C}\to\mathbb{C}$ with period $\{\omega_1,\omega_2\}$ descends a meromorphic $f:\mathbb{C}/\Lambda\to\mathbb{C}$ so that $f\circ\pi=g$. This finishes the proof. ∎

Problem 3.1: Prove that $\nu_p(f)$ is well-defined.

Problem 3.2.: If C is the Riemann sphere or a complex torus then $\sum_{p\in C}\nu_p(f)=0$.

Hint: It is easier to prove this for any compact Riemann Surface C using the degree theory. Namely, for any smooth map $f: C \to S = \mathbb{CP}^1$ the degree of f is computed as the number of preimages of any point $p \in S$.

Now $\#(\text{zero of } f) = \#(\text{poles of } f)$

$$\Rightarrow \#(\text{zeros of } f) - \#(\text{poles of } f) = 0$$

$$\Rightarrow \sum_{p \in C} \nu_p(f) = \sum_{p = 0, \infty} \nu_p(f) = \#(\text{zeros of } f) - \#(\text{poles of } f) = 0.$$

Definition 3.8. Suppose C, C' are Riemann Surfaces with $\{(U_i, z_i)\}$ and $\{(U'_a, z'_a)\}$ as their coordinate coverings, respectively. Then a holomorphic mapping $f: C \to C'$ is by definition a family of continuous mappings $f_i: U_i \to C'$, $\forall i$, such that

a) $f_i = f_j$ on $U_i \cap U_j$ for $U_i \cap U_j \neq \emptyset$

b) $z'_a \circ f_i \circ z_i^{-1}$ is a holomorphic function on $f_i^{-1}(U'_a) \cap U_i$, whenever $f_i^{-1}(U'_a) \cap U_i \neq \emptyset$.

Remark 3.9. Any meromorphic function of a Riemann Surface is just a holomorphic function to the Riemann sphere.

Problem 3.3. Suppose C, C' are Riemann Surfaces and $f: C \to C'$ is a holomorphic mapping with $p \in C, q \in C'$ and $f(p) = q$. Prove that there exist local coordinates z and w defined near p and q, respectively so that $z(p) = 0 = w(q)$ and in these coordinates f becomes

$z \longmapsto w = f(z) = z^\nu$ for some $\nu \in \mathbb{Z}^+$. Moreover, this integer ν is well defined and $\nu - 1$ is called the ramification index of f at the point p.

§4. Holomorphic and Meromorphic Differentials:

Definition 4.1. Suppose C is a Riemann surface. Then a holomorphic differential (resp., meromorphic differential) ω is by definition a family $\{(U_i, z_i, \omega_i)\}$

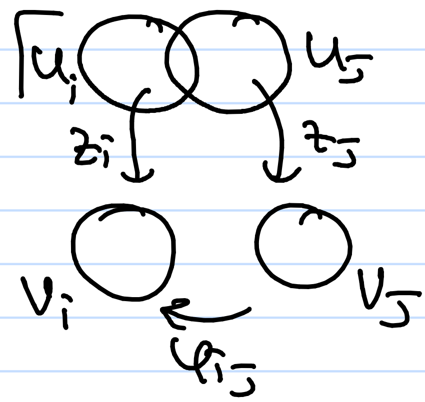
such that

a) $\{(U_i, z_i)\}$ is a holomorphic covering of C, and $\omega_i = f_i(z_i)\,dz_i$, where $f_i \in \mathcal{O}(U_i)$ (resp., $K(U_i)$);

b) If $z_i = \varphi_{ij}(z_j)$ is the coordinate transformation on $U_i \cap U_j (\neq \emptyset)$, then

$$f_i(\varphi_{ij}(z_j)) \frac{d\varphi_{ij}(z_j)}{dz_j} = f_j(z_j),$$

i.e., $f_i(\varphi_{ij}(z_j))\, d\varphi_{ij}(z_j) = f_j(z_j)\, dz_j$.



$$z_i = \varphi_{ij}(z_j) \Rightarrow dz_i = d\varphi_{ij}(z_j)$$

$$\Rightarrow f_i(z_i)\, dz_i = f_i(\varphi_{ij}(z_j))\, d\varphi_{ij}(z_j)$$

$$= f_j(z_j)\, dz_j,$$

$$\Rightarrow \omega_i = \omega_j \quad \text{on } U_i \cap U_j \neq \emptyset. \quad \lrcorner$$

$\Omega^1(C)$ (resp. $K^1(C)$) will denote the set of holomorphic (resp. meromorphic) differentials on C.

Clearly, we have

<u>Proposition 4.2.</u>: Suppose ω_0, ω_1 are meromorphic differentials on C, $\omega_0 \neq 0$. Then ω_1/ω_0 can be regarded as a function on C.

<u>Proof</u>: $\omega_0 = f_i^0(z_i)\,dz_i$ and $\omega_1 = f_i^1(z_i)\,dz_i$ and hence

$$\omega_1/\omega_0 = f_i^1/f_i^0. \quad \blacksquare$$

<u>Example 1</u>: If $r(z)$ is a rational function then $r(z)dz$ is a meromorphic differential on the Riemann sphere.

<u>Problem 4.1</u> Prove that any meromorphic 1-form on the Riemann sphere is of the form $r(z)dz$, when $r(z)$ is a rational function.

<u>Example 2.</u>: Suppose $\Lambda = \{m_1\omega_1 + m_2\omega_2 \mid m_i \in \mathbb{Z}\}$ is a lattice in $\mathbb{C}$,

and f is a doubly periodic meromorphic function with period (ω_1, ω_2), then $f(w)\,dw$ is a meromorphic differential defined on $C = \mathbb{C}/\Lambda$.

Problem 4.2.: Prove that $\omega = dz$ yields a holomorphic differential on $C = \mathbb{C}/\Lambda$.

Example 3: Suppose C is a Riemann surface, with

$f = \{(U_i, z_i, f_i(z_i))\} \in K(C)$. Then the following expression defines a meromorphic differential $df \in K^1(C)$,

$$df = \left\{\left(U_i, z_i, df_i(z_i) = \frac{df_i(z_i)}{dz_i}\, dz_i\right)\right\}.$$

Definition 4.3.: We call df, as defined above, the differential of the meromorphic function f.

Definition 4.4.: Suppose C is a Riemann surface with $\omega = \{(U_i, z_i, f_i(z_i)\,dz_i)\} \in K^1(C)$, $p \in U_i \cap U_j$, then

$$\nu_p(f_i) = \nu_p\left(f_i(\varphi_{ij}(z_j))\frac{d\varphi_{ij}(z_j)}{dz_j}\right) = \nu_p(f_j),\ \text{because}$$

$d\varphi_{ij}(z_j)/dz_j$ is a meromorphic function having no zeros or poles.

Thus, we can define $\nu_p(\omega) \doteq \nu_p(f_i)$, $p \in U_i$.

If $\nu_p(\omega) > 0$ then p is called a zero of ω; if $\nu_q(\omega) < 0$, then q is called a pole of ω.

Definition 4.5.: Suppose C is a Riemann surface with $\omega = \{(U_i, z_i, f_i(z_i)\,dz_i)\} \in K^1(C)$, γ is a piecewise smooth curve on C not containing the poles of ω, and $\gamma = \bigcup_i \gamma_i$ is any partition of γ satisfying $\gamma_i \subseteq U_i$. We define the integral

$$\int_\gamma \omega = \sum_i \int_{\gamma_i} f_i(z_i)\, dz_i .$$

This definition is well defined because if $\gamma = \cup \gamma_i'$ is another partition of γ satisfying $\gamma_i' \in U_i$, then by the change of variable formula for $z_i = \varphi_{ij}(z_j)$

$$\int_{\gamma_i \cap \gamma_j'} f_i(z_i)\, dz_i = \int_{\gamma_i \cap \gamma_j'} f_i(\varphi_{ij}(z_j))\, d\varphi_{ij}(z_j) = \int_{\gamma_i \cap \gamma_j'} f_j(z_j)\, dz_j ,$$

so that
$$\sum_i \int_{\gamma_i} f_i(z_i) dz_i = \sum_i \sum_j \int_{\gamma_i \cap \gamma_j'} f_i(z_i)\, dz_i$$
$$= \sum_j \sum_i \int_{\gamma_i \cap \gamma_j'} f_j(z_j)\, dz_j$$
$$= \sum_j \int_{\gamma_j'} f_j(z_j)\, dz_j . \quad \blacksquare$$

Theorem 4.6. (Stokes' Theorem for Holomorphic Differentials)

Suppose C is a Riemann surface with $\Omega \subseteq C$ an open set, $\bar{\Omega}$ compact, $\partial\Omega = \gamma$ a piecewise smooth curve, and ω a holomorphic differential defined on an open set containing $\bar{\Omega}$. Then
$$\int_{\partial\Omega} \omega = 0.$$

Proof: Subdivide Ω into Ω_i's so that each $\Omega_i \subseteq U_i$ and $\Omega_i \cap \Omega_j = \phi$ and each $\partial\Omega_i$ a piecewise smooth curve. Using the local coordinate representations we get
$$\int_{\partial\Omega_i} \omega = \int_{\partial\Omega_i} f(z)\, dz = \int_{\partial\Omega_i} (u(x,y) + i v(x,y))(dx + i dy)$$

$\int f dx + g dy$

$= \int g_x - f_y$

$$= \int_{\partial\Omega_i} (u(x,y)dx - v(x,y)dy) + i \int_{\partial\Omega_i} u(x,y)dy + v(x,y)dx$$
$$= \int_{\Omega_i} (-v_x - u_y)\, dx\, dy + i \int_{\Omega_i} (u_x - v_y)\, dx\, dy$$

$= 0 + i0 = 0$. Hence, $\int_{\partial\Omega} \omega = \sum \int_{\partial\Omega_i} \omega = 0$, because the contributions of the boundaries of Ω_i cancel out other than those arcs contributing to $\partial\Omega$. $\blacksquare$

<u>Problem 4.3.</u> Prove that there are no holomorphic differentials on the Riemann sphere other than the trivial one.

<u>Suggestion:</u> If ω is a holomorphic differential on the Riemann sphere S, then the funtion

$f: S^2 \to \mathbb{C}$, $f(p) = \int_q^p \omega$, $p \in S$, where q is any fixed point is holomorphic. Hence f must be constant and thus $\omega = df = 0$.

<u>Definition 4.7.:</u> Let C be a Riemann surface with $\omega \in K'(C)$, $p \in C$, γ_p a small circle around the point p, and ω having no poles other than p on the disc surrounded by γ_p (p itself may not be a pole). Then we define the residue of the point p of ω to be

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i} \int_{\gamma_p} \omega.$$

By the Stokes' Theorem we have just proved this definition is independent of the choice of the small circle γ_p. Also if locally ω is given by $\omega = f_j(z_j)\,dz_j$ in U_j, $p \in U_j$, then

$$\mathrm{Res}_p(\omega) = \frac{1}{2\pi i} \int_{\gamma_p} \omega = \frac{1}{2\pi i} \int_{\gamma_p} f_j(z_j)\,dz_j = \mathrm{Res}_p(f_j(z_j)\,dz_j).$$

<u>Theorem 4.8.</u> (Residue Theorem)

Suppose C is a compact Riemann surface. For $\omega \in K'(C)$, we have

$$\sum_{p \in C} \mathrm{Res}_p(\omega) = 0.$$

Proof: Since C is compact ω has finitely many poles on C say $p_1, p_2, \dots, p_m$. Choose small discs $\Delta_1, \dots, \Delta_m$ so that each contains a distinct p_i and each satisfies the conditions of Definition 4.7. Let $\Omega = C \setminus \bigcup_{i=1} \Delta_i$ and orient Ω and Δ_i's so that $\partial\Omega = -\bigcup_i \partial\Delta_i$.
Then by the Stokes' Theorem

$$2\pi i \sum_{p\in C} \operatorname{Res}_p(\omega) = 2\pi i \sum_{i=1}^{m} \operatorname{Res}_{p_i}(\omega) = \sum_{i=1}^{m} \int_{\partial\Delta_i} \omega = -\int_{\partial\Omega} \omega = 0.$$

<u>Theorem 4.9.</u>: Let C be a compact Riemann surface. If $f \in K(C)$ is not a constant function, then $\sum_{p\in C} \nu_p(f) = 0$.

This implies in particular that the number of zeros of f is equal to the number of poles of f (counting multiplicity).

$$\#\{\text{zeros of } f\} = \#\{\text{poles of } f\}.$$

<u>Proof</u>: Choose $\omega = df/f$ and apply the Residue Theorem.

<u>Corollary 4.10.</u> If $f \in K(C)$ is not constant, then for any $a \in \mathbb{C}$, we have

$$\# f^{-1}(a) = \#\{\text{poles of } f\}.$$

Here, $\# f^{-1}(a)$ is the number of points p such that $f(p) = a$, counting each such point with suitable multiplicity.

<u>Proof</u>: $f(z)$ and $f(z) - a$ have the same poles.

<u>Remark</u>: Considering any $f \in K(C)$ and a holomorphic map $f : C \to \mathbb{CP}^1$ the above corollary states that the function $p \mapsto \# f^{-1}(p)$, $p \in \mathbb{CP}^1$ is constant.

§5. <u>Differential Forms</u>: Writing any complex number $z = x + iy$ we identify $\mathbb{C}$ with $\mathbb{R}^2$ and any function $f : \mathbb{C} \to \mathbb{C}$ may be regarded as a mapping $\mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (u(x,y), v(x,y))$, $f(z) = u(x,y) + iv(x,y)$.

Since $z=x+iy$ we may write $dz=dx+idy$ and for $\bar{z}=x-iy$, $d\bar{z}=dx-idy$. Then $\partial/\partial z$ and $\partial/\partial\bar{z}$ becomes

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-i\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial\bar{z}}=\frac{\partial}{\partial x}+i\frac{\partial}{\partial y} \quad \text{so that}$$

$$dz\left(\frac{\partial}{\partial z}\right)=1,\ dz\left(\frac{\partial}{\partial\bar{z}}\right)=0,\ d\bar{z}\left(\frac{\partial}{\partial z}\right)=0 \text{ and } d\bar{z}\left(\frac{\partial}{\partial\bar{z}}\right)=1.$$

So, $\dfrac{\partial f}{\partial z}=\dfrac{\partial}{\partial z}(f)=\left(\dfrac{\partial}{\partial x}-i\dfrac{\partial}{\partial y}\right)(f)=\dfrac{\partial f}{\partial x}-i\dfrac{\partial f}{\partial y}$ and

$$\frac{\partial f}{\partial\bar{z}}=\frac{\partial}{\partial\bar{z}}(f)=\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(f)=\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}.$$

Also we can write, $df=f_x\,dx+f_y\,dy=\dfrac{\partial f}{\partial z}dz+\dfrac{\partial f}{\partial\bar{z}}d\bar{z}$.

Then, $\partial f/\partial\bar{z}=0$ is equivalent to the Cauchy-Riemann equations

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}\\[2mm] \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}.\end{cases}$$

Hence, f is holomorphic if and only if $\partial f/\partial\bar{z}=0$.

Also note that $dz\wedge d\bar{z}=(dx+idy)\wedge(dx-idy)=-2i\,dx\wedge dy$.

<u>Definition 5.1.</u>: Let C be a Riemann surface with holomorphic coordinate covering $\{(U_i,z_i)\}$. A differential one-form λ on C is given by a family of local expressions

$$\lambda_i=f_i(z_i,\bar{z}_i)\,dz_i+g_i(z_i,\bar{z}_i)\,d\bar{z}_i,$$

where f_i, g_i are smooth functions which obey the transformation laws:

$$f_i(\varphi_{ij}(z_j),\overline{\varphi_{ij}(z_j)})\,\frac{d\varphi_{ij}(z_j)}{dz_j}=f_j(z_j,\bar{z}_j),$$

$$g_i(\varphi_{ij}(z_j),\overline{\varphi_{ij}(z_j)})\,\frac{\overline{d\varphi_{ij}(z_j)}}{dz_j}=g_j(z_j,\bar{z}_j).$$

Definition 5.2.: The exterior derivative of a differentiable 1-form $\lambda = \{f_i dz_i + g_i d\bar{z}_i\}$ is defined as follows:

$$d\lambda = \{df_i \wedge dz_i + dg_i \wedge d\bar{z}_i\} = \left\{\left(\frac{\partial g_i}{\partial z_i} - \frac{\partial f_i}{\partial \bar{z}_i}\right) dz_i \wedge d\bar{z}_i\right\}$$

One can easily check that $d\lambda$ is well-defined.

Definition 5.3.: A differential 1-form λ on C is called closed if $d\lambda = 0$. If there exits a function f such that

$$\lambda = df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z},$$ then λ is called an exact 1-form.

Remark 5.4. Since $\dfrac{\partial^2 f}{\partial z \partial \bar{z}} = \dfrac{\partial^2 f}{\partial \bar{z} \partial z}$ any exact 1-form is exact:

$$d(df) = d\left(\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}\right) = \frac{\partial^2 f}{\partial \bar{z} \partial z} d\bar{z} \wedge dz + \frac{\partial^2 f}{\partial z \partial \bar{z}} dz \wedge d\bar{z}$$

$$= \frac{\partial^2 f}{\partial \bar{z} \partial z} (d\bar{z} \wedge dz + dz \wedge d\bar{z}) = 0.$$

Theorem 5.5. (Stokes' Thm for Differential Forms)

Suppose C is a Riemann surface, Ω is an open set in C, and λ is a differential 1-form defined on an open set containing $\bar{\Omega}$. Then

$$\int_{\partial \Omega} \lambda = \iint_{\Omega} d\lambda.$$

Corollary 5.6. If λ is a differentiable 1-form on a compact Riemann surface C, then

$$\int_C d\lambda = 0.$$

Proof. $\displaystyle\int_C d\lambda = \int_{\partial C} \lambda = \int_{\emptyset} \lambda = 0.$ ∎

§6. The Poincaré-Hopf Formula:

Suppose C is a compact Riemann surface. By §4 we know that for any $f \in K(C)$, $\sum_{p\in C} \nu_p(f) = 0$.

It is then natural to consider a similar problem: For any $\omega \in K^1(C)$ compute $\sum_{p\in C} \nu_p(\omega)$.

To answer this question we need some preparations.

Definition 6.1.: Suppose $f: S^1 \to \mathbb{R}^2 \setminus \{0\}$ is a smooth mapping. Then its winding number is by definition

$$\frac{1}{2\pi}\int_{S^1} f^* \frac{u\,dv - v\,du}{u^2+v^2} = \frac{1}{2\pi}\int_{S^1} d\arg(f)$$

The winding number counts how many times $f(z)$ goes around the origin $0 \in \mathbb{R}^2$ as z goes around S^1. So it is always an integer. Moreover it is homotopy invariant: namely, if $f_t: S^1 \times [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ is a homotopy from f_0 to f_1 (i.e., f_t is continuous on $S^1 \times [0,1]$) then winding number of f_0 and f_1 are the same. This follows from Stokes' Theorem:

$$F: S^1 \times [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}, \quad F(z,t) = f_t(z).$$

Let $\omega = F^*\left(\frac{u\,dv - v\,du}{u^2+v^2}\right) = F^*(d\theta)$, where $\theta = \tan^{-1} v/u$.

It follows that $d\theta$ is closed but not exact on $\mathbb{R}^2 \setminus \{(0,0)\}$. So, ω is closed. Now

$$0 = \int_{S^1\times[0,1]} d\omega = \int_{\partial(S^1\times[0,1])} \omega = \int_{S^1\times\{1\} - S^1\times\{0\}} F^*\left(\frac{u\,dv - v\,du}{u^2+v^2}\right)$$

$$= \int_{S^1\times\{1\}} f_1^*\left(\frac{u\,dv - v\,du}{u^2+v^2}\right) - \int_{S^1\times\{0\}} f_0^*\left(\frac{u\,dv - v\,du}{u^2+v^2}\right)$$

$$\Rightarrow \frac{1}{2\pi}\int_{S^1} f_1^*\left(\frac{u\,dv - v\,du}{u^2+v^2}\right) = \frac{1}{2\pi}\int_{S^1} f_0^*\left(\frac{u\,dv - v\,du}{u^2+v^2}\right).$$

Suppose $U \subseteq \mathbb{R}^2$ is an open set containing the origin 0, with $f: U \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ a smooth mapping.

If $\gamma_t : S^1 \times [0,1] \to U \setminus \{0\}$ is a smooth family of maps then the winding of f computed any γ_t is independent of t.

Let $\omega = f^*\left(\frac{u\,dv - v\,du}{u^2+v^2}\right)$. Then $d\omega = 0$ and thus

$$0 = \int_{S^1 \times [0,1]} \gamma_t^*(d\omega) = \int_{S^1 \times [0,1]} d(\gamma_t^* \omega) = \int_{\partial(S^1 \times [0,1])} \gamma_t^* \omega = \int_{S^1 \times \{1\}} \gamma_1^* \omega - \int_{S^1 \times \{0\}} \gamma_0^* \omega$$

$$\Rightarrow \frac{1}{2\pi} \int_{\gamma_1} \omega - \frac{1}{2\pi} \int_{\gamma_0} \omega = 0,$$ which is the desired result.

Definition 6.2.: (Index of Differentiable Forms)

Suppose U is an open set around the origin 0 in $\mathbb{R}^2$, and $\lambda = a(x,y)\,dx + b(x,y)\,dy$ is a differential form which is defined in $U \setminus \{0\}$ and nowhere equal to 0. Consider a small circle S^1 around the origin 0 in U and the mapping $g: S^1 \to \mathbb{R}^2 \setminus \{0\}$, $p \mapsto (a(p), b(p))$.

The index of the differential form λ at the point 0 is then defined to be the winding number of the mapping g:

$$\mathrm{Ind}_0 \lambda \doteq \frac{1}{2\pi} \int_{S^1} g^*\left(\frac{u\,dv - v\,du}{u^2+v^2}\right).$$

One can show in a similar fashion then this is a well-defined integer.

Theorem 6.3.: (The Poincaré-Hopf Index Formula for Real Differentiable Forms)

Suppose C is a smooth orientable two dimensional compact manifold and λ is a differentiable 1-form which is smooth except at a finite number of isolated singularities $p_1, p_2, \ldots, p_m$.

Then $\sum_{j=1}^{m} \operatorname{Ind}_{p_j} \lambda = \chi(C)$, where $\chi(C) = 2-2g$ is the Euler characteristic of C.

Remark 6.4.: The usual Poincaré-Hopf formula for vector fields is equivalent to this.

Theorem 6.5.: (The Poincaré-Hopf Index formula for meromorphic differentials)

Suppose C is a compact Riemann surface, and $\omega \in K^1(C)$, then
$$\sum_{p \in C} \nu_p(\omega) = -\chi(C).$$

Proof: Consider the real differential form $\lambda = \operatorname{Re}\omega$. In a neighborhood of the given singularity p of ω we may write
$$\omega = z^{\nu} dz, \quad \nu \in \mathbb{Z}.$$

Then
$$\begin{aligned}\lambda = \operatorname{Re}\omega &= r^{\nu}(\cos\nu\theta\, dx - \sin\nu\theta\, dy) \\ &= r^{\nu}(\cos(-\nu\theta)\, dx + \sin(-\nu\theta)\, dy).\end{aligned}$$

By direct computation we see that $\operatorname{Ind}_p \lambda = -\nu_p(\omega)$.

Thus $\sum_{p \in C} \nu_p(\omega) = -\sum_{p\in C} \operatorname{Ind}_p \lambda = -\chi(C)$. ∎

§7. Complex Manifolds:

Definition 7.1.: Suppose $U \subseteq \mathbb{C}^n$ is an open set. A complex function $f: U \to \mathbb{C}$ is said to be holomorphic, if for any $z_0 = (z_0^1, \dots, z_0^n) \in U$, there exists an $\epsilon > 0$ such that on the polydisc $P(z_0, \epsilon) = \{z = (z^1, \dots, z^n) \in \mathbb{C}^n \mid |z^i - z_0^i| < \epsilon,\ i = 1, \dots, n\} \subseteq U$, f can be represented by a convergent power series in n variables
$$f(z) = \sum_{i_1, \dots, i_n \geq 0} a_{i_1 \dots i_n} (z^1 - z_0^1)^{i_1} \cdots (z^n - z_0^n)^{i_n}.$$

The set of all holomorphic functions on $U \subseteq \mathbb{C}^n$ is denoted by $\mathcal{O}(U)$.

Definition 7.2.: Suppose $U \subseteq \mathbb{C}^n$ and $V \subseteq \mathbb{C}^m$ are open sets. A mapping $f: U \to V$ is said to be holomorphic if every component $f^j(z)$ $(j=1,\dots,m)$ of $f(z) = (f^1(z), \dots, f^m(z))$ is a holomorphic function. The mapping is said to be biholomorphic if $m=n$, f is both one-to-one and onto mapping, and $f^{-1}: V \to U$ is also holomorphic.

Definition 7.3.: A complex manifold X is a connected Hausdorff space, which together with an open covering $\{U_\alpha\}$ and a family of mappings $z_\alpha : U_\alpha \to \mathbb{C}^n$, $\forall \alpha$, satisfy the conditions:

a) each z_α is a homeomorphism from U_α onto an open set of $\mathbb{C}^n$;
b) $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \longrightarrow z_\beta(U_\alpha \cap U_\beta)$ is a biholomorphism whenever $U_\alpha \cap U_\beta \neq \phi$.

As in the case Riemann surfaces, each (z_α, U_α) is called a local holomorphic coordinate and $\{(U_\alpha, z_\alpha)\}$ as a holomorphic coordinate covering.

Definition 7.4.: Suppose X and Y are complex manifolds and $\{(U_\alpha, z_\alpha)\}$ and $\{(V_i, w_i)\}$ their respective holomorphic coordinate coverings. A continuous mapping $f: X \to Y$ is called holomorphic, if its coordinates

$$w_i \circ f \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap f^{-1}(V_i)) \longrightarrow w_i(V_i), \text{ are holomorphic.}$$

f is biholomorphic if f is a bijection and its inverse f^{-1} is also holomorphic. In this case, we say that X and Y are complex isomorphic manifolds. Any isomorphism of X onto itself is called an automorphism of X and the set of automorphisms, denoted as $\mathrm{Aut}(X)$, forms a group under composition of maps.

Example 1: All Riemann surfaces are complex manifolds (the case $n=1$).

Example 2.: If $C_1, \dots, C_n$ are Riemann surfaces then $C_1 \times \dots \times C_n$ is a complex manifold

Example 3: Let $\omega_1, \dots, \omega_{2n} \in \mathbb{C}^n$ be an $\mathbb{R}$-linearly independent set.

Then $\Lambda = \{m_1\omega_1 + \dots + m_{2n}\omega_{2n} \mid m_i \in \mathbb{Z}, i = 1, \dots, n\}$ is a lattice in $\mathbb{C}^n$. It is easy to see that $\mathbb{C}^n/\Lambda$ is a complex manifold.

Example 4.: $\mathbb{CP}^n = \mathbb{P}^n\mathbb{C} = \mathbb{C}^{n+1} \setminus \{(0,\dots,0)\}/\sim$, where

$(\xi^0, \dots, \xi^n) \sim \lambda(\xi^0, \dots, \xi^n)$, $\lambda \in \mathbb{C} \setminus \{0\}$ is a complex manifold. The equivalence class of $(\xi^0, \dots, \xi^n)$ will be denoted as $[\xi^0, \dots, \xi^n]$.

Note that $\mathbb{CP}^n = U_0 \cup \dots \cup U_n$, where $U_i = \{[\xi^0, \dots, \xi^n] \mid \xi^i \neq 0\}$ and $z_i : U_i \to \mathbb{C}^n$, given by

$z_i([\xi^0, \dots, \xi^n]) = \left(\frac{\xi^0}{\xi^i}, \dots, \widehat{\frac{\xi^i}{\xi^i}}, \dots, \frac{\xi^n}{\xi^i}\right)$ form a holomorphic coordinate covering.

Example 5: The curve $C \subseteq \mathbb{CP}^2 = \mathbb{P}^2\mathbb{C}$ given by $xy = z^2$ describes a complex manifold isomorphic to $\mathbb{CP}^1$:

$\varphi : \mathbb{CP}^1 \longrightarrow C \subseteq \mathbb{CP}^2$, $\varphi([u,v]) = [u^2, v^2, uv]$ is a biholomorphism.

Example 6: The algebraic curve C in $\mathbb{CP}^2$ given by $zy^2 - x^3 + xz^2 = 0$ is a complex manifold.

Definition 7.5.: An invertible linear mapping $T : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ induces an invertible mapping $T : \mathbb{CP}^n \to \mathbb{CP}^n$. Clearly, $T \in \mathrm{Aut}(\mathbb{CP}^n)$. Linear maps in $\mathrm{Aut}(\mathbb{CP}^n)$ forms a subgroup of $\mathrm{Aut}(\mathbb{CP}^n)$ (in fact, they are equal!).

Definition 7.6.: A subset of $\mathbb{CP}^n$, $V = \{\xi \in \mathbb{CP}^n \mid L^i(\xi) = 0, i = 1, \dots, k\}$, defined by linear mappings $L^i : \mathbb{C}^{n+1} \to \mathbb{C}$, $\xi \mapsto \sum_{j=1}^{k} \ell^i_j \xi^j$ is called a projective linear subspace of $\mathbb{CP}^n$.

Remark 7.7.: If $\mathrm{rank}\begin{pmatrix} \ell^1_0 & \dots & \ell^1_n \\ \vdots & & \vdots \\ \ell^k_0 & \dots & \ell^k_n \end{pmatrix} = k$, then by a linear change of variables we may assume that $L^i(\xi) = \xi^i$, $i = 1, \dots, k$, so that V becomes

$V = \{\xi = (\xi^0, \dots, \xi^n) \in \mathbb{CP}^n \mid \xi^i = 0, i = 1, \dots, k\}$. In other words, V is isomorphic to the $n-k$ dimensional projective space:

$\varphi: \mathbb{CP}^{n-k} \longrightarrow \mathbb{CP}^n$, $\xi = [\xi_0, \dots, \xi_{n-k}] \longmapsto [\xi_0, \dots, \xi_{n-k}, 0, \dots, 0]$,

$\varphi(\mathbb{CP}^{n-k}) = V$, which is a biholomorphism onto its image with inverse map $\varphi^{-1}: V \longrightarrow \mathbb{CP}^{n-k}$, $[\xi, 0] \longmapsto [\xi]$.

Definition 7.8. k points $p_1, \dots, p_k$ in $\mathbb{CP}^n$ are said to be in general position if any l of these points, $l \leq n+1$, $p_{i_1}, \dots, p_{i_l}$ do not lie in an $(l-2)$-dimensional projective linear subspace of $\mathbb{CP}^n$.

Example 7. Two points in $\mathbb{CP}^2$ are in general position if and only if they do not coincide; three points are in general position if and only if they are not colinear, any four points p_1, p_2, p_3, p_4 are in general position if and only if any three of them are in general position.

Problem 7.1. Verify $\mathrm{Aut}(\mathbb{P}^1\mathbb{C}) = \{z \longmapsto \frac{az+b}{cz+d},\ ad-bc \neq 0\}$.

Problem 7.2. Given four points $\xi_0, \xi_1, \xi_2, \xi_3$ in general position in $\mathbb{CP}^2$, prove that there exists a unique linear automorphism $T: \mathbb{CP}^2 \to \mathbb{CP}^2$ so that $\varphi(\xi_i) = \eta_i$, where $\eta_0 = [1,0,0]$, $\eta_1 = [0,1,0]$, $\eta_2 = [0,0,1]$ and $\eta_3 = [1,1,1]$.

§8. Algebraic Varieties: First let's introduce some notation.

$R = \mathbb{C}[x^1, x^2, \dots, x^n]$ the ring of polynomials with complex coefficients in the variables $x^1, \dots, x^n$.

$J = \{f_1, \dots, f_k\}$, $f_i \in R$, $i = 1, \dots, k$, the ideal in R generated by the polynomials $f_1, \dots, f_k$.

S^d the linear space spanned by the homogeneous polynomials of degree d in $(n+1)$ variables $\xi^0, \xi^1, \dots, \xi^n$, and

$$S = \bigoplus_{d \geq 0} S^d, \text{ the graded algebra.}$$

$I = \{F_1, \dots, F_k\}$ the homogeneous ideal, $F_i \in S^{d_i}$, $i = 1, \dots, k$.

Definition 8.1. An affine algebraic variety is a subset of $\mathbb{C}^n$ defined by $V_0 = \{x \in \mathbb{C}^n \mid f_i(x) = 0, i = 1, \dots, k\}$, $f_i \in R$, $i = 1, \dots, k$. If $k = 1$, then V_0 is called an affine algebraic hypersurface, and the degree of V_0 is $\deg(f_1)$. An affine algebraic variety in $\mathbb{C}^2$ is called an affine algebraic plane curve.

Definition 8.2.: A projective algebraic variety is a subset of $\mathbb{CP}^n$ defined by

$$V = \{\xi \in \mathbb{CP}^n \mid F_i(\xi) = 0, i = 1, \dots, k\}, \quad F_i \in S^{d_i},$$

$i = 1, \dots, k$.

Since $F_i(\lambda\xi) = \lambda^{d_i} F(\xi)$ this is well defined.

If $I = \{F_1, \dots, F_k\}$, the ideal generated by $F_1, \dots, F_k$, then we can write

$$V = \{\xi \in P\mathbb{C}^n \mid F(\xi) = 0, \forall F \in I\}.$$

When $k = 1$, V is called a projective hypersurface, and the degree of V is $\deg F_1$. A plane projective algebraic curve, or simply a plane algebraic curve is a projective hypersurface in $P^2\mathbb{C}$.

Consider the canonical embedding of $\mathbb{C}^n$ into $P^n\mathbb{C} = \mathbb{C}^n \cup P^{n-1}\mathbb{C}$,

$$\mathbb{C}^n \longrightarrow P\mathbb{C}^n, \ (x_1, \dots, x_n) \longmapsto [1, x_1, \dots, x_n].$$

Theorem 8.3.: Suppose $\mathbb{C}^n$ is embedded in $P^n\mathbb{C}$ as above. Then an affine algebraic variety V_0 in $\mathbb{C}^n$ uniquely determines a projective variety V in $P^n\mathbb{C}$ such that $V \cap \mathbb{C}^n = V_0$.

Conversely, a projective variety $V \subseteq P^n\mathbb{C}$ determines uniquely an affine variety V_0 in $\mathbb{C}^n$, such that $V_0 = V \cap \mathbb{C}^n$.

Proof: Suppose $V_0 = \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_k(x) = 0\}$. Let

$F_i(\xi^0, \dots, \xi^n) = (\xi^0)^{\deg f_i} f_i\left(\frac{\xi^1}{\xi^0}, \dots, \frac{\xi^n}{\xi^0}\right)$. Then if $V = \{\xi \in P^n\mathbb{C} \mid F_i(\xi) = 0 \ \forall i\}$, then $V \cap \mathbb{C}^n = V_0$.

Conversely, if $V = \{\xi \in P^n\mathbb{C} \mid F_i(\xi) = 0, i = 1, \dots, k\}$ then let $f_i(x^1, \dots, x^n) = F(1, x^1, \dots, x^n)$ and $V_0 = \{x \in \mathbb{C}^n \mid f_i(x) = 0, i = 1, \dots, k\}$.

Then one can easily see that $V_0 = V \cap \mathbb{C}^n$. ∎

Remark 8.4. If we exclude the variety $\mathbb{P}\mathbb{C}^{n-1} = \{\xi \mid \xi_1 = 0\}$, then the above correspondence is bijective.

Problem 8.1.: Draw the real part of the affine plane curve given by
$$y^2 = (x-a_1)\cdots(x-a_{2g+2}),$$ where $a_i \in \mathbb{R}$ so that $a_1 < a_2 < \cdots < a_{2g+2}$.

Definition 8.5. Given a projective hypersurface in $\mathbb{P}^n\mathbb{C}$
$C = \{\xi \in \mathbb{P}^n\mathbb{C} \mid F(\xi) = 0\}$, $(F \in S^d)$ we can factor $F = F_1^{m_1} \cdots F_\ell^{m_\ell}$ where each F_i are also homogeneous irreducible polynomials. Then we write
$$C = m_1 C_1 + \cdots + m_\ell C_\ell, \text{ where } C_i = \{F_i = 0\},$$
$i = 1, \dots, \ell$. Each C_i is called a component of C and the m_i is called the multiplicity of C_i. If F is already irreducible then C is called an irreducible hypersurface.

Example 1. Consider the plane algebraic curve C given by $F(\xi) = 0$ with degree d.

a) If $d = 1$ then C is a line.

b) If $d = 2$ then there are two possibilities.

i) If F is irreducible then C is conic. Moreover, by classification of conics any two such C are isomorphic by a linear automorphism T of $\mathbb{C}\mathbb{P}^2$.

ii) If F is reducible then $F = F_1 F_2$, where $\deg F_i = 1$, $i = 1, 2$, so that C is a union of two lines (possible the same).

Problem 8.2. Prove the assertion in part (b.i) above.

Theorem 8.6. Suppose C is a degree d plane algebraic curve. Then for any generic line L, $\#(C \cap L) = d$, where $\#(C \cap L)$ represents the number of intersection points of C and L counted with multiplicities.

Proof. By a linear change of coordinates we may assume that L is given by $\xi^0 = 0$ in $\mathbb{C}\mathbb{P}^2 = \{[\xi^0, \xi^1, \xi^2]\}$. Since L is "generic" we may assume that L is not a component of C and hence F is not divisible by ξ^0, where $C = \{F = 0\}$. Using a coordinate transformation of the form $\begin{cases} \xi^0 = \xi'^0 \\ \xi^1 = \xi'^1 + \lambda \xi'^2 \\ \xi^2 = \xi'^2 \end{cases}$, where λ is a suitably chosen complex number

we may assume that $F(\xi^0, \xi^1, \xi^2) = (\xi^2)^d +$ terms of lower degree containing ξ^2.

Now the points of intersection of C and L are given by $F(0, \xi^1, \xi^2) = 0$ and we have

$$\#\{F(0,\xi^1,\xi^2)=0\} = \#\{F(0,\xi^1,\xi^2)=0, \xi^1 \neq 0\} + \#\{F(0,0,\xi^2)=0\}.$$

but $f(x) \doteq F(0,1,x) = x^d +$ term of lower degree containing x, and

$\#\{F(0,\xi^1,\xi^2)=0, \xi^1 \neq 0\} = \#\{x \in \mathbb{C} \mid f(x) = 0\} = d$ and

since $F(0,0,\xi^2) = (\xi^2)^d \neq 0$ (we are in the projective plane so that $\xi^2 \neq 0$ since $\xi^0 = \xi^1 = 0$), it follows that

$\#\{F(0,0,\xi^2)=0\} = 0$. Thus, $\#(C \cap L) = d$. $\blacksquare$

§ 9. Smooth Points, Tangent Spaces and the Implicit Function Thm.

Definition 9.1. (The Affine Case) Suppose

$V_0 = \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_k(x) = 0\}$ is an affine algebraic variety. A point $p \in V_0$ is called a smooth point of V_0 if there exists a neighborhood W of p and $\{f_{i_1}, \dots, f_{i_l}\} \subseteq \{f_1, \dots, f_k\}$ satisfying

a) $V_0 \cap W = \{x \in W \mid f_{i_1}(x) = \dots = f_{i_l}(x) = 0\}$

b) $\operatorname{rank}\left(\dfrac{\partial f_{i_\nu}}{\partial x^j}\right) = l$.

In this case, l and $(n-l)$ are called respectively the complementary dimension, or codimension, and dimension of the algebraic variety V_0 at the point p.

For a smooth point $p \in V_0$, we call the affine subspace of $\mathbb{C}^n$, $T_p(V_0) = \{y \in \mathbb{C}^n \mid \sum_{j=1}^{n} \frac{\partial f_{i_\nu}}{\partial x^j}(p)\,(y^j - p^j) = 0, \nu = 1, \dots, l\}$

the tangent space of V_0 at p.
The points of V_0 which are not smooth are called singular points.
V_0 is called smooth if it has no singular points.

If $V_0 = \{x \in \mathbb{C}^n / f(x) = 0\}$ is an affine algebraic hypersurface then a point $p \in V_0$ is smooth if and only if $\partial f / \partial z_j (p) \neq 0$ for some $j = 1, \dots, n$.

Exercise 9.1. Find conditions on a and b so that the curve given by $y^2 = 4x^3 + ax + b$ is smooth.

Definition 9.2. (The Projective Case)

Suppose $V = \{\xi \in \mathbb{CP}^n \mid F_1(\xi) = \dots = F_k(\xi) = 0\}$ is a projective algebraic variety. A point p of V is called smooth if there exists a neighborhood W of p and $\{F_{i_1}, \dots, F_{i_l}\} \subseteq \{F_1, \dots, F_k\}$ such that

a) $V \cap W = \{\xi \in W / F_{i_1}(\xi) = \dots = F_{i_l}(\xi) = 0\}$,

b) $\operatorname{rank}\left(\frac{\partial F_{i_\nu}}{\partial \xi^j}(p)\right) = l$.

For a smooth point $p \in V$, we call the projective linear subspace of $\mathbb{P}^n\mathbb{C}$

$$T_p(V) = \{\eta \in \mathbb{P}^n\mathbb{C} \mid \sum_{j=1}^{n} \frac{\partial F_{i_\nu}}{\partial \xi^j}(p)\, \eta^j = 0, \nu = 1, \dots, l\}$$ the tangent space of V at p. (By the rank condition one can see that $T_p(V) \cong \mathbb{P}^{n-l}(\mathbb{C})$.)

If $V = \{F(\xi) = 0\}$ is a projective hypersurface then a point $p \in V$ is smooth if and only if $\partial F / \partial \xi_j (p) \neq 0$ for some $j = 1, \dots, n$.
Moreover, the singular points of V are the common solutions of
$F(q) = \frac{\partial F}{\partial \xi^0}(q) = \dots = \frac{\partial F}{\partial \xi^n}(q) = 0$. However, from the Euler formula for homogeneous functions, $\sum_{j=0}^{n} \xi^j \frac{\partial F(\xi)}{\partial \xi^j} = \deg(F) F(\xi)$,

we deduce that $\frac{\partial F}{\partial \xi^j}(q)=0 \ \forall j=1,\dots,n \Rightarrow F(q)=0$.

Hence, $q\in V$ is a singular point if and only if $\frac{\partial F}{\partial \xi^j}(q)=0$, for all $j=1,\dots,n$.

Exercise 9.2.: Prove that the algebraic variety,

$$\xi^0\xi^3-\xi^1\xi^2=0, (\xi^1)^2-\xi^0\xi^2=0, (\xi^2)^2-\xi^1\xi^3=0, \text{ is smooth.}$$

Theorem 9.3.: Suppose $V\subseteq P^n\mathbb{C}$ is a projective variety and $V_0=V\cap\mathbb{C}^n$ is its affine piece. Suppose $p\in V_0$ is a point point, then $[1,p]\in V$ is a smooth point and

$$T_pV_0=T_{[1,p]}V\cap\mathbb{C}^n.$$

Proof: For simplicity we'll consider only the case of hypersurfaces. So let

$$V_0=\{x\in\mathbb{C}^n \mid f(x)=0\}$$
$$V=\{\xi\in\mathbb{C}P^n \mid F(\xi)=0\}, \text{ where}$$

$f(x^1,\dots,x^n)=F(1,x^1,\dots,x^n)$. Then $\frac{\partial f}{\partial x^i}(x)=\frac{\partial F}{\partial \xi^i}(\xi)\Big|_{(1,x^1,\dots,x^n)}$, $i=1,\dots,n$.

If $p=(p^1,\dots,p^n)\in V_0$, then $[1,p]=[1,p^1,\dots,p^n]\in V$. So,

$T_{[1,p]}V=\{\eta\in P^n\mathbb{C} \mid \sum_{i=0}^n \frac{\partial F(\xi)}{\partial \xi^i}\Big|_{(1,p)}\eta^i=0\}$. Here we'll write $\eta=[1,y]=[1,y^1,\dots,y^n]$. Then we have

$$T_{[1,p]}V\cap\mathbb{C}^n=\{y\in\mathbb{C}^n \mid \frac{\partial F}{\partial \xi^0}(1,p)\cdot 1+\sum_{j=1}^n\frac{\partial F(\xi)}{\partial \xi^j}(1,p)y^j=0\}.$$

However,
$$\frac{\partial F}{\partial \xi^0}(1,p)+\sum_{j=1}^n\frac{\partial F(\xi)}{\partial \xi^j}y^j=\left[\frac{\partial F}{\partial \xi^0}(1,p)+\sum_{j=1}^n\frac{\partial F}{\partial \xi^j}(1,p)p^j\right]+\sum_{j=1}^n\frac{\partial F}{\partial \xi^j}(1,p)(y^j-p^j),$$

By the Euler's formula the term in brackets is $\deg(F)\cdot F(1,p^1,\dots,p^n)$ which is equal 0 since $p\in V_0$.

Thus, $\frac{\partial F}{\partial \xi^0}(1,p) + \sum_{j=1}^{n} \frac{\partial F}{\partial \xi^j}(1,p)\, y^j = \sum_{j=1}^{n} \frac{\partial F(\xi)}{\partial \xi^j}(1,p)\,(y^j - p^j)$

$= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(p)\,(y^j - p^j)$, and we obtain

$$T_{(1,p)} V \cap \mathbb{C}^n = \{ y \in \mathbb{C}^n / \sum_{j=1}^{n} \frac{\partial f(p)}{\partial x^j}(y^j - p^j) = 0 \} = T_p V_0 .$$

Next, we'll see that any variety at a smooth point looks, locally as $\mathbb{C}^k$, $k \leq n$.

<u>Notation:</u> Recall the notation $dz^j = dx^j + i\,dy^j$, $d\bar{z}^j = dx^j - i\,dy^j$,

$\partial/\partial z^j = \frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}$ and $\partial/\partial \bar{z}_j = \partial/\partial x_j + i\, \partial/\partial y_j$ and

$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j = \sum_j \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j .$$

<u>Lemma 9.4.</u> (Osgood) A necessary and sufficient condition for a continuous f to be holomorphic in an open set $U \subseteq \mathbb{C}^n$ is that it be holomorphic with respect to every variable z^j $(j = 1, 2, \ldots, n)$.

<u>Proof:</u> The necessary part is obvious. For the other part, let $z_0 = (z_0^1, \ldots, z_0^n) \in U$. Since f is holomorphic in each variable by the successive applications of the Cauchy Integral formula to each variable we obtain

$$f(z) = \frac{1}{(2\pi i)^n} \int \cdots \int_{T(z_0, \epsilon)} \frac{f(\xi^1, \ldots, \xi^n)}{(\xi^1 - z^1) \cdots (\xi^n - z^n)} d\xi^1 \ldots d\xi^n, \text{ where}$$

$T(z_0, \epsilon) = \{ \xi \in \mathbb{C}^n / |\xi^j - z_0^j| = \epsilon,\ j = 1, \ldots, n \}$ is a polycylinder.

Since $\frac{1}{\xi^j - z^j} = \frac{1}{(\xi^j - z_0^j) - (z^j - z_0^j)}$

$$= \frac{1}{\xi^j - z_0^j} \frac{1}{1 - \frac{z^j - z_0^j}{\xi^j - z_0^j}} = \sum_{k=0}^{\infty} \frac{1}{(\xi^j - z_0^j)^{k+1}} (z^j - z_0^j)^k$$

(provided that $|z^j - z_0^j| < \epsilon' < \epsilon = |\xi^j - z_0^j|$, $j=1,\dots,n$.)

So, we get a power series representation for f as

$f = \sum_{i_1,\dots,i_n \geq 0} a_{i_1\dots i_n} (z^1 - z_0^1)^{i_1} \dots (z^n - z_0^n)^{i_n}$, which finishes the proof.

Corollary 9.5. A necessary and sufficient condition for a continuously differentiable function f to be holomorphic in an open set $U \subseteq \mathbb{C}^n$ is $\frac{\partial f}{\partial \bar{z}^j} = 0$ ($j=1,\dots,n$) at all points of U.

Notation: $\mathcal{O}_n$ will denote the set of functions which are holomorphic in some neighborhood of $(0,\dots,0) \in \mathbb{C}^n$.

Theorem 9.6.: (Holomorphic Implicit Function Theorem)

Suppose $f_1,\dots,f_k \in \mathcal{O}_n$ satisfy $\det\left(\frac{\partial f_i(0)}{\partial z^j}\right)_{1\leq i,j\leq k} \neq 0$.

Then there exist $w_1,\dots,w_k \in \mathcal{O}_{n-k}$ such that in a neighborhood of $(0,\dots,0) \in \mathbb{C}^n$, we have

$f_1(z) = \quad = f_k(z) = 0 \iff z^i = w_i(z^{k+1},\dots,z^n)$, $i=1,\dots,k$.

Proof: By the C^∞-Implicit Function Theorem we obtain C^∞ functions $w_1,\dots,w_k$ to satisfy the last condition. So, we just need to show that each w_i is holomorphic.

Let $z' = (z^{k+1},\dots,z^n)$. For $k+1 \leq \alpha \leq n$, we have (since f_j is holomorphic)

$$0 = \frac{\partial}{\partial \bar{z}^\alpha} f_j(w(z'), z') \quad , \quad w(z') = (w_1(z'),\dots,w_k(z'))$$

$$= \frac{\partial f_j(w(z'),z')}{\partial \bar{z}^\alpha} + \sum_\ell \left(\frac{\partial f_j(w(z'),z')}{\partial w_\ell} \cdot \frac{\partial w_\ell(z')}{\partial \bar{z}^\alpha}\right)$$

$$+ \sum_\ell \frac{\partial f_j(w(z'),z')}{\partial \bar{w}_\ell} \frac{\partial \bar{w}_\ell(z')}{\partial \bar{z}^\alpha}$$

$$\Rightarrow 0 = \sum_{\ell} \frac{\partial f_j(w(z'), z')}{\partial w_\ell} \frac{\partial w_\ell(z')}{\partial \bar{z}^\alpha}$$

(since f_j is holomorphic, $\partial f_j/\partial \bar{z}^\alpha$ and $\partial f_j/\partial \bar{w}_\ell$ are both zero for $\alpha = k+1, \dots, n$ and $\ell = 1, \dots, k$.)

However, in a neighborhood of $0 \in \mathbb{C}^n$ the "coefficient matrix" has non zero determinant:

$$\det \left(\frac{\partial f_j(w(z'), z')}{\partial w_\ell} \right)_{1 \le j, \ell \le k} \neq 0.$$

Hence, we deduce that $\frac{\partial w_\ell(z')}{\partial \bar{z}^\alpha} = 0$ for all $\ell = 1, \dots, k$, $\alpha = k+1, \dots, n$.

This finishes the proof. $\blacksquare$

Exercise: Write the above proof for $n=2$, $k=1$.

Proposition 9.7. Suppose X is a compact, complex manifold and Y is a closed, connected subset of X. Then Y is an $(n-l)$-dimensional compact complex manifold if there exists a covering $\{W_i\}$ of Y, each W_i being a local coordinate neighborhood of X, and $f_i^1, \dots, f_i^l \in \mathcal{O}(W_i)$ such that

a) $Y \cap W_i = \{p \in W_i \mid f_i^1(p) = \dots = f_i^l(p) = 0\}$;

b) The Jacobian condition holds:

$$\operatorname{rank} \left(\frac{\partial(f_i^1, \dots, f_i^l)}{\partial(z_i^1, \dots, z_i^n)} \right)(p) = l, \ \forall\, p \in W_i.$$

Proof: By refining W_i's we may assume that $W_i = U_i \times V_i \subseteq \mathbb{C}^{n-l} \times \mathbb{C}^l$, and there exist holomorphic mappings $g_i : U_i \to V_i$, which satisfy $(u_i, v_i) \in Y \cap W_i \Leftrightarrow v_i = g_i(u_i)$.

Now, just choose u_i to be local coordinate function of Y in $Y \cap W_i$. The rest is left as an exercise below. $\blacksquare$

Exercise 9.3. Prove that for the u_i above, the functions $u_j \circ u_i^{-1}$ are holomorphic so that the above proof is completed.

Definition 9.8. The Y described above is called a complex submanifold of X.

We'll prove in the next chapter that an irreducible variety is always connected. Hence, any smooth irreducible algebraic curve (a one-dimensional smooth, irreducible algebraic variety) is necessarily a compact Riemann Surface.

Exercise 9.4. Suppose that the power series in two variables

$f = \sum_{m,n \geq 0} a_{mn} x^m y^n$ converges in a neighborhood of $(0,0) \in \mathbb{C}^2$, and $f(0,0) = a_{00} = 0$, $\left|\frac{\partial f}{\partial y}(0,0)\right| = a_{01} \neq 0$.

a) Verify that there is a unique formal power series $y = \sum_{j=1}^{\infty} b_j x^j$ such that $f(x, y(x)) = 0$.

b) Verify that the series $\sum_{j=1}^{\infty} b_j x^j$ converges in a sufficiently small ball of radius $\rho > 0$, $\{x \in \mathbb{C} \mid |x| < \rho\}$.

§10. Holomorphic mappings from compact Riemann Surfaces into complex Projective Spaces:

Theorem 10.1. Any $(n+1)$ linearly independent meromorphic functions $f_0, f_1, \ldots, f_n$ on a compact Riemann surface C give rise to a nondegenerate holomorphic mapping from C to $\mathbb{CP}^n$ (i.e., a holomorphic mapping whose image is not contained in an $(n-1)$-dimensional projective linear subspace). Conversely, any non-degenerate holomorphic mapping from C to $\mathbb{CP}^n$ is induced by $(n+1)$ such functions in this manner.

First we'll study some examples.

Example 1 (Rational Canonical Curve)

On the Riemann sphere $S = \mathbb{CP}^1$ choose the $(n+1)$ linearly independent meromorphic functions $f_0(z) = 1, \ldots, f_n(z) = z^n$. So we get $f: S \to \mathbb{CP}^n$, $f(z) = [1, z, \ldots, z^n]$, $z \in S$, and $f(\infty) = [0, \ldots, 0, 1]$. In homogeneous coordinates f becomes

$$f([\xi^0, \xi^1]) = [(\xi^0)^n, (\xi^0)^{n-1}\xi^1, \ldots, (\xi^1)^n].$$

If $n=1$, $f: S \to S$ is the identity map. If $n=2$, then $f: S \to \mathbb{CP}^2$,

$f([\xi^0, \xi^1]) = [(\xi^0)^2, \xi^0\xi^1, (\xi^1)^2]$. If $\zeta^0 = (\xi^0)^2$, $\zeta^1 = \xi^0\xi^1$ and $\zeta^2 = (\xi^1)^2$, then the image of f in $\mathbb{CP}^2$ is described by the equation $\zeta^0\zeta^2 - (\zeta^1)^2 = 0$, a conic. Indeed, $f: \mathbb{CP}^1 \to \mathbb{CP}^2$ is a injective and a biholomorphism onto its image. Thus, any smooth conic in $\mathbb{CP}^2$ is isomorphic to the Riemann sphere S^2.

When $n=3$, $f([\xi^0, \xi^1]) = [(\xi^0)^3, (\xi^0)^2\xi^1, \xi^0(\xi^1)^2, (\xi^1)^3]$. Letting $\zeta^0 = (\xi^0)^3$, $\zeta^1 = (\xi^0)^2\xi^1$, $\zeta^2 = \xi^0(\xi^1)^2$ and $\zeta^3 = (\xi^1)^3$, the image of f satisfies the equations

$\zeta^0\zeta^3 - \zeta^1\zeta^2 = 0$, $(\zeta^1)^2 - \zeta^0\zeta^2 = 0$ and $(\zeta^2)^2 - \zeta^1\zeta^3 = 0$. This algebraic variety is also smooth.

<u>Exercise 10.1</u>. Suppose $f: \mathbb{CP}^1 \to \mathbb{CP}^n$ is a rational canonical curve. Prove that the image in $\mathbb{CP}^n$ of any collection of mutually distinct points in $\mathbb{CP}^1$ is in general position.

<u>Exercise 10.2</u>. Prove a) The rational curve for $n=3$ is a parametric representation of the algebraic variety in $\mathbb{P}^3$.

b) No two equations of the algebraic variety above suffice to describe it.

<u>Definition 10.2</u>. Given a lattice in $\mathbb{C}$, $\Lambda = \{m_1\omega_1 + m_2\omega_2 \mid m_i \in \mathbb{Z}, i = 1,2\}$ the Weierstrass $\wp$ function with double period ω_1, ω_2 is defined to be the unique meromorphic function which satisfies the following conditions:

a) doubly periodic : $\wp(\omega + \omega_i) = \wp(\omega)$, $\forall \omega \in \mathbb{C}$, $i = 1,2$.

b) the pole condition: the poles of $\wp$ are all on Λ, and the Laurent expansion of $\wp$ at the pole 0 is

$\wp(\omega) = \omega^{-2} + h(\omega)$, where $h(\omega)$ is holomorphic in a neighborhood of 0, and $h(0) = 0$.

It is already easy to see the uniqueness of such $\wp$: If $\mathcal{Q}(\omega)$ is another such function then $\wp - \mathcal{Q}$ would be a holomorphic and doubly periodic and hence a bounded function which implies that it is constant. Finally, the condition $h(0) = 0$ would finish the proof ∎

Exercise 10.3. Prove that a) $\wp(-\omega)=\wp(\omega)$ and
b) $\wp'^2 = 4\wp^3 + a\wp + b$ for some $a,b\in\mathbb{C}$.

Hint: Choose a so that $\wp'^2 - 4\wp^3 - a\wp$ has no poles. Then by doubly periodicity it must be a constant.

Example 2. For the lattice Λ above $C=\mathbb{C}/\Lambda$ is a compact Riemann surface. Consider the mapping $f: C \to \mathbb{CP}^2$, $f([\omega]) = [1, \wp(\omega), \wp'(\omega)]$. By the above exercise the image of f is described by the equation $y^2 = 4x^3 + ax + b$. Adding the point at infinity $[0,0,1]$ it becomes a cubic algebraic curve $zy^2 = 4x^3 + axz^2 + bz^3$ in $\mathbb{CP}^2$.

Definition 10.3. Suppose X is a complex manifold, and $\{(U_i, g_i, h_i)\}$ satisfies the following conditions:

a) $\{U_i\}$ is an open covering of X;
b) $g_i, h_i \in \mathcal{O}(U_i)$, $h_i \not\equiv 0$;
c) compatibility: for $U_i \cap U_j \neq \emptyset$ we have $g_i h_j = g_j h_i$ on $U_i \cap U_j$.

Then the formal expression $\varphi = g_i/h_i$ (on U_i) becomes a well defined function on X, and it is called the rational function (meromorphic function) given by $\{(U_i, g_i, h_i)\}$.

The set of all meromorphic functions on X is denoted by $K(X)$, which is clearly a field.

When X is a Riemann surface this definition coincides with the earlier one.

Remark 10.4. When $\dim X = n > 1$, a meromorphic function on X is in general not a holomorphic function to $\mathbb{CP}^1$. For example, $\varphi = y/x$ on $\mathbb{C}^2$. φ does not have a limit value at $(0,0)$ not even the value ∞. To make it well defined one has to "change the topology" of $\mathbb{C}^2$ near $(0,0)$, namely "to blow up the plane at the origin".

Example 3. $P(\xi), Q(\xi) \in S^d$, $Q \not\equiv 0$ be two degree d homogeneous polynomials in $n+1$ variables $\xi^0, \xi^1, \dots, \xi^n$. Then $\varphi = P(\xi)/Q(\xi)$ defines a meromorphic function of $\mathbb{CP}^n$.

Example 4. In the above example if we choose $P(\xi)=\xi^i$ and $Q(\xi)=\xi^0$, then we obtain the meromorphic function $x^i=\xi^i/\xi^0$ on $P^n\mathbb{C}$.

Proposition 10.5: Suppose C is a Riemann surface, X is a complex manifold, $\varphi \in K(X)$ is given by $\{(U_i, g_i, h_i)\}$ and $f: C \to X$ is a holomorphic mapping with $f(C) \not\subset \bigcup \{h_i=0\}$. Then, $f^*\varphi \doteq \varphi \circ f \in K(C)$.

Proof: Setting $W_i = f^{-1}(U_i)$, then $f^*\varphi = \varphi \circ f$ is the meromorphic function given by $\{(W_i, f^*g_i, f^*h_i)\}$ so that on W_i,

$$f^*\varphi = \varphi \circ f = \frac{g_i \circ f}{h_i \circ f} = \frac{f^*(g_i)}{f^*(h_i)}.$$

Note that $f(C) \not\subset \bigcup\{h_i=0\}$ guarantees that $f^*h_i \not\equiv 0$.

Now we can give the proof of Theorem 10.1.:

Suppose $f_0, f_1, \dots, f_n$ are $(n+1)$ linearly independent meromorphic functions in C. Consider the mapping from C into $P^n\mathbb{C}$

$$f(p) = [f_0(p), f_1(p), \dots, f_n(p)].$$

Then except at finitely many points (zeros or poles of f_i's) this function is well defined and locally holomorphic. Observe that f can be extended to these exceptional points holomorphically: Namely, let q be one such exceptional point. Then there is a local holomorphic coordinate function z around q so that for each $i=0, \dots, n$, there is an integer ν_i so that $f_i = z^{\nu_i} h_i(z)$, $h_i(0) \neq 0$, (h_i being holomorphic on the coordinate chart).

Let $\nu = \min\{\nu_0, \nu_1, \dots, \nu_n\}$. Then for $z \neq 0$ we get

$$f(z) = [z^{-\nu} f_0(z), \dots, z^{-\nu} f_n(z)]$$
$$= [z^{\nu_0-\nu} h_0(z), \dots, z^{\nu_n-\nu} h_n(z)].$$

Hence, f is not only defined at q but it is also holomorphic. For instance, if $\nu=\nu_0$ then

$$f(z) = [h_0(z), z^{\nu_1-\nu_0} h_1(z), \ldots, z^{\nu-\nu_n} h_n(z)]$$
$$= [1, z^{\nu_1-\nu_0} h_1(z)/h_0(z), \ldots, z^{\nu-\nu_n} h_n(z)/h_0(z)].$$

The linear independence of $f_0, f_1, \ldots, f_n$ guarantees that the image set of f will not lie in an $(n-1)$-dimensional linear subspace, so that f is not non-degenerate.

Conversely, given a non-degenerate holomorphic mapping f from $\mathbb{C}$ to $\mathbb{CP}^n$, consider the meromorphic functions $x^1, \ldots, x^n$ in $\mathbb{CP}^n$: $x^i = \xi^i/\xi^0$.

Since $\{\xi \in \mathbb{CP}^n \mid \xi^0 = 0\} = \mathbb{CP}^{n-1}$ and f is non degenerate, we see that $f(\mathbb{C}) \not\subset \{\xi \in P^n\mathbb{C} \mid \xi^0 = 0\}$.

Thus $f_i = f^* x^i = x^i \circ f$ are all meromorphic functions on $\mathbb{C}$ $(i = 1, \ldots, n)$ and $1, f_1, \ldots, f_n$ are linearly independent (otherwise the image would lie in some $(n-1)$-dimensional subspace). $\blacksquare$

CHAPTER 2: The Normalization Theorem and Its Applications

§1. Singularities of the Plane Algebraic Curves.

Let $C = \{[z,x,y] \in \mathbb{P}^2\mathbb{C} \mid F(z,x,y)=0\}$ be a plane algebraic curve. Recall that a point $p \in \mathbb{P}^2\mathbb{C}$ is singular if and only if $\partial F/\partial z\,(p) = \partial F/\partial x\,(p) = \partial F/\partial y\,(p) = 0$ (these conditions imply that $p \in C$).

Consider $\mathbb{C}^2$ as a subset of $\mathbb{P}^2\mathbb{C}$ via the canonical embedding $\mathbb{C}^2 \to \mathbb{CP}^2$, $(x,y) \mapsto [1,x,y]$. Let $f(x,y) = F(1,x,y)$ and write $f(x,y)$ as the sum of homogeneous polynomials:

$$f(x,y) = f_k(x,y) + f_{k+1}(x,y) + \cdots + f_d(x,y), \text{ where } f_j(x,y) \in S^j$$

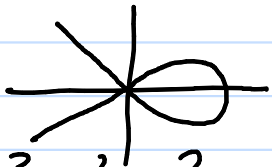
and $f_k(x,y) \neq 0$.

Assume $f(0,0) = F(1,0,0) = 0$, then $k \geq 1$.

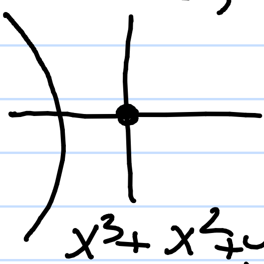
If $k=1$, then $f_1(x,y) = ax+by \not\equiv 0$. Hence, $\frac{\partial f}{\partial x}(0,0) \neq 0$ or $\frac{\partial f}{\partial y}(0,0) \neq 0$ so that $(0,0)$ is a smooth point. Hence, in order $(0,0)$ to be a singular point we must have $k \geq 2$. Note that in this case, since $f_k(x,y)=0$ has k-many lines as solutions (counted with multiplicity) we see that C has k-many tangent lines. In this case, we call the point $p=(0,0)$ a k-tuple point of C.

Definition 1.1. A k-tuple singular point of C is called ordinary if the k-many tangent lines are all distinct.

Example 1. $x^3 - x^2 + y^2 = 0$, $f_0 = f_1 = 0$, $f_2 = -x^2+y^2 = (y-x)(y+x)$ so that it has two tangent lines, namely $y=x$ and $y=-x$.



$x^3 - x^2 + y^2 = 0$

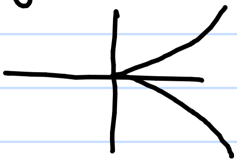


(Isotropic Curve)

$x^3 + x^2 + y^2 = 0$

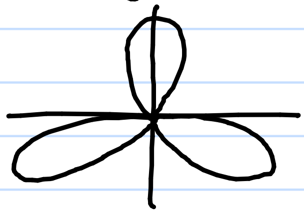
Example 2. $x^3 + x^2 + y^2 = 0$, $f_0 = f_1 = 0$, $f_2 = x^2+y^2 = (x+iy)(x-iy)$ so that the curve has again two tangent lines at the origin both nonreal and conjugate to each other.

Example 3. Consider the curve C: $x^3-y^2=0$.



Cuspidal curve

$x^3-y^2=0$



Exercise 1.1. Prove that $(0,0)$ is an ordinary triple point of the curve given by $(x^2+y^2)^2+3x^2y-y^3=0$.

§2 The Connectedness of the Irreducible Plane Algebraic Curves

First we'll show that the set of singular points of an algebraic curve is finite. We need some preparations.

Lemma 2.1. Suppose D is a unique factorization domain, and $f(x)=a_0x^m+\dots+a_m$, $g(x)=b_0x^n+\dots+b_n$, $a_0\neq0$, $b_0\neq0$, are polynomials over D. Then a necessary and sufficient condition for these polynomials to have a common factor is that there exists two polynomials $F,G\in D[x]$, not both equal to zero, which satisfy $\deg F<m$, $\deg G<n$ and $fG=gF$.

Proof: Since D is a U.F.D. by the Gauss lemma of algebra $D[x]$ is also a U.F.D. If h is a non trivial common factor of f and g then we can write $f=Fh$, $g=Gh$, $F,G\in D[x]$ and $\deg F<m$ and $\deg G<n$. Moreover,
$fG=FhG=gF$.
Conversely, if there are $F,G\in D[x]$ so that $FG\neq0$, $\deg F<m$ $\deg G<n$ and $fG=gF$, then the nontrivial factors of f cannot be all the factors of F because $\deg F<\deg f$. So, since $D[x]$ is a U.F.G. some factors of f must divide g. ∎

Let's write the polynomials F and G in the above lemma explicitly:

$$F(x)=A_0x^{m-1}+\dots+A_{m-1}$$
$$G(x)=B_0x^{n-1}+\dots+B_{n-1}.$$

Now, from the equation $fG=gF$ we obtain

$$a_0 B_0 = b_0 A_0$$
$$a_1 B_0 + a_0 B_1 = b_1 A_0 + b_0 A_1$$
$$\vdots$$
$$a_m B_{n-1} = b_n A_{m-1}.$$

We may regard these as a linear system in the variables $A_0, \dots, A_{m-1}, B_0, \dots, B_{n-1}$, so that for the existence of a nonzero solution the coefficient matrix should have zero determinant.

$$\underbrace{\begin{vmatrix} a_0 & & & b_0 & & \\ a_1 & a_0 & & b_1 & b_0 & \\ \vdots & a_1 & & \vdots & b_1 & \\ \vdots & \vdots & a_0 & \vdots & \vdots & b_0 \\ \vdots & \vdots & a_1 & \vdots & \vdots & b_1 \\ a_m & \vdots & \vdots & b_n & \vdots & \vdots \\ & a_m & \vdots & & b_n & \vdots \\ & & a_m & & & b_n \end{vmatrix}}_{n\text{-column} \quad m\text{-column}}{}_{(m+n)\times(m+n)} = 0, \quad R(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & a_m & & \\ & a_0 & a_1 & \cdots & a_m & \\ & & \vdots & & & \\ & & a_0 & a_1 & \cdots & a_m \\ b_0 & b_1 & \cdots & b_n & & \\ & b_0 & b_1 & \cdots & b_n & \\ & & \vdots & & & \\ & & b_0 & b_1 & \cdots & b_n \end{vmatrix}$$

Theorem 2.2. Suppose D is a U.F.D., and f and g are as above. Then f and g have a nontrivial factor if and only if the determinant $R(f,g)$ above is zero.

Definition 2.3. The determinant $R(f,g)$ is called the eliminant (or resultant) of f and g.

Corollary 2.4. Under the hypothesis of Theorem 2.2, there exists polynomials $\alpha, \beta \in D[x]$ such that $\deg \alpha < n$, $\deg \beta < m$ and
$$\alpha(x) f(x) + \beta(x) g(x) = R(f,g).$$

Proof: In the eliminant $R(f,g)$ let A_i denote the cofactor of the element in the $m+n$ th column and the ith row. Let $\alpha(X) = A_1 X^{n-1} + \dots + A_n$ and $\beta(X) = A_{n+1} X^{m-1} + \dots + A_{n+m}$. Now let's compute $\alpha f + \beta g$:

$$\alpha(x) f(x) + \beta(x) g(x) = (A_1 X^{n-1} + \dots + A_n)(a_0 X^m + \dots + a_m) + (A_{n+1} X^{m-1} + \dots + A_{n+m})(b_0 X^n + \dots + b_n)$$
$$= (A_1 a_0 + A_{n+1} b_0) X^{m+n-1} + \dots + (A_n a_m + A_{n+m} b_m)$$

$= R(f,g)$, because all the non-constant terms are zero: For example the coefficient of x^{m+n-1}!

$A_1 a_0 + A_{n+1} b_0 =$ determinant of the matrix, which is obtained from the one that gave $R(f,g)$, by replacing the $m+n^{th}$ column by the first column. This is clearly zero since the first and $m+n^{th}$ column are now equal.

Similarly, the coefficient of x^{m+n-2} is

$A_1 a_1 + A_2 a_0 + A_{n+1} b_1 + A_{n+2} b_0 =$ determinant of the matrix, obtained frome the one that gave $R(f,g)$, by replacing the last column by the second column. Of course it is again zero since it has two identical columns.

These observations finish the proof. ▪

__Definition 2.5.__ Suppose D is U.F.D.. Then the eliminant of $f \in D[X]$ and its derivative $f' \in D[X]$, denoted by

$$D(f) = R(f,f') \text{ is called the discriminant of } f.$$

__Corollary 2.6.__ Suppose D is U.F.D.. Then a necessary and sufficient condition for $f \in D[X]$ to have multiple factors is that its discriminant be equal to zero

$$D(f) = R(f,f') = 0.$$

__Lemma 2.7.__ Suppose C is a plane algebraic curve of degree n. Then it is possible to choose a coordinate system in $P^2\mathbb{C}$ such that C possesses an affine equation of the form

$$f(x,y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x) = 0, \text{ where } a_i(x) \in \mathbb{C}[X]$$

with $\deg(a_j(x)) \leq j$ or $a_j(x) = 0$.

__Proof:__ Let C be given by the polynomial $g(x,y)$, $\deg g = n$.
If g is __not__ of the necessary form, then we make the variable change $x = x' + \lambda y'$, $y = y'$, (λ to be determined later).

The coefficient of y'^n in $g(x'+\lambda y', y')$ is a polynomial in λ, say $b(\lambda)$. So $b(\lambda)$ is zero for only finitely many values of λ. We choose λ so that $b(\lambda) \neq 0$. Then we write
$f(x', y') = (1/b(\lambda))\, g'(x'+\lambda y', y')$ and in the affine coordinate system (x', y'), the equation of C is $f(x', y') = 0$, which has the desired form. ∎

<u>Theorem 2.8.</u> An irreducible plane curve C has at most finitely many singular points.

<u>Proof</u>: Changing the coordinate system by a linear isomorphism we may assume that C is given by a polynomial equation $f(x,y)=0$, where $f \in \mathbb{C}[X][Y]$ is given as ($D = \mathbb{C}[X]$ which is a U.F.D.)

$$f(x,y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x).$$

Then the discriminant of f, $D(f) = R(f, f_y)$ is an element of $\mathbb{C}[X]$. Since f is irreducible, $D(f)(x) \not\equiv 0$.

Let S denote the singular points of C. Clearly, $S \cap \mathbb{C}^2 \subseteq \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = f_y(x,y) = 0\}$, and by Theorem 2.2, the projection of this set into the x-axis is $D = \{x \in \mathbb{C} \mid D(f)(x) = 0\}$. This is the zero set of a non zero polynomial and thus it must be finite. Moreover, for each $x_0 \in D = \{x \in \mathbb{C} \mid D(f)(x) = 0\}$ there can be a finite set of y values such that $f(x_0, y) = 0$. Hence, $S \cap \mathbb{C}^2$ is a finite set.

On the other hand, the irreducible curve C and the line at infinity L_∞ may intersect at finitely many points so that $L_\infty \cap S$ is also finite.
Hence, $S = (S \cap \mathbb{C}^2) \cup (S \cap L_\infty)$ is finite. ∎

Next we'll show that C and $C^* = C \setminus S$ are both connected. We'll use the concept of analytic continuation: An analytic function element is a pair (Δ, f), which consists of an open disc $\Delta \subseteq \mathbb{C}$ and an analytic function f defined on this disc. Two analytic function elements (Δ_i, f_i),

$i=1,2$, are said to be direct analytic continuations of each other if $\Delta_1 \cap \Delta_2 \neq \emptyset$ and $f_1 \equiv f_2$ on $\Delta_1 \cap \Delta_2$.

An analytic continuation chain is a collection of analytic function elements (Δ_i, f_i), $i=1,\dots,N$, in which any pair of successive elements are direct analytic continuations of each other. Suppose γ is a connected continuous curve in $\mathbb{C}$, whose starting point and ending point are labelled as a and b, respectively, and suppose (Δ_0, f_0) is an analytic function element which satisfies $a \in \Delta_0$. Then we say that (Δ_0, f_0) can be analytically continued along the path γ, if there exits a partition of γ, $\gamma = \bigcup_{j=0}^{N} \gamma_j$, $a = x_0 < x_1 < \dots < x_{N+1} = b$, $\gamma_j = \gamma[x_j, x_{j+1}]$,

an analytic continuation chain begins at (Δ_0, f_0):
$(\Delta_0, f_0), \dots, (\Delta_N, f_N)$ such that $\gamma_j \subseteq \Delta_j$, $j=1,2,\dots,N$.

An important result on this subject is the

Riemann Monodromy Theorem (Theorem 2.9)

Suppose $\Omega \subseteq \mathbb{C}$ is a simply connected open set. If an analytic function element (Δ, f) can be analytically continued along any path inside Ω, then this analytic function element can be extended to be a single-valued holomorphic function defined on the whole of Ω.

Now we turn back to the connectedness of C and C^*. Since $C \setminus C^* = S$ and $C \cap L_\infty$ are both finite,

$$\overline{C^* \cap \mathbb{C}^2} = C.$$

Recall the topological fact that if $A \subseteq B \subseteq \overline{A}$ and A is connected then B is connected. So it suffices to show that C^* is connected.

For simplicity of notation we'll use C^* and C to denote the affine parts $C^* \cap \mathbb{C}^2$ and $C \cap \mathbb{C}^2$.

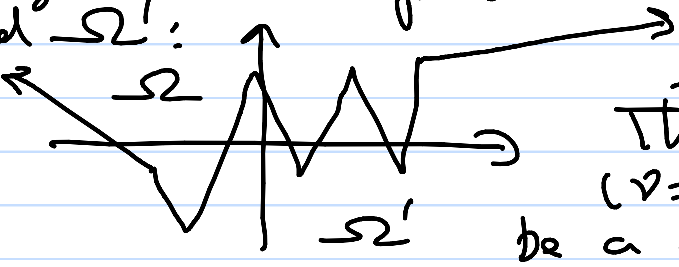
Suppose C is given by the equation $f(x,y)=0$. We may assume that $f(x,y)$ has the form $y^n + a_1(x) y^{n-1} + \dots + a_n(x)$ (Lemma 2.7).

Consider the discriminant $D(f)$ of f. Let D denote its zero set: $D = \{x \in \mathbb{C} \mid D(f)(x) = 0\}$.

Let $\pi: C \to \mathbb{C} \equiv \mathbb{C}_x$ be the projection from C to the x-axis. The proof of Theorem 2.8 show that $\pi^{-1}(D)$ is a finite set. For $x \in \mathbb{C} \setminus D$, we just have n distinct points $(x, y_\nu(x))$ in $\mathbb{C} \setminus \pi^{-1}(D)$ $(\nu = 1, \ldots, n)$ such that $f(x, y_\nu(x)) = 0$.

Moreover, for the points in $C \setminus \pi^{-1}(D)$, $f_y \neq 0$, and from the Implicit Function Theorem, every $y_\nu(x)$ can be regarded as an analytic function element defined on a disc.

Let Λ be a simple broken line which connects the points in D and goes to infinity so that $\mathbb{C} \setminus \Lambda$ is the disjoint union of two simply connected regions Ω and Ω':



From the Riemann Monodromy Theorem all n functions $y_\nu(x)$ $(\nu = 1, 2, \ldots, n)$, can be extended to be a single-valued holomorphic functions defined over the whole of Ω, and we denote these extended functions still by $y_\nu(x)$ $(\nu = 1, 2, \ldots, n)$. From the Identity theorem of analytic functions, the extended $y_\nu(x)$ must still satisfy $f(x, y_\nu(x)) = 0$.

(This is called the Heredity Property of functional relations under analytic continuations.)

Now continue $y_\mu(x)$ $(1 \leq \mu \leq n)$ along a path γ which crosses $\Lambda \setminus D$. The extended function $y_\mu^*(x)$ must still satisfy the equation $f(x, y_\mu^*(x)) = 0$ (principle of heredity) and thus must be one of $y_1(x), \ldots, y_n(x)$.

Note that if the original $y_\mu(x) \neq y_{\mu'}(x)$ then after extension we still have $y_\mu^*(x) \neq y_{\mu'}^*(x)$ (think of going along $-\gamma$).

Moreover, any $y_\mu^*(x)$ extends to all $\mathbb{C} \setminus D$ because the lower part of the broken line is also simply connected. Call the functions extended to whole $\mathbb{C} \setminus D$ again $y_1(x), \ldots, y_n(x)$. Take any point $x_0 \in \Omega$ and a path γ in $\mathbb{C} \setminus D$ starting and ending at x_0. Take any $y_\mu(x)$ and continue it along γ, say we end at $y_\nu(x)$. Then we say that $y_\mu \sim y_\nu$. In fact, this is just the action of the fundamental group $\pi_1(\mathbb{C}\setminus D, x_0)$ on the set $\{y_1(x), \ldots, y_n(x)\}$. So, $y_\mu \sim y_\nu$ if and only if they are in the same orbit. Let $E_1, \ldots, E_\ell$ be the orbits of this action.

We'll show that for any E_j

$$\prod_{y_\nu(x) \in E_j} (y - y_\nu(x)) \in \mathbb{C}[x,y] \text{ and,}$$

$$f(x,y) = \prod_{j=1}^{\ell} \prod_{y_\nu(x) \in E_j} (y - y_\nu(x)).$$

Since $f(x,y)$ is irreducible this would imply that $\ell = 1$ and hence there is only one orbit. This also implies that any two points $(x_0, y_\mu(x_0))$ and $(x_1, y_\nu(x_1))$ in $C \setminus \pi^{-1}(D)$ is connected. Hence, C^* and C are also connected.

Now let's prove the above assertion:

<u>Lemma 2.10.</u> For any orbit E of the above action $\prod_{y_\nu(x)} (y - y_\nu(x))$ is a polynomial in $\mathbb{C}[x,y]$.

<u>Proof</u>: Without loss of generality we may assume that $E = \{y_1(x), \ldots, y_m(x)\}$. Then

$$\prod_{y_\nu(x) \in E} (y - y_\nu(x)) = \prod_{i=1}^{m} (y - y_i(x)) = y^m + b_1(x) y^{m-1} + \ldots + b_m(x),$$

where

$$b_1(x) = -\sum y_\mu(x)$$
$$b_2(x) = \sum_{\mu \neq \nu} y_\mu(x) y_\nu(x)$$
$$\vdots$$
$$b_m(x) = (-1)^m y_1(x) \cdots y_m(x).$$

So this continuation along any path in $\mathbb{C}\setminus D$ leads to a permutation of y_μ's so that $b_i(x)$ are kept fixed. Hence, each $b_i(x)$ is a holomorphic function on $\mathbb{C}\setminus D$.

By Rouché's Theorem, if the coefficients of the polynomial $y^n + a_1 y^{n-1} + \ldots + a_n$ satisfy $|a_j| \leq M$ ($j = 1, \ldots, n$) then every root of this polynomial must satisfy $|y_\nu| \leq 1+M$ $(\nu=1,..,n)$. Therefore, every $b_\lambda(x)$ is bounded in a neighborhood of every point of D. So by Riemann's Extension Theorem, each $b_\lambda(x)$ is holomorphic on whole $\mathbb{C}$.

To finish the proof, next we'll show that each $b_\lambda(x)$ has a pole at infinity. This will imply that that $b_\lambda(x)$ is a polynomial.

In the original polynomial $f(x,y)=y^n+a_1(x)y^{n-1}+\dots+a_n(x)$ make the variable change $x=1/x'$ and $y=y'/x'$ to obtain

$$x'^n f(1/x', y'/x') = y'^n + (x' a_1(1/x'))\, y'^{n-1} + \dots + x'^n a_n(1/x').$$

Since, we are assuming that $\deg a_\nu \leq \nu$ or $a_\nu = 0$ $(\nu=1,..,n)$ we have

$$x'^\nu a_\nu(1/x') \in \mathbb{C}[x'],$$ and therefore

$y'^n + (x' a_1(1/x'))\, y'^{n-1} + \dots + x'^n a_n(1/x') \in \mathbb{C}[x', y']$. Fixing x'

we may consider the above polynomial as an element of $\mathbb{C}[y']$. Then $r \equiv r(x')$ is a root of this polynomial in y' if and only if $x'^n f(1/x', y'/x')=0 \Leftrightarrow f(x, r/x')=0 \Leftrightarrow$ $r/x' = y_\nu(x)$ for some $\nu \in \{1,2,\dots,n\}$. Moreover, this is possible if and only if $r = x' y_\nu(1/x')$. Hence, the set

$\{x' \mid 1/x' \in \Omega \equiv \mathbb{C}\setminus\Lambda\}$, the roots of the above polynomial in y' give rise to n holomorphic functions $y'_\nu(x') = x' y_\nu(1/x')$ $(\nu=1,2,\dots,n)$, and m of these

$\{y'_1(x'),\dots,y'_m(x')\}$, get permuted among themselves when analytically continued along any path which avoids the set

$$\{x' \mid x'=0 \text{ or } 1/x' \in D\}.$$

Now observe that for the same reason as above, each $y'_\nu(x')$ is bounded in a neighborhood of $x'=0$. Furthermore,

$$x' b_1(1/x') = -x' \sum_{\lambda=1}^{m} y_\lambda(1/x') = -\sum_{\lambda=1}^{m} y'_\lambda(x')$$

$$x'^2 b_2(1/x') = x'^2 \sum_{1\leq\lambda<\mu\leq m} y_\lambda(1/x')\, y_\mu(1/x') = \sum_{1\leq\lambda<\mu\leq m} y'_\lambda(x')\, y'_\mu(x'),$$

$$\vdots$$

$$x'^m b_m(1/x') = (-1)^m x'^m y_1(1/x') \cdots y_m(1/x')$$
$$= (-1)^m y_1'(x') \cdots y_m'(x').$$

Hence, $x'^\nu b_\nu(1/x')$, $\nu = 1, 2, \ldots, m$, are all bounded holomorphic functions in a neighborhood of $x' = 0$. This means that each $b_\lambda(1/x')$ has a pole at $x'=0$ of multiplicity at most λ, so that $b_\lambda(x)$ is a polynomial of degree at most λ. ■

So we've proven the important result:

Theorem 2.11. Let C be an irreducible plane curve in $\mathbb{P}^2\mathbb{C}$. Then C and C^* are both connected in $\mathbb{P}^2\mathbb{C}$.

Corollary 2.12. C^* is a Riemann surface (not necessarily compact).

§3. The Concept of Normalization:

Definition 3.1. Suppose C is an irreducible plane algebraic curve, and S is the set of its singular points. If there exists a compact Riemann surface $\widetilde{C}$ and a holomorphic mapping $\sigma : \widetilde{C} \to \mathbb{P}^2\mathbb{C}$ such that

a) $\sigma(\widetilde{C}) = C$
b) $\sigma^{-1}(S)$ is a finite set
c) $\sigma : \widetilde{C} \setminus \sigma^{-1}(S) \to C \setminus S$ is bijective, then we

call $(\widetilde{C}, \sigma)$ the normalization (or desingularization) of C.

Lemma 3.2. Suppose $\widetilde{C}$ and $\widetilde{C}'$ are Riemann surfaces and $h : \widetilde{C} \to \widetilde{C}'$ is a surjective holomorphic mapping which is injective on a open dense subset of $\widetilde{C}$. Then h is a biholomorphic mapping.

Proof: We know from Exercise 3.3 of Chapter 1 that h can be written locally as $w = z^\nu$ for some positive integer ν. If $\nu > 1$ then h cannot be holomorphic on a dense open set. Hence, $\nu = 1$, meaning that h is locally biholomorphic. ∎

Theorem 3.3. The normalization of an algebraic curve C is unique in the sense that if $(\tilde{C}, \sigma)$ and $(\tilde{C}', \sigma')$ are two normalizations of C, then there exists an isomorphism $\tau : \tilde{C} \to \tilde{C}'$ so that $\sigma = \sigma' \circ \tau$.

Proof: Let $S \subset C$ be the set of singular points. Then $\tilde{C} \setminus \sigma^{-1}(S) \xrightarrow{\sigma} C \setminus S \xrightarrow{(\sigma')^{-1}} \tilde{C}' \setminus (\sigma')^{-1}(S)$ is a biholomorphic map. It can be extended continuously to a map $\tilde{C} \to \tilde{C}'$, because it is bounded near every point of $\sigma^{-1}(S)$. Now by Lemma 3.2 this extended mapping is a biholomorphism. This finishes the proof. ∎

The main goal of this chapter is to prove the existence of normalization. Indeed, we'll show that every algebraic curve is obtained from an orientable surface by pinching and/or identifying a finite number of points.

§4. The Weierstrass Polynomial:

Let $\mathbb{C}\{x\}$ or $\mathbb{C}\{x,y\}$ denote the ring of holomorphic functions defined on an open set around the origin. So they are just ring of convergent power series.

Definition 4.1. An element $\omega \in \mathcal{O}$ is said to be a Weierstrass polynomial with respect to y, if

$$\omega = y^d + a_1(x) y^{d-1} + \cdots + a_d(x),$$

$a_j(x) \in \mathbb{C}\{x\}$ and $a_j(0) = 0$, $j = 1, \ldots, d$.

Let $f \in \mathcal{O}$ is not a unit and $f(0,y) \not\equiv 0$. Then

$f(0,y) = b y^d + b_1 y^{d+1} + \cdots$, $b \neq 0$, $d \geq 1$. Since the zeros of $f(0,y)$ are isolated we can assume that when $|y| \leq \varepsilon$ $f(0,y)$ contains no zeros other than $y = 0$. Thus on the circle $|y|=\epsilon$ we have $|f(0,y)| \geq c > 0$, for some $c>0$.

So for sufficiently small $\rho>0$, we have $|f(x,y)| \geq c/2 > 0$ for all (x,y) with $|x|<\rho$ and $|y|=\epsilon$.

Lemma 4.2. Under the above conditions ~~stated~~ above, for any $|x|<\rho$, $f(x,y)$ as a function in y has the same number of zeros in $|y|<\epsilon$ (the number of zeros $=d$).

Proof: According to the argument principle, for a fixed x, where $|x|<\rho$, the number $n(x)$ of zeroes of $f(x,y)$ as a function in y in $|y|<\epsilon$, is

$$n(x) = \frac{1}{2\pi i} \int_{|y|=\epsilon} f_y(x,y)/f(x,y)\, dy.$$

Clearly, this is a continuous function of x. Since it is integer valued it must be constant. Hence, $n(x)=n(0)=d$, for any $|x|$ with $|x|<\rho$. ∎

For any fixed, x ($|x|<\rho$) suppose $y_\nu(x)$ ($\nu=1,\dots,d$) are the zeros of $f(x,y)=0$. Construct the polynomial

$$\omega(x,y) = \prod_{\nu=1}^{d} (y-y_\nu(x)) = y^d + a_1(x)y^{d-1} + \dots + a_d(x), \text{ where } a_\nu(x)$$

are the elementary symmetric functions $y_\nu(x)$, $\nu=1,2,\dots,d$. Note that $a_\nu(0)=0$ because $f(0,y)$ contains no zeros other than $y=0$, so $y=0$ must be the multiple zero of order d.

Lemma 4.3. The $\omega(x,y)$ constructed above is a Weierstrass polynomial.

Proof: We just need to show that each $a_\nu(x)$ ($\nu=1,2,\dots,d$) is holomorphic. First note that each elementary symmetric function can be written as a polynomial of Newton symmetric polynomials:

$a_1(x) = -\sigma_1(x)$

$a_2(x) = \frac{1}{2}(\sigma_1(x))^2 - \sigma_2(x))$

$\vdots$

, where $\sigma_i(x) = \sum_{\nu=1}^{d} y_\nu^i(x)$.

Hence, one has to prove that each $\sigma_k(x)$ is holomorphic. However, this is clear by the Residue Theorem:

$$\sigma_k(x) = \frac{1}{2\pi i} \int_{|y|=\epsilon} y^k \frac{f_y(x,y)}{f(x,y)}\, dy.$$

<u>Lemma 4.4.</u> (The same notation as Lemma 4.3) The function defined by

$$u(x,y) = \frac{f(x,y)}{\omega(x,y)}$$ can be extended to a holomorphic function in the region $|x|<\rho$, $|y|<\epsilon$ with $u(0,0)\neq 0$.

<u>Proof:</u> We claim that $u(x,y) = f(x,y)/\omega(x,y)$ is holomorphic in the region $|x|<\rho$, $|y|\leq\epsilon$, whenever $\omega(x,y)\neq 0$. For fixed x ($|x|<\rho$), $\omega(x,y)$ and $f(x,y)$ have the same zeros and thus $u(x,y)$ is holomorphic for $|y|\leq\epsilon$. Now fix y, with $|y|<\epsilon$. Then

$$u(x,y) = \frac{1}{2\pi i} \int_{|\eta|=\epsilon} \frac{u(x,\eta)}{\eta - y}\, d\eta.$$

The R.H.S. is clearly holomorphic in x, and thus $u(x,y)$ is holomorphic for $|x|<\rho$. The same formula shows that $u(x,y)$ is continuous in $|x|<\rho$, $|y|<\epsilon$. Now by the Osgood Lemma (Chapter 1, Lemma 9.4) we see that $u(x,y)$ is holomorphic for $|x|<\rho$, $|y|<\epsilon$. ∎

Now we are ready to prove

<u>Theorem 4.5.</u> (Weierstrass Preparation Theorem)

If $f\in\mathcal{O}$ and $f(0,y)$ is not identically zero, then in a suitable neighborhood of $(0,0)$, f has a unique representation

$f(x,y) = u(x,y)\,\omega(x,y)$, where $\omega(x,y)$ is a Weierstrass polynomial and $u(x,y)$ is a unit of $\mathcal{O}$, i.e., an invertible element in $\mathcal{O}$, which is to say,

$1/u(x,y) \in \mathcal{O}$, or equivalently $u(0,0)\neq 0$.

Proof: Existence is already proved and hence we'll prove only the uniqueness part.

Since $u(x,y) \neq 0$ in a neighborhood of $(0,0)$, $f(x,y)$ and $w(x,y)$ has the same zeros. Thus $w(x,y)$ can be written as

$w(x,y) = \prod_{\lambda=1}^{d} (y - y_\lambda(x))$, where $y_\lambda(x)$ ($\lambda = 1 \dots d$) are the zeros of $f(x,y)$, for a fixed x. This implies that

$w(x,y) = y^d + a_1(x) y^{d-1} + \dots + a_d(x)$ is uniquely determined and has coefficients

$$a_1(x) = -\sum_{\nu=1}^{d} y_\nu(x), \quad a_2(x) = \sum_{1 \leq \lambda < \mu \leq d} y_\lambda(x) y_\mu(x), \dots$$

Moreover, $u(x,y) = f(x,y)/w(x,y)$.

Corollary 4.6. $\mathcal{O}$ is a U.F.D.

Proof: Step 1: $\mathbb{C}\{x\}$ is a U.F.D. This is clearly since any $f(x) \in \mathbb{C}\{y\}$ can be written as
$f(x) = x^\nu h(x)$, $h(0) \neq 0$ so that $h(x) \in \mathbb{C}\{y\}$ is a unit.

Step 2: $\mathbb{C}\{x\}[y]$ is a U.F.D. Since $\mathbb{C}\{x\}$ is a U.F.D. this follows from Gauss's Lemma.

Step 3: If $f(0,y) \equiv 0$, then f can be written uniquely as
$f(x,y) = x^\nu f_1(x,y)$, $f_1(0,y) \not\equiv 0$. Hence, without loss of generality, we may assume that $f(0,y) \not\equiv 0$.

Step 4: $f(0,y) \not\equiv 0$. Now by the Weierstrass preparation theorem, f can be written uniquely as

$f(x,y) = u(x,y)\, w(x,y)$, where $u \in \mathcal{O}$ is a unit and w is a Weierstrass polynomial.

Now by Step 2, we can write $w = h_1 \dots h_L$, where each h_j is irreducible in $\mathbb{C}\{x\}[y]$. Since $w(0,y) \not\equiv 0$ each $h_j(0,y) \not\equiv 0$. So again by the Weierstrass Preparation

Theorem, $h_j = u_j \omega_j$, where u_j is a unit in $\mathcal{O}$ and ω_j is a Weierstrass polynomial. Since each h_j is irreducible in $\mathbb{C}\{x\}[y]$, so is ω_j. Now we have

$\omega = (u_1 \cdots u_L)(\omega_1 \cdots \omega_L)$, where $u_1 \cdots u_L$ is still a unit in $\mathcal{O}$ and $\omega_1 \cdots \omega_L$ is still a Weierstrass polynomial. From the uniqueness part of the Preparation theorem, we see that $u_1 \cdots u_L = 1$ and hence $\omega = \omega_1 \cdots \omega_L$.

Thus, ω can be factored into a product of irreducible Weierstrass polynomials $\omega_1, \ldots, \omega_L$ (which are irreducible in $\mathbb{C}\{x\}[y]$). We now prove that ω_j's are also irreducible in $\mathbb{C}\{x,y\}$:

Suppose that $\omega_j = v' v''$, where v' and v'' are nontrivial factors in $\mathbb{C}\{x,y\}$. Since $\omega_j(0,y) \not\equiv 0$ both $v'(0,y) \not\equiv 0$ and $v''(0,y) \not\equiv 0$. Now, by the Weierstrass Preparation Theorem, we have $v' = u'\omega'$ and $v'' = u''\omega''$, where u' and u'' are units in $\mathbb{C}\{x,y\}$, and ω' and ω'' are Weierstrass polynomials. Then $\omega_j = (u'u'')(\omega'\omega'')$, where $u'u''$ is a unit and thus the uniqueness part of the Weierstrass Preparation Theorem $u'u'' \equiv 1$ and hence $\omega_j = \omega'\omega''$, which contradicts to the irreducibility of ω_j in $\mathbb{C}\{x\}[y]$.

In particular, we have proven that every ω_j is irreducible and $f = (u\omega_1)\omega_2 \cdots \omega_L$.

<u>Step 5:</u> Next we'll show that the factorization in the above line is unique upto multiplication by units. In order to do that, let $f = f_1 \cdots f_{L'}$ be another factorization. From the W.P.Th., we have $f_j = u_j' \omega_j'$, where $u_j' \in \mathcal{O}$ is a unit and ω_j' is a Weierstrass polynomial ($j = 1, \ldots, L'$). So, $f = u'\omega'$, where $u' = u_1' \cdots u_{L'}'$ and $\omega' = \omega_1' \cdots \omega_{L'}'$. Compare this with the earlier factorization $f = u\omega$, we get (by the Uniqueness Part of W.P.Th), $u' = u$ and $\omega = \omega'$.

By Step 2, $\mathbb{C}\{x\}[y]$ is a UFD and hence $L' = L$, and $\omega_j' = \omega_j$, for $j = 1, \ldots, L'$. This finishes the proof. ∎

<u>Remark 4.7.</u> The Step 4 of the above proof indeed shows that being irreducible in $\mathbb{C}\{x\}[y]$ and in $\mathbb{C}\{x,y\}$ are one and the same thing.

Exercise 4.1.: Let $f(x,y) = x^3 - x^2 + y^2 \in \mathbb{C}[x,y]$.

a) Verify that it is irreducible in $\mathbb{C}[x,y]$.

b) Verify that it is irreducible in $\mathbb{C}\{x\}[y]$.

Exercise 4.2.: Prove that $g(x,y) = x^3 - y^2$ is irreducible in $\mathbb{C}\{x\}[y]$.

§5. The Local Structure of Plane Algebraic Curves.

Lemma 5.1.: Suppose C is a plane algebraic curve with $p \in C$. We can choose a coordinate system such that $p = (0,0) \in \mathbb{C}^2 \subseteq \mathbb{CP}^2$ and the affine equation of C, $f(x,y) = 0$ satisfies

$$f(x,y) = y^n + a_1(x) y^{n-1} + \dots + a_n(x), \text{ where } a_i(x) \in \mathbb{C}[x]$$

and $\deg a_i(x) \leq i$, $i = 1, \dots, n$.

Proof: By a linear change of coordinates we may assume that $p = (0,0)$. Then Lemma 2.7 finishes the proof. ∎

Let $C \subseteq \mathbb{CP}^2$ be a plane algebraic curve given by an equation as in the above lemma:

$$f(x,y) = y^n + a_1(x) y^{n-1} + \dots + a_n(x) = 0.$$

If C is irreducible, then f is irreducible in $\mathbb{C}[x,y] = \mathbb{C}[x][y]$. Near the point $(0,0)$ we may regard $f \in \mathbb{C}\{x\}[y]$. Assume that we can write f in $\mathbb{C}\{x\}[y]$ as $f = f_1 \cdots f_e$, where each f_i is irreducible in $\mathbb{C}\{x\}[y]$.

Since f is irreducible in $\mathbb{C}[x][y]$, $D(f) \not\equiv 0$. However $D(f)$ is independent regardless of $f \in \mathbb{C}[x][y]$ or $\mathbb{C}\{x\}[y]$. Hence, f has <u>no</u> multiple factors in $\mathbb{C}\{x\}[y]$ also.

Definition 5.2. Suppose $f \in \mathcal{O}$, $f(0,0) = 0$, then

$$V = \{(x,y) \in \mathbb{C}^2 \,/\, |x| < \rho, \ |y| < \epsilon, \ f(x,y) = 0\} \text{ is called}$$

the local analytic curve in the neighborhood of $p = (0,0)$.

If f is irreducible in $\mathcal{O} = \mathbb{C}\{x,y\}$, then V is called an irreducible local analytic curve.

b) Suppose f has the following factorization in $\mathcal{O}$

$$f = f_1^{m_1} \cdots f_l^{m_l}, \text{ where } f_j \text{ is irreducible in } \mathcal{O} = \mathbb{C}\{x,y\}.$$

Then we can write $V = m_1 V_1 + \cdots + m_l V_l$, where

$$V_j = \{(x,y) \in \mathbb{C}^2 \mid |x| < \rho,\ |y| < \epsilon,\ f_j(x,y) = 0\} \quad (j = 1, \ldots, l).$$

is called an irreducible local analytic curve component of V.

Thus an irreducible plane algebraic curve can be locally factored into several analytic curve components which pass locally through the same point

$$f = f_1 \cdots f_l, \quad V = V_1 + \cdots + V_l.$$

Next result tells that each V_j is homeomorphic to a disc.

<u>Theorem 5.3.</u> For every irreducible local analytic curve defined as above,

$$V_j = \{(x,y) \in \mathbb{C}^2 \mid |x| < \rho, |y| < \epsilon, f_j(x,y) = 0\}, \text{ there exists}$$

a disc $\Delta = \{t \in \mathbb{C} \mid |t| < \delta\}$ and a holomorphic mapping $g_j : \Delta \to \mathbb{C}^2$ such that this mapping is injective from Δ onto V_j.

<u>Remark 5.4.</u> From $f(0,y) = y^n$, $f = f_1 \cdots f_l$ we deduce that every f_j must satisfy $f_j(0,y) \not\equiv 0$. Then by the W.P.Th. $f_j = u_j w_j$, where u_j is a unit in $\mathcal{O}$ and w_j is a Weierstrass polynomial. Then V_j can be also defined as

$$V_j = \{(x,y) \in \mathbb{C}^2 \mid |x| < \rho,\ |y| < \epsilon,\ w_j(x,y) = 0\}.$$

So, in the following we may assume that f is a Weierstrass polynomial.

Lemma 5.5. Suppose f is an irreducible Weierstrass polynomial, $f(x,y) = y^k + a_1(x)y^{k-1} + \dots + a_k(x)$. Then there exists a disc,

$$D = \{x \in \mathbb{C} \mid |x| < \rho\},$$ such that for each $x \neq 0$ in D, $f(x,y)$ as a polynomial in y has only simple roots.

Proof: Since f is an irreducible (Weierstrass) polynomial $D(f)(x) \not\equiv 0$. So $D(f)(x)$ can have only isolated zeros. Moreover, since $f(0,y) = y^k$ has multiple roots $D(f)(0) = 0$. So, $x = 0$ is an isolated zero of $Df(x)$ and thus, there is a disc $D = \{x \in \mathbb{C} \mid |x| < \rho\}$ such that for every $x \neq 0$ in D, we have $Df(x) \neq 0$. So for these values of x, f as a polynomial in y has only simple zeros. ∎

A Weierstrass polynomial as in the above theorem can be written as

$$f(x,y) = y^k + a_1(x)y^{k-1} + \dots + a_k(x) = \prod_{\nu=1}^{k} (y - y_\nu(x)),$$ where $y_\nu(x)$'s $(\nu = 1, \dots, k)$ are the roots of $f = 0$.

For each $x \neq 0$ in D, so that $D(f)(x) \neq 0$, which implies $f_y(x,y) \neq 0$.

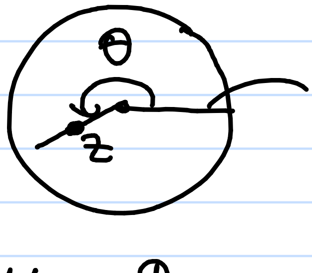
From the Implicit Function Theorem, every $y_\nu(x)$ is a locally defined holomorphic function. Along the direction of a ray (such as the positive real axis) cut open D and analytically continues $y_\nu(x)$ inside the disc-with-the-cut.

Example:

$$y^2 = x \Rightarrow y = \pm\sqrt{x}, \quad x = re^{i\theta}$$

$$y_1 = \sqrt{r}\, e^{i\theta/2}, \quad y_2 = -\sqrt{r}\, e^{i\theta/2} = \sqrt{r}\, e^{i(\theta+2\pi)/2}$$

$$y_1(\theta + 2\pi) = y_2(\theta) \text{ and } y_2(\theta + 2\pi) = y_1(\theta).$$



cut

The group of two elements permutes the solutions y_1 and y_2.

From the Riemann Monodromy Theorem we get k analytic functions $y_\nu(x)$, $(\nu = 1, \dots, k)$, all of which satisfy $f(x, y_\nu(x)) = 0$, $\nu = 1, \dots, k$. Now analytically continue $y_\nu(x)$ across the cut around $x=0$. The $y_\nu^*(x)$ obtained after continuation must still satisfy $f(x, y_\nu^*(x)) = 0$ (principle of heredity), therefore $y_\nu^*(x)$ is one of $y_1(x), \dots, y_k(x)$, with the correspondence between $y_\nu(x)$ and $y_\nu^*(x)$ being bijective. Therefore by virtue of analytic continuation around $x=0$, $y_1(x), \dots, y_k(x)$ undergo a permutation τ.

Lemma 5.6.: In order for a Weierstrass polynomial f to be irreducible, it is necessary and sufficient for the aforementioned permutation τ to be transitive.

Proof: Suppose $(y_{11}(x), \dots, y_{1s_1}(x)), \dots, (y_{\ell 1}(x), \dots, y_{\ell s_\ell}(x))$

be the cycle decomposition of the permutation σ. We want to show that $\ell = 1$.

Every cycle gives rise to a polynomial

$$f_i = \prod_{j=1}^{s_i} (y - y_{ij}(x)) = y^{s_1} + b_1(x) y^{s_1 - 1} + \dots + b_{s_1}(x), \text{ where}$$

$$b_1(x) = -\sum_{\lambda=1}^{s_1} y_{i\lambda}(x), \dots, b_{s_1}(x) = (-1)^{s_1} y_{i_1}(x) \cdots y_{i s_1}(x).$$

These coefficients are invariant under the permutation and thus they are well-defined holomorphic functions on $D \setminus \{0\}$. They are bounded in a neighborhood of $0 \in D$, they are holomorphic on the whole of D.

Hence, each $f_i(x) \in \mathbb{C}\{x\}[y]$, $i = 1, \dots, \ell$. But

$f = f_1 \cdot f_2 \cdots f_\ell$ is irreducible and hence we must have $\ell = 1$ and $s_1 = k$, which finishes the proof.

Theorem 5.7. (Local Normalization)

Under the hypothesis of Lemma 5.5 define

$g: \Delta \to \mathbb{C}^2$, $g(t) = (t^k, y_\nu(t^k))$, on the disc $\Delta = \{t \in \mathbb{C} \,/\, |t| < \rho^{1/k}\}$.

We then have

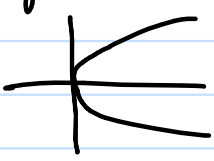
a) g is a well-defined holomorphic mapping in Δ;

b) g is an injective mapping from Δ into the local analytic curve
$$V = \{(x,y) \in \mathbb{C}^2 \mid |x| < \rho,\ |y| < \epsilon,\ f(x,y) = 0\};$$
furthermore, g is a biholomorphism from $\Delta \setminus \{0\}$ onto $V \setminus \{(0,0)\}$, where the latter is regarded as a Riemann surface.

Proof a) When t wraps once around the origin $0 \in \Delta$, t^k wraps k times around the origin and hence the value of $y_\nu(t^k)$ remains unchanged. So, $y_\nu(t^k)$ defines a single valued holomorphic function for $0 < |t| < \rho^{1/k}$. Since $y_\nu(t^k)$ is bounded in a neighborhood of origin, it is holomorphic on the whole disc Δ. Now it is easy to ~~see~~ that $g: \Delta \to \mathbb{C}^2$, $t \mapsto (t^k, y_\nu(t^k))$ is a holomorphic mapping on Δ.

b) If $g(t') = (t'^k, y_\nu(t'^k)) = (t^k, y_\nu(t^k)) = g(t)$, then
$$t' = e^{2\pi i l/k}\, t \quad \text{and} \quad y_\nu(e^{2\pi i l}\, t^k) = y_\nu(t^k), \text{ where } e^{2\pi i l}\, t^k$$
denote the value of t^k after t^k wraps around the origin l times. By Lemma 5.6, when and only when the independent variable x wraps around the origin km times does the value $y_\nu(x)$ remain unchanged. So we must have $l = km$ for some $m \in \mathbb{Z}$. This proves the injectivity of g. Moreover, when t varies inside Δ, $y_\nu(t^k)$ can assume all the possible values of $y_1(x), \ldots, y_k(x)$, $(|x| < \rho)$, again by Lemma 5.6. Hence, g maps Δ onto V.

From the Implicit Function Theorem, $V \setminus \{(0,0)\}$ can be regarded as a Riemann surface with local coordinate x. The mapping $g: \Delta \setminus \{0\} \to V \setminus \{(0,0)\}$ is holomorphic because its local representative is $x = t^k$.

[Example $f(x,y) = y^2 - x$,  $t \mapsto (t^2, t)$ holomorphic mapping.]

Now by Lemma 3.2, the injective and surjective mapping g from $\Delta \setminus \{0\}$ to $V \setminus \{(0,0)\}$ is a biholomorphism. ∎

§6. The Completion of the Proof of the Normalization Theorem.

For simplicity we will assume that our curve has only one singular point q. Suppose that there are m irreducible local analytic curve components passing through the point q.

By the methods of the previous section there are m open discs Δ_j $(j=1,\dots,m)$ together with m local normalization mappings $g_j : \Delta_j \setminus \{0\} \to C^*$ is a biholomorphic mapping onto the image set. Now introduce the set

$$\tilde{C} = C^* \underset{g_1}{\cup} \Delta_1 \underset{g_2}{\cup} \Delta_2 \cup \cdots \underset{g_m}{\cup} \Delta_m,$$ where $C^* \cup \Delta_1$ is defined as follows.

Consider the disjoint union $C^* \cup \Delta_i$ and then introduce the quotient space defined by the identification $p \sim g_i(p)$, $p \in \Delta_i \setminus \{0\}$. Since g_i's are holomorphic mapping $\tilde{C}$ has a Riemann surface structure.

If the singular set S has more points, say $S = \{q_1, \dots, q_l\}$ then we do the same for each singular point.

Since the original curve C is compact and S is a finite set $\tilde{C}$ is a compact Riemann surface.

Finally, the desired mapping $\sigma : \tilde{C} \to P\mathbb{C}$ is given by

$$\sigma(p) = \begin{cases} p, & p \in C^* \\ g_{rs_r}(p), & p \in \Delta_{rs} \quad (1 \le r \le l,\ 1 \le s_r \le m_r). \end{cases}$$

§7. Divisors, Intersection Numbers and Bezout's Theorem

Definition 7.1. Suppose $\tilde{C}$ is a Riemann surface. A divisor of $\tilde{C}$ is a formal finite sum $D = m_1 p_1 + \cdots + m_l p_l$, where $m_j \in \mathbb{Z}$ and $p_j \in \tilde{C}$ $(j=1,\dots,l)$. The degree of the divisor is defined as $\deg(D) = \sum_j m_j$.

Under the obvious definition of addition among divisors, the set of all divisors of $\tilde{C}$ forms an Abelian group, called the group of divisors of $\tilde{C}$ and denoted by $\mathrm{Div}(\tilde{C})$.

The degree defines a homomorphism $\deg : \mathrm{Div}(\tilde{C}) \to \mathbb{Z}$.
Each $f \in K(\tilde{C})$ induces a divisor (f) defined by

$$(f) = \sum_{p \in C} \nu_p(f)\, p.$$

<u>Exercise 7.1.</u> Prove $(fg) = (f) + (g)$, $(1/f) = -(f)$.

Now Theorem 4.9 of Chapter 1 can be restated as

<u>Theorem 7.2.</u>: If $f \in K(\tilde{C})$ is not constant, then

$$\deg(f) = \sum_{p \in \tilde{C}} \nu_p(f) = 0.$$

<u>Definition 7.3.</u> Suppose the plane affine algebraic curves

$$V = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\},$$

and

$$W = \{(x,y) \in \mathbb{C}^2 \mid h(x,y) = 0\}$$

intersect at p. By Lemma 5.1. we may assume $p = (0,0)$ and that $f(x,y) = y^k + a_1(x) y^{k-1} + \cdots + a_k(x)$, $a_j(x) \in \mathbb{C}[x]$, $\deg a_j(x) \leq j$ or $a_j(x) \equiv 0$.

If V is locally irreducible in the neighborhood of $(0,0)$ i.e., f is irreducible in $\mathbb{C}\{x\}[y]$, we can construct a local normalization

$$g : t \mapsto (t^k, y_\nu(t^k)),$$

and define the intersection number of V and W at $p = (0,0)$ to be

$$(V.W)_{p=(0,0)} = \nu_0(g^* h) = \nu_0(h(t^k, y_\nu(t^k)))$$

(the multiplicity of $h(t^k, y_\nu(t^k))$ with respect to t and $t = 0$.

In the general case, suppose in a neighborhood of $p = (0,0)$ V factors into irreducible factors

$$V = m_1 V_1 + \cdots + m_l V_l$$

(corresponding to the factorization $f = f_1^{m_1} \cdots f_l^{m_l}$ of f into irreducible elements in $\mathbb{C}\{x\}[y]$).

In this case, we define the intersection of V and W at $p=(0,0)$ as
$$(V.W)_p = \sum_{j=1}^{l} m_j (V_j . W)_p .$$

Exercise 7.2. Prove that $(V.W)_p = (W.V)_p$.

Examples: 1) $f(x,y) = x$, $g(x,y) = y - x^2$

$f=0 \Rightarrow$ y-axis, $t \mapsto (0,t)$, $g(0,t) = t \Rightarrow (V.W)_p = 1$.

2) $f(x,y) = y$, $g(x,y) = y - x^2$

$f=0 \Rightarrow$ x-axis, $t \mapsto (t,0)$, $g(t,0) = 0 - t^2 = -t^2$
$\Rightarrow (V.W)_p = 2$.

OR 1') $g=0 \Rightarrow y = x^2$, $t \mapsto (t,t^2)$, $f(t,t^2) = x = t$
$\Rightarrow (V.W)_p = 1$.

2') $f(t,t^2) = y = t^2$, $(V.W)_p = 2$.

Exercise 7.3. Suppose $p=(0,0)$ is a point of multiplicity k and m, respectively, of the plane algebraic curves
$$V = \{f(x,y)=0\} \text{ and } W = \{h(x,y)=0\}.$$
i.e., $f = f_k + f_{k+1} + \cdots$ and $h = h_m + h_{m+1} + \cdots$, where each f_i or h_i is homogeneous of degree i.

Then, prove that $(V.W)_p \geq km$.

Exercise 7.4. Let $V = \{y - \lambda x = 0\}$ and $W = \{y^2 - x^3 = 0\}$. Compute $(V.W)_{(0,0)}$ for different values of λ.

Definition 7.4. For algebraic curves V and W in $\mathbb{P}^2\mathbb{C}$ let, $(V.W) \doteq \sum_{p \in V \cap W} (V.W)_p$

Theorem 7.5. (Bezout)

Suppose two plane algebraic curves C and E have no common curve components (i.e. the polynomials defining C and E have no common factor.) Then

$$(C \cdot E) = \sum_{p \in C \cap E} (C \cdot E)_p = \deg C \deg E.$$

Proof: Step 1: If $C = m_1 C_1 + \cdots + m_\ell C_\ell$, where C_j is an irreducible algebraic curve, then

$$(C \cdot E) = \sum_{j=1}^{\ell} m_j (C_j \cdot E) \text{ and } \deg C = \sum m_j \deg C_j$$

which implies $\deg C \deg E = \sum_j m_j \deg C_j \deg E$.

Hence, it is enough to the theorem in case where C is an irreducible curve.

Step 2. If $C = L$ is a straight line in $P^2\mathbb{C}$, then Theorem 8.6 of Chapter 1 can be rephrased as

$$\sum_{p \in L \cap E} (L \cdot E)_p = \deg E \left(= \deg L \deg E\right).$$

Step 3. Suppose C is irreducible but not a straight line. Choose any $k = \deg E$ straight lines,

$(j = 1, \ldots, k)$ $\qquad L_j = \{\xi \in P^2\mathbb{C} \mid L_j(\xi) = 0\}$.

Then $(C \cdot L_j) = \sum_{q \in C \cap L_j} (C \cdot L_j)_q = \deg C$, $(j = 1, \ldots, k)$.

Let $E = \{\xi \in P^2\mathbb{C} \mid H(\xi) = 0\}$; then $\varphi \equiv H(\xi)/L_1(\xi) \cdots L_k(\xi)$ is a meromorphic function on $P^2\mathbb{C}$.

Now let, $g : \tilde{C} \to P^2\mathbb{C}$ be the normalization of C.

Then $g^*\varphi = \varphi \circ g \in K(\tilde{C})$, $\deg(\varphi \circ g) = \deg \Big(\sum_{p \in C \cap E} (C \cdot E)_p \, p$

$$- \sum_{j=1}^{k} \sum_{q \in C \cap L_j} (C \cdot L_j)_q \, q \Big).$$

Note that $\sum_{q\in C\cap L_j}(C\cdot L_j)_q = \deg C$ $(j=1,\dots,k)$ and thus using Theorem 7.2, we obtain $0=\deg(\varphi\circ g)$

$$= \sum_{p\in C\cap E}(C\cdot E)_p - \sum_{j=1}^{k}\sum_{q\in C\cap L_j}(C\cdot L_j)_q$$

$$= \sum_{p\in C\cap E}(C\cdot E)_p - k\deg C$$

$$= \sum_{p\in C\cap E}(C\cdot E)_p - (\deg E)(\deg C).$$

Example 1. An irreducible curve $C\subseteq \mathbb{P}^2\mathbb{C}$ is said to be rational if $\tilde{C}=\mathbb{CP}^1$.

We know that lines and conics are rational. Now using Bezout's theorem one can prove the following:

An irreducible cubic with exactly one singular point, which is an ordinary double point is rational.
Proof is left as an exercise.

Exercise 7.5. If an n^{th} degree curve has $[n/2]+1$ singular points on a straight line L, then L is necessarily a curve component of this curve.

Exercise 7.6. Prove that a fourth degree curve having four singular points is reducible.

(Hint: Just choose a conic passing through 5 points of C, four of which are these singular points!)

Exercise 7.7. Prove that a fourth degree curve with three distinct singular points is rational.

§8. Ramification Divisors and the Riemann-Hurwitz Formula

Let C, C' be Riemann surfaces, and let $f: C \to C'$ be a holomorphic mapping. For the meromorphic differential ω on C', given by $\{U'_\alpha, w_\alpha, g_\alpha(w_\alpha)dw_\alpha\}$ select a holomorphic coordinate covering for C $\{(U_i, z_i)\}$ such that $f(U_i) \subseteq U'_{\alpha(i)}$.

Suppose that the local representation of f is $w_\alpha = f_{\alpha i}(z_i)$. Then one can see that the data

$$\{U_i, z_i, g_\alpha(f_\alpha(z_i)) \frac{df_{\alpha i}(z_i)}{dz_i} dz_i\} \text{ defines a meromorphic}$$

differential on C (Exercise 8.1. below). It is called the pull back of ω via f and will be denoted as $f^*(\omega)$.

Exercise 8.1. Verify the statement above.

Suppose now, $f: C \to C'$ is non-constant and $f(p) = q$. Choose local coordinates around p and q, say z and w so that $z(p) = 0$, $w(q) = 0$ and f has the form

$$w = z^\nu, \quad z \in \Delta, \ w \in \Delta', \ \nu \geq 1.$$

Then for any $w \neq 0$, $w \in \Delta'$, $f^{-1}(w) = \{\nu$ distinct points in $C\}$ and $f^{-1}(0) = \nu \cdot 0$ (0 counted ν times).

We call $\nu = \nu_f(p)$ the multiplicity of f at p.

For any $q \in C'$ consider $\sum_{f(p)=q} \nu_f(p)$, which is a finite sum. This sum is locally constant for $q \in C'$; i.e., for $q' \in C'$ in a neighborhood of q, we have

$$\sum_{f(p')=q'} \nu_f(p') = \sum_{f(p)=q} \nu_f(p).$$

However, C' is connected and hence it is constant on the whole of C'.

Definition: a) $f^{-1}(q) \doteq \sum_{f(p)=q} \nu_f(p) p \in \mathrm{Div}(C)$;

b) $\deg f \doteq \deg f^{-1}(q) = \sum_{f(p)=q} \nu_f(p)$.

$\deg f$ will be called the degree of f.

Note that from the above paragraphs the set of points p with $\nu_f(p) > 1$ is discrete (i.e., such points are isolated) and hence $\{p \in C \mid \nu_f(p) > 1\}$ is a finite set, because C and C' are given to be compact.

<u>Definition 8.3.</u> We call $R = \sum_{p \in C} (\nu_f(p) - 1)p \in \mathrm{Div}(C)$ the Ramification divisor of f.

<u>Definition 8.4.</u> Suppose C is a compact Riemann surface, and $\omega \in K^1(C)$ is a nontrivial meromorphic differential. Then we define (cf. Definition 4.4 of Chapter 1)

$$(\omega) = \sum_{p \in C} \nu_p(\omega) \cdot p.$$

<u>Theorem 8.5.</u> (Riemann–Hurwitz Formula)

Suppose C and C' are compact Riemann surfaces with $\mathrm{genus}(C) = g$ and $\mathrm{genus}(C') = g'$. Let $f : C \to C'$ be a non-constant holomorphic mapping with degree $\deg f = n$, and let R be the ramification divisor of f,

$$R = \sum_{p \in C} (\nu_f(p) - 1)p.$$

Then $\deg R = 2(g + n - ng' - 1)$.

<u>Proof</u>: We will make use of the yet unproven fact that on any compact Riemann surface there exists a non trivial meromorphic differential.

So let's choose a meromorphic (nontrivial) differential on C', $\eta \in K^1(C')$. We'll compute $f^*(\eta)$.

Represent f locally as $w = z^{\nu}$, and $\eta = g(w)dw$.

Then, $f^{*}\eta = \nu\, g(z^{\nu})\, z^{\nu-1}\, dz$ and

$$\nu_{f^{*}\eta}(p) = \nu_f(p)\, \nu_\eta(f(p)) + \nu_f(p) - 1.$$

Thus,
$$f^{*}(\eta) = \sum_p \nu_f(p)\, \nu_\eta(f(p))\, p + R$$
$$= \sum_q \nu_\eta(q) \sum_{f(p)=q} \nu_f(p) p + R$$
$$= f^{-1}(\eta) + R, \text{ where}$$

$$f^{-1}((\omega)) = f^{-1}\Big(\sum_q \nu_\eta(q) q\Big)$$
$$= \sum_q \nu_\eta(q) f^{-1}(q)$$
$$= \sum_q \nu_\eta(q) \Big(\sum_{f(p)=q} \nu_f(p) p\Big).$$

It follows then, $\deg(f^{*}\eta) = \deg f^{-1}((\eta)) + \deg R$.

From the Poincaré-Hopf formula (Theorem 6.5 of Chap 1)

$\deg(f^{*}\eta) = 2g-2$ and $\deg(\omega) = 2g'-2$.

Then a direct computation gives

$$\deg(f^{-1}(\eta)) = \sum_q \nu_\omega(q) \Big(\sum_{f(p)=q} \nu_f(p)\Big)$$

$$= \deg(\omega) \deg f$$
$$= (2g'-2)\, n, \text{ and hence}$$

$2g-2 = (2g'-2)n + \deg R$, and the proof finishes.

Exercise 8.2. If $n \geq 1$, prove that $g \geq g'$.

Exercise 8.3. Suppose $\mathbb{C} \to \mathbb{CP}^1$ is the canonical embedding. If relative to the z-coordinate z on $\mathbb{C}$, a holomorphic map $f: \mathbb{CP}^1 \to \mathbb{CP}^1$ has the representation
$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n,$$ what is $\deg f$?

What is R?

Exercise 8.4. Suppose $\Lambda = \{m_1\omega_1 + m_2\omega_2 \mid m_i \in \mathbb{Z}\}$ is a lattice in $\mathbb{C}$ and $C = \mathbb{C}/\Lambda$. Suppose that there is a complex number $a \in \Lambda$ such that $a\Lambda \subseteq \Lambda$. Then a is called a complex multiplier of C. Consider the mapping $a: C \to C$, $a[z] \doteq [az]$. Clearly, this mapping is holomorphic. Show that $R = 0$ and $\deg a = \#(\Lambda/a\Lambda)$, the order of elements in the finite group $\Lambda/a\Lambda$.

Exercise 8.5. Suppose C is a compact Riemann surface, and $x, y: C \to \mathbb{CP}^1$ are holomorphic mappings with $\deg x = d$. Thus for all $\xi \in \mathbb{CP}^1$, $x^{-1}(\xi)$ is a divisor of degree d. We write

$$a_1(\xi) = -\sum_i y(P_i(\xi))$$

$$a_2(\xi) = \sum_{i<j} y(P_i(\xi))\, y(P_j(\xi))$$

$$\vdots$$

$$a_d(\xi) = (-1)^d y(P_1(\xi)) \cdots y(P_d(\xi)).$$

Put $f(\xi, \eta) = \eta^d + a_1(\xi)\eta^{d-1} + \cdots + a_n(\xi) = \prod_{i=1}^{d} (\eta - y(P_i(\xi)))$.

Prove that

a) $a_i(\xi)$ is well-defined and $a_i(\xi) \in K(P^1\mathbb{C})$.

b) $f(\xi, \eta) \in K(P^1\mathbb{C})[\eta]$.

c) $f(x, y) = 0$, i.e. $f(x(p), y(p)) = 0$, $\forall p \in C$.

Deduce that the image of any holomorphic mapping $g: \mathbb{C} \to \mathbb{CP}^2$ is an algebraic curve.

§9. The Genus Formula:

An ordinary double point on an algebraic curve is a point where two irreducible smooth branches cross transversally.

Theorem 9.1. Suppose $C \subseteq \mathbb{CP}^2$ is an irreducible algebraic curve of degree d, with δ singular points all being ordinary double points, and suppose also that $\sigma: \tilde{C} \to C$ is its normalization, $g(\tilde{C}) = g$. Then we have the following genus formula:

$$g = \frac{(d-1)(d-2)}{2} - \delta.$$

Proof of the genus formula: We can choose a coordinate system in $\mathbb{CP}^2$ such that

a) L_∞ (the line at infinity) and C intersect in d distinct points (hence L_∞ is neither a tangent of C, nor does it pass through the singular points of C);

b) No tangents to C at the double points $p_1, \dots, p_\delta$ passes through $[0,0,1]$;

c) $[0,0,1] \notin C$.

Step 1: Let's consider another curve $E = \{\frac{\partial f(x,y)}{\partial y} = 0\}$, where $f(x,y)$ is chosen as above. Then by Bezout's theorem $(E \cdot C) = d(d-1)$.

Let $\pi: \mathbb{CP}^2 \setminus \{[0,0,1]\} \to \mathbb{CP}^1$ be the projection from $[0,0,1]$ onto the x-axis and consider the mapping

$$x = \pi \circ \sigma : \tilde{C} \to P^1\mathbb{C}.$$

Step 2. Suppose C has a vertical tangent at the point p. Then $\partial f(x,y)/\partial y\,(p) = 0$.

By part (b) of the conditions we've choosen the coordinate system the point p is necessarily a smooth point. Thus $\partial f(x,y)/\partial x\,(p) \neq 0$. So by the Implicit Function Theorem, in a neighborhood of p, we can solve $f(x,y)=0$ as $x = x(y)$. Then, since

$f(x(y),y) = 0\ \forall y$, we get $0 = \frac{\partial f}{\partial x}(x(y),y)\ x'(y) + \frac{\partial f}{\partial y}(x(y),y)$

in that neighborhood. Hence,

$$\text{order}\ \frac{\partial f(x(y),y)}{\partial y} = \text{order}\ x'(y) = \text{order}\ x(y) - 1, \text{ because}$$

$\partial f/\partial x\,(x(y),y) \neq 0$ in the same neighborhood.

Therefore, $(E\cdot C)_p = \nu_x(p) - 1$. Now summing over all points p, at which there is a vertical tangent, we obtain
$$\sum_p (E\cdot C)_p = \deg R_x.$$
This is because, $E\cdot C$ is finite and contained in $\mathbb{C}^2 = \mathbb{CP}^2 \setminus L_\infty$. Moreover, it is equal to the points of ramification of the map $x = \pi\circ\sigma : \tilde{C} \to \mathbb{P}^1\mathbb{C}$.

If we write $D = \{D(f)(x) = 0\}$, then on $\mathbb{C}\setminus D$, x is a d to 1 mapping, and so $\deg x = d$.

By the Riemann-Hurwitz formula

$$\sum_p (E\cdot C)_p = \deg R = 2(g+d-1).$$

<u>Step 3</u>. Suppose p_j is one of the double points. By a suitable transformation of coordinates we may assume $p = (0,0)$. Then $f(x,y) = ax^2 + bxy + cy^2 + f_3(x,y) + \cdots$ and

$$\partial f/\partial y = 2bx + 2cy + \cdots$$

Since p is an ordinary double point, and that C has no vertical tangent at $p_j = (0,0)$ we have then $ac - b^2 \neq 0$, $c \neq 0$.

Using the implicit function theorem, from $\frac{\partial f}{\partial y}(x,y) \neq 0$ we can solve for y as a function of x,

$$y = y(x) = -b/c \, x + \cdots .$$

A direct calculation then gives

$$f(x, y(x)) = ax^2 + 2bx\,y(x) + c\,(y(x))^2 + \cdots$$
$$= \frac{ac-b^2}{c} x^2 + \cdots \qquad (ac-b^2) \neq 0, \text{ which is to say}$$

then $(E \cdot C)_{P_j} = 2$. For the sum of all double points, we obtain

$$\sum_{j=1}^{\delta} (E \cdot C)_{P_j} = 2\delta,$$ and combined with the results obtained in Step 2, we get

$$d(d-1) = \sum_{P} (E \cdot C)_P + \sum_{j=1}^{d} (E \cdot C)_{P_j}$$
$$= 2(g+d-1) + 2\delta, \text{ which gives } g = \frac{(d-1)(d-2)}{2} - \delta.$$

CHAPTER II. The Riemann–Roch Theorem

§1. Preliminaries:

<u>Definition 1.1</u> Let $D = \sum_{i=1}^{k} n_i p_i$ be a divisor on a compact Riemann Surface C. If all $n_i \geq 0$ then D is called a effective divisor and we write $D \geq 0$.

Clearly, any divisor D on C can be written as a difference of two effective divisors.

We write $\mathcal{L}(D) = \{f \in K(C) \mid (f) + D \geq 0\}$ and $l(D) = \dim \mathcal{L}(D)$

Note that if $f \in \mathcal{L}(D)$, then for any $n_i < 0$ in $D = \sum n_i p_i$ the order of zero of f at p_i is at least $-n_i$ and if $n_i \geq 0$ the pole of (f) at p_i has order at most $-n_i$.

Also we write $K'(D) = \{\omega \in K'(C) \mid (\omega) \geq D\}$ and $i(D) = \dim K'(D)$.

Note that if $D \geq 0$ then $\mathcal{L}(D)$ contain the vector space of constant functions $\mathbb{C}$ on C. We'll also write $\Omega'(D)$ for $K'(D)$.

If $D \geq 0$ then clearly, $K'(D) = \Omega'(D) \subseteq \Omega'(C)$ is a linear subspace of holomorphic differentials on C.

<u>Definition 1.2.</u> Suppose $D, E \in \mathrm{Div}(C)$. D and E are said to be linearly equivalent (denoted $D \sim E$) if there exists $f \in K(C)$ such that $(f) = D - E$.

<u>Proposition 1.3.</u> If $D \sim E$, then $\mathcal{L}(D) \cong \mathcal{L}(E)$, $K'(D) \cong K'(E)$, $\deg D = \deg E$.

<u>Proof.</u> Since $D \sim E$, there exists an $f \in K(C)$ such that $(f) = D - E$.

Let $g \in \mathcal{L}(D)$ then $(g) + D \geq 0$ and hence

$$(fg) + E = (f) + (g) + E = D - E + (g) + E = (g) + D \geq 0.$$

So we get a linear map $\mathcal{L}(D) \to \mathcal{L}(E)$, $g \mapsto fg$. Moreover, $\mathcal{L}(E) \to \mathcal{L}(D)$, $g \mapsto g/f$ is its inverse. Hence, $\mathcal{L}(D) \cong \mathcal{L}(E)$.

Similarly, one can prove $K^1(D) \cong K^1(E)$.

Finally, if $f \in K(C)$, then $\sum_{p \in C} \nu_p(f) = 0$ (Thm 4.9 of Chap 1) and hence

$$\sum \nu_p(f) = \deg(f) = \deg(D) - \deg(E),$$ which implies then $\deg D = \deg E$. ∎

<u>Proposition 1.4.</u> (Brill–Noether Reciprocity)

Suppose $\omega \in K^1(C)$ and $(\omega) = D + E$. Then, $\mathcal{L}(D) \cong K^1(E)$ and $\mathcal{L}(E) \cong K^1(D)$.

<u>Proof</u>: By the definition, if $f \in \mathcal{L}(D)$ then $(f) + D \geq 0$ and hence $(f) \geq -D$. So

$(f\omega) = (f) + (\omega) \geq -D + D + E = E$. Hence, we have a mapping $\mathcal{L}(D) \to K^1(E)$, $f \mapsto f\omega$. Similarly, $K^1(E) \to \mathcal{L}(D)$, $\varphi \mapsto \varphi/\omega$ is its inverse. This finishes the proof. ∎

<u>Definition 1.5.</u> Suppose $\sum_{j=-n}^{\infty} a_j z^j$ is a Laurent series. Then its singular part $\sum_{j=-n}^{-1} a_j z^j$ is called its Laurent principal part, or just principal part. In case $a_{-n} \neq 0$, n is called the order of this Laurent principal part.

Suppose C is a Riemann surface and D is a divisor with $D \geq 0$. Suppose z_i $(i = 1, 2, \ldots, k)$ is a local coordinate near p_i and $z_i(p_i) = 0$ $(D = \sum n_i p_i)$. If for every p_i, there is given a Laurent principal part of order n_i,

$$\eta_i = \frac{a_{in_i}}{z_i^{n_i}} + \cdots + \frac{a_{i1}}{z_i},$$

then one can ask the natural question: Under what conditions does there exist an $f \in K(C)$ such that in a neighborhood of p_i f has η_i as its Laurent principal part? This is the so-called Mittag-Leffler problem. It is closely related to the Riemann-Roch Theorem.

Note that a meromorphic function on a compact Riemann surface is uniquely determined by its Laurent principal parts, up to constant, because if there were two functions with the same Laurent principal parts then their difference is a holomorphic function on a compact Riemann surface, which must be a constant function.

This observation allows us to identify $L(D)/\mathbb{C}$ with a subspace of $\mathbb{C}^d$, where $d = \deg D$:

Namely, let $D = \sum_{i=1}^{k} n_i p_i \geq 0$. Then if $f \in L(D)$, then as above the Laurent principal part of f at n_i must be of the form

$$\eta_i = \frac{a_{in_i}}{z_i^{n_i}} + \cdots + \frac{a_{i_2}}{z_i^2} + \frac{a_{i_1}}{z_i}, \text{ where}$$

$a_{in_i}, \ldots, a_{i_1} \in \mathbb{C}$. Now let P_i be the n_i-dimensional vector space over $\mathbb{C}$ spanned by $\{1/z_i^{n_i}, \ldots, 1/z_i\}$. Then the map

$$L(D)/\mathbb{C} \longrightarrow P_1 \oplus P_2 \oplus \cdots \oplus P_k, \quad [f] \longmapsto (\eta_1, \eta_2, \ldots, \eta_k), \text{ is}$$

well-defined. Note that, clearly, $P_1 \oplus \cdots \oplus P_k \cong \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_k} \cong \mathbb{C}^d$.

§2. The Dimension of $\Omega^1(C)$.

Theorem 2.1. Suppose C is a compact Riemann surface of genus g. Then $\dim \Omega^1(C) = g$.

Remark 2.2. The case $g = 0$ (i.e., C is the Riemann sphere) is already proved (Chapter 1, Exercise 4.3).

We'll prove the theorem in two steps.

Proposition 2.3. Suppose C is a compact Riemann surface of genus g. Then $\dim \Omega^1(C) \geq g$.

Proof: The proof is based on the following general fact, whose proof will be omitted: For any compact Riemann surface C there is a holomorphic mapping $\pi : C \to \mathbb{CP}^2$ such that $\pi(C)$ is a degree d curve with only double point singularities, say $p_1, p_2, \dots, p_\delta \in C' \doteq \pi(C) \subseteq \mathbb{CP}^2$.

By the genus formula if $d = 1, 2$ then $g = 0$ and this is already done. So we may assume $d \geq 3$. By changing coordinates linearly if necessary, we may assume that C' is not tangent to L_∞, the line at infinity, L_∞ does not pass any of $p_1, \dots, p_\delta$ and C' does not pass through $[0,0,1]$. (More precisely choose any line L which intersects C' at d distinct points. Then L is not tangent to C'. Now apply a linear coordinate change which maps L to L_∞!)

Suppose the affine equation of C' in this coordinate system is $f(x,y) = 0$. Write $\Gamma = p_1 + p_2 + \dots + p_\delta$ and let S^{d-3} represent the set of homogeneous polynomials with complex coefficients of degree $d-3$ in three variables.

Also let $S^{d-3}(-\Gamma) = \{ G(\xi^0, \xi^1, \xi^2) \in S^{d-3} \mid G(p_i) = 0, i = 1, 2, \dots, \delta \}$.

They are both complex vector spaces with so that

$$\dim S^{d-3} = \frac{1}{2}(d-1)(d-2) \text{ and } \dim S^{d-3}(-\Gamma) \geq \frac{1}{2}(d-1)(d-2) - \delta = g,$$

(by the degree-genus formula), because each condition $G(p_i) = 0$ imposes a linearly independent condition on S^{d-3} (since the points are distinct!).

Next we'll construct an injective homomorphism from $S^{d-3}(-\Gamma)$ into $\Omega^1(C)$ so that $\dim \Omega^1(C) \geq \dim S^{d-3}(-\Gamma)$.

For any $G \in S^{d-3}(-\Gamma)$ let $g(x,y) = G(1,x,y)$.

As before let $U_0 = \{[1,x,y] \in \mathbb{CP}^2\}$ and $U_1 = \{[u,1,v] \in \mathbb{CP}^2\}$.

Now define

$$\alpha = \begin{cases} \dfrac{g(x,y)}{\partial f/\partial y}\,dx\Big|_{C'}, & \text{on } C'\cap U_0 \text{ and } \partial f/\partial y|_{C'}\neq 0, \\ \dfrac{-g(x,y)}{\partial f/\partial x}\,dy\Big|_{C'}, & \text{on } C'\cap U_0 \text{ and } \partial f/\partial y|_{C'}=0, \\ -\dfrac{g(1/u, v/u)}{\partial f/\partial y}\Big|_{C'}, & \text{on } C'\cap U_1 \end{cases}$$

Note that on U_0 the affine equation of C' is $f(x,y)=0$, and hence

$$\partial f/\partial x\, dx + \partial f/\partial y\, dy = 0 \text{ which implies}$$

$$\frac{dx}{\partial f/\partial y\,(x,y)}\Big|_{C'} = -\frac{dy}{\partial f/\partial x\,(x,y)}\Big|_{C'}.$$

In other words, the value of α when $\partial f/\partial y|_{C'}=0$ is a continuation of that for α when $\partial f/\partial y\neq 0$.

It is easy to see that $\pi^*(\alpha)$ is a meromorphic differential on C. So we have a mapping

$$S^{d-3}(-\Gamma)\longrightarrow K^1(C),\ G\longmapsto \omega_G \doteq \pi^*(\alpha).$$

Clearly, this is an injective linear map over $\mathbb{C}$: To see this, if $\omega_G=0$ then C' is contained in the curve $g(x,y)=0$, which is impossible since $\deg G = d-3$.

So, to finish the proof we just need to show that ω_G is a holomorphic differential on C.

Note that a point p on $\pi^{-1}(C'\cap U_0)$ can become a pole of ω_G only if

$$\partial f/\partial y|_{p'} = \partial f/\partial x|_{p'} = 0, \text{ where } p'=\pi(p).$$

In this case, p' is a double point of C' and hence $p'\in\Gamma$, so that $g(x,y)|_{p'}=0$ (by the choice of G and hence of g).

However, p' is a second-order zero of F, and hence a first order zero of $\partial f/\partial x$ so $\alpha|_{p'} = (-g(x,y)dy/\partial f/\partial x)|_{p'} \neq \infty$.

This means that ω_G is holomorphic at q.

Now let's study the behavior of ω_G on $\pi^{-1}(C' \cap L_\infty)$. We have, by the definition of α,

$$\alpha = -\frac{u^{d-3}\, g(1/u, v/u)\, du}{u^{d-1}\, \partial f/\partial y\, (1/u, v/u)}\Bigg|_{C'},$$

where the numerator and the denominator are polynomials in u and v.

Let $h(u,v)$ denote the denominator. We next need to show that $h(u,v) \neq 0$ on $C' \cap L_\infty$. Note that the equation of C' on U_1 is $k(u,v) = u^d f(1/u, v/u) = 0$. Clearly,

$$\frac{\partial k}{\partial v}(u,v) = u^d \frac{\partial f}{\partial y}(1/u, v/u)\frac{1}{u} = h(u,v).$$

Suppose $p' \in C' \cap L_\infty$ is such that $h(u,v)|_{p'} = 0$, then since p' is not a double point of C' we have

$$\partial k/\partial u\,|_{p'} \neq 0.$$

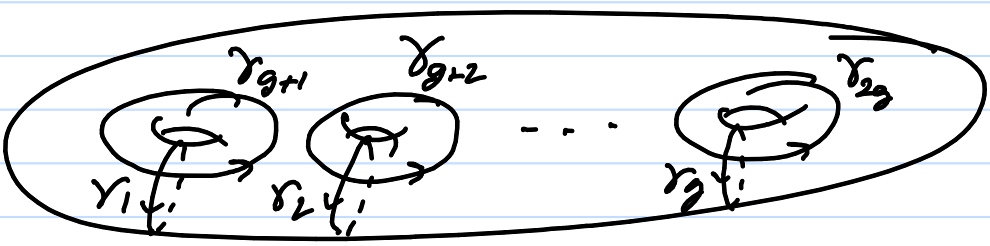
Since $p' \in L_\infty$, the coordinates of p' are of the form $[0,1,v]$ and we may write then the tangent line to C' at p' as

$$\partial k/\partial u\,|_{p'}\,(u-0) = 0,$$ i.e., the tangent line is just L_∞.

This contradicts to our choice of coordinate system and hence the proof concludes. ∎

Now we'll show that $\dim \Omega^1(C) \leq g$.

Consider the following basis for $H_1(C;\mathbb{Z})$:



By the definition of homology, any closed curve γ on C

can be written as $\gamma = \sum_{i=1}^{2g} n_i \gamma_i + \partial\Omega$, for some $n_i \in \mathbb{Z}$ and $\Omega \subseteq C$ is a region in C.

<u>Proposition</u>: If λ is a closed differential 1-form on C, then the mapping

$$\eta_\lambda : H_1(C;\mathbb{Z}) \longrightarrow \mathbb{C},\ [\gamma] \longmapsto \int_\gamma \lambda,$$

is well defined.

<u>Proof</u>: If $[\gamma] = [\gamma']$ then $\gamma - \gamma' = \partial\Omega$ for some region on Ω, so that

$$\eta_\lambda(\gamma) - \eta_\lambda(\gamma') = \int_{\gamma-\gamma'} \lambda = \int_{\partial\Omega} \lambda = \int_\Omega d\lambda = 0. \quad \blacksquare$$

<u>Proposition 2.5.</u> If λ is a closed differential one-form on C, and if for all i we have

$$\int_{\gamma_i} \lambda = 0,$$ then λ is exact.

<u>Proof</u>: Fix a point p_0 on C and define a function $f: C \to \mathbb{C}$ as $f(p) = \int_\gamma \lambda$, where γ is any path from p_0 to p. By the previous proposition $f(p)$ is independent of the choice of γ and thus $f(p)$ is well defined.

Then f is C^∞ function on C thus $df = \lambda$. $\blacksquare$

<u>Proposition 2.6.</u> Suppose C is a compact Riemann surface and $\omega, \varphi \in \Omega^1(C)$. If $\omega + \bar{\varphi} = df$ for some C^∞-function f then $\omega = \varphi = 0$.

<u>Proof</u> Suppose $\omega = h(z)dz$ and $\varphi = g(z)dz$. Then clearly, $\omega \wedge \varphi = 0$, and

$\frac{i}{2}\,\varphi\wedge\overline{\varphi} = |g(z)|^2 dz\wedge d\bar{z} = |g(z)|^2 du\wedge dv.$

Now assume on the contrary that $\varphi\neq 0$. Then

$$\frac{i}{2}\int_C \varphi\wedge\overline{\varphi} = \int_C |g(z)|^2 du\wedge dv > 0.$$

However, $\varphi\wedge\overline{\varphi} = \varphi\wedge\omega + \varphi\wedge\overline{\varphi} = \varphi\wedge(\omega+\overline{\varphi}) = \varphi\wedge df = -d(f\varphi)$.
This implies that $0 = \int_C -d(f\varphi) = \int_C \varphi\wedge\overline{\varphi} \neq 0$, a clear contradiction. ∎

<u>Proposition 2.7.</u> For any compact Riemann surface C, we have

$$\dim \Omega^1(C) \leq g.$$

Proof: Suppose on the contrary that $\omega_1, \ldots, \omega_{g+1} \in \Omega^1(C)$ are linearly independent elements. Then for any γ_i $(i=1,\ldots,2g)$ consider the equations,

$$\int_{\gamma_i} \sum_{j=1}^{g+1} \lambda_j\omega_j + \mu_j\overline{\omega_j} = 0,$$ where λ_j and μ_j are unknowns.

Since the number of unknowns is $2g+2$ and there are $2g$ equations there is a nontrivial solution, say

$\lambda_1^0, \ldots, \lambda_{g+1}^0, \mu_1^0, \ldots, \mu_{g+1}^0 \in \mathbb{C}$.

Let $\omega = \sum_j \lambda_j^0\omega_j$ and $\varphi = \sum \overline{\mu_j^0}\,\omega_j$.

Then $$\int_{\gamma_i} \omega + \overline{\varphi} = \int_{\gamma_i} \sum_{j=1}^{g+1} \lambda_j^0\omega_j + \mu_j^0\overline{\omega_j} = 0.$$

By Proposition 2.5 there is then some $f\in C^\infty(C)$ so that $\omega+\overline{\varphi} = df$. But then by Proposition 2.6 $\omega=\varphi=0$. However, this contradicts to the assumption that $\omega_1,\ldots,\omega_{g+1}$ are linearly independent. ∎

§3. Two Important Theorems.

Definition 3.1. Suppose C is a compact Riemann surface. The first cohomology group of C is

$$H^1(C,\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(H_1(C,\mathbb{Z}),\mathbb{C}).$$

Definition 3.2. The first De Rham cohomology group of C (denoted $H^1_{DR}(C)$) is defined to be the quotient group of all the closed differential one-forms on C modulo all the exact differential one-forms on C.

Note that we have a natural homomorphism

$$H^1_{DR}(C) \longrightarrow H^1(C,\mathbb{C}),\quad \lambda \longmapsto \eta_\lambda : H_1(C;\mathbb{Z}) \longrightarrow \mathbb{C}$$
$$[\gamma] \longmapsto \int_\gamma \lambda.$$

Proposition 2.5 shows that this homomorphism is injective. Moreover Proposition 2.7 of the preceding section the homomorphism

$$\Omega^1(C) \oplus \overline{\Omega^1(C)} \longrightarrow H^1_{DR}(C)$$
$$\omega + \overline{\varphi} \longmapsto \omega + \overline{\varphi}$$

is also injective.

So we have a chain of homomorphisms:

$$\Omega^1(C) \oplus \overline{\Omega^1(C)} \longrightarrow H^1_{DR}(C) \longrightarrow H^1(C,\mathbb{C}),$$ where the

two ends are complex vector spaces of the same dimension $2g$. Hence, both injections are indeed isomorphisms. In particular, we have proved special cases of Hodge and De Rham Theorems.

Theorem 3.3. (Hodge) For a compact Riemann surface we have

$$\Omega^1(C) \oplus \overline{\Omega^1(C)} \cong H^1_{DR}(C).$$

Theorem 3.4. (De Rham) For a compact Riemann surface C, we have

$$H^1_{DR}(C) \cong H^1(C,\mathbb{C}).$$

Definition 3.5. Suppose C is a compact Riemann surface of genus g with $\omega_1, \ldots, \omega_g$ forming a basis for $\Omega^1(C)$, and $\gamma_1, \ldots, \gamma_{2g}$ forming a $\mathbb{Z}$-basis for $H_1(C, \mathbb{Z})$.

We call the following $2g$ g-dimensional vectors

$$\pi_i = \begin{pmatrix} \int_{\gamma_i} \omega_1 \\ \vdots \\ \int_{\gamma_i} \omega_g \end{pmatrix} \quad (i = 1, \ldots, 2g)$$

a system of period vectors of C, and we call the matrix $\Omega = (\pi_1, \ldots, \pi_{2g})_{g \times 2g}$ a period matrix of C.

Proposition 3.6. The $2g$ period vectors given above are $\mathbb{R}$-linearly independent.

Proof: We prove this by contradiction. Suppose $\pi_1, \ldots, \pi_{2g}$ are $\mathbb{R}$-dependent. Then there exist real numbers, $a_1, \ldots, a_{2g}$ not all zero, such that

$a_1 \pi_1 + \cdots + a_{2g} \pi_{2g} = 0$. Since a_i's are real, taking the conjugate we obtain $a_1 \overline{\pi_1} + \cdots + a_{2g} \overline{\pi_{2g}} = 0$, where

$$\overline{\pi_i} = \begin{pmatrix} \int_{\gamma_i} \overline{\omega_1} \\ \vdots \\ \int_{\gamma_i} \overline{\omega_{2g}} \end{pmatrix}.$$ Consider the matrix $\Omega^* = \begin{pmatrix} \pi_1 & \cdots & \pi_{2g} \\ \overline{\pi_1} & \cdots & \overline{\pi_{2g}} \end{pmatrix}_{2g \times 2g}$.

Equations (3.1) and (3.2) together imply that the linear $2g$ columns of Ω^* with coefficients $a_1, \ldots, a_{2g}$ is equal to zero, and hence the rank of Ω^* is smaller than $2g$, so there exist complex numbers, $\lambda_1, \ldots, \lambda_g, \eta_1, \ldots, \eta_g$, not all zero, such that the linear combination of the $2g$ rows Ω^* with these as coefficients is also equal to zero, i.e.,

$$\int_{\gamma_i} \left(\sum_{j=1}^{g} \lambda_j \omega_j + \sum_{j=1}^{g} \eta_j \overline{\omega_j} \right) = 0, \quad (j = 1, \ldots, 2g).$$

Let $\omega = \sum_{j=1}^{g} \lambda_j \omega_j$, $\varphi = \sum_{j=1}^{g} \eta_j \omega_j$, $\int_{\gamma_i} (\omega + \overline{\varphi}) = 0$

for all $i = 1, \ldots, 2g$. Now Prop. 2.5 and Prop 2.6 imply that $\omega = \varphi = 0$, which contradicts the linear independence of $\omega_1, \ldots, \omega_g$. ∎

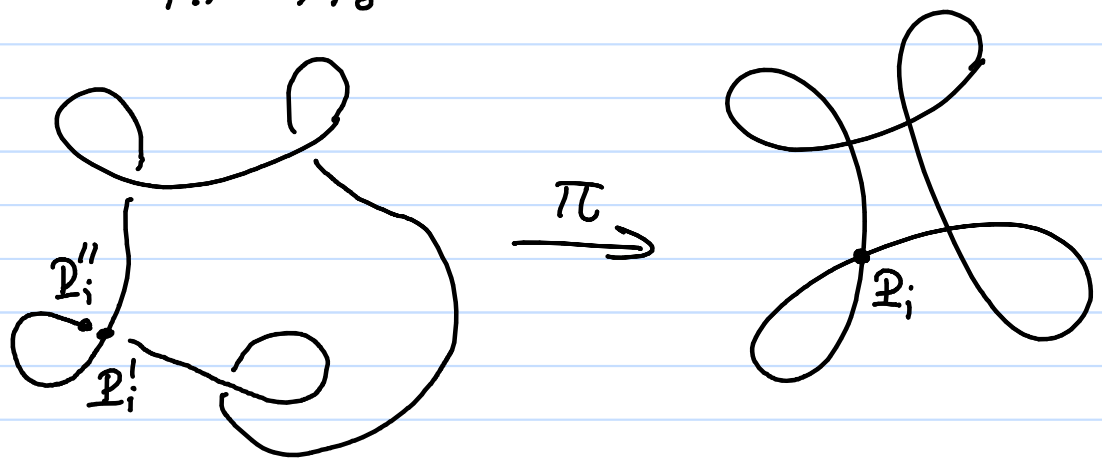
§4. The Riemann Inequality

Proposition 4.1. (The Riemann Inequality)

Suppose C is a compact Riemann surface of genus g. $D \in \mathrm{Div}(C)$ and $\deg D = d$. Then

$$\ell(D) \geqslant d - g + 1.$$

Proof: As in Proposition 2.3 we'll assume that there exists a holomorphic mapping $\pi: C \longrightarrow C' \subseteq \mathbb{CP}^2$ such that C is the normalization of C' and where C' is an algebraic curve containing only δ ordinary double points $p_1, \dots, p_\delta$.



Let $\pi^{-1}(p_i) = P_i' + P_i''$ $(i = 1, 2, \dots, \delta)$, and write

$$\Delta = \sum_{i=1}^{\delta} (P_i' + P_i'') \in \mathrm{Div}(C).$$

We may assume that neither $\pi(D)$ nor $\pi(\Delta)$ contains the point at infinity.

Suppose C' is given by the equation $F(\xi^0, \xi^1, \xi^2) = 0$ with $\deg F = m$. Let $D = D' - D''$, where D' and $D'' \geq 0$ and let $\deg D' = d'$ and $\deg D'' = d''$.

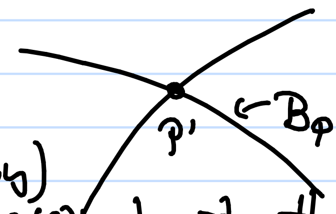
Let S^n the set of complex polynomials in three variables of degree n. Then S^n is a $\mathbb{C}$-vector space of dimension $1/2\,(n+1)(n+2)$.

Let $G(\xi^0, \xi^1, \xi^2) \in S^n$ satisfy the following two conditions:

a) $F \nmid G$
b) $G \cdot C \geq D' + \Delta$, where $G \cdot C = \pi^*(G(\xi^0, \xi^1, \xi^2))$ (i.e., the divisor induced by the zeros of $\pi^* G$ on C.

Let us study $G \cdot C \in \mathrm{Div}(C)$ more closely: For any $p \in C$ let $p' = \pi(p) \in C'$. By a suitable coordinate transformation we may assume that $p' = [1, 0, 0]$. This transformation maps G to G_p. Since the multiple points of C' are all ordinary double points, the equation of the curve component B_p of C' containing p', near p', is given by an equation of the form

$$y = a_0 x + a_1 x^2 + \cdots.$$



Substituting this expression in $G(1, x, y)$ we get a power series in x. Let $n(p)$ denote the degree of the lowest term, then

$$G \cdot C = \sum_{p \in C} n(p) p.$$

This is a finite sum because $F \nmid G$, F is irreducible, C' and $\{G = 0\}$ have no common curve components and thus $C' \cap \{G = 0\}$ is a finite set; moreover only if p belongs to the inverse image under π of this finite set do we have $n(p) \neq 0$. Also note that the definition of $G \cdot C$ is independent of the choice of the coordinate transformation on $\mathbb{P}^2$ used above.

Claim: The condition $G \cdot C \geq D' + \Delta$ implies that the coefficients of G must satisfy $d' + \delta$ linear equations.

Proof: Suppose $D' = \sum_{i=1}^{\delta} (n_i' p_i' + n_i'' p_i'') + \sum_{j=1}^{\gamma} m_j q_j$, where p_i', p_i'' and q_j are all mutually distinct, and $n_i', n_i'', m_j \geq 0$. Then

$$D' + \Delta = \sum_{i=1}^{\delta} ((n_i' + 1) p_i' + (n_i'' + 1) p_i'') + \sum_{j=1}^{\gamma} m_j q_j.$$

For any fixed i, the condition that $G\cdot C \geq (n_i'+1)p_i'$ is equivalent to the coefficients of G satisfying $(n_i'+1)$ linear equations. This is because the coefficients $l_0', l_1', \dots, l_{n_i'}'$ of the terms of degree 0, degree 1, ..., deg n_i', respectively, of the power series in x obtained by substituting the local equation of $B_{p_i'}$ into $G_{p'}(1,x,y)$ are all equal to zero, and each l_j' is a linear function in the coefficient of G. Similarly, $G\cdot C \geq (n_i''+1)p_i''$ is equivalent to the coefficients of G having to satisfy $n_i''+1$ linear equations, which we write as $l_k''=0$ $(k=0,1,\dots,n_i'')$.

However, $\pi(p_i')=\pi(p_i'')=0$ which implies that $l_0'=0$ and $l_0''=0$ are both equivalent to $G(p_i)=0$. Hence, $l_0'=0$ if and only if $l_0''=0$.

Thus, $G\cdot C \geq (n_i'+1)p_i' + (n_i''+1)p_i''$ is equivalent to the coefficients of G satisfying $n_i'+n_i''+1$ linear equations.

As for q_j, since $q_j \notin \sum_{i=1}^{\delta}(p_i'+p_i'')$, $\pi(q_j)$ is not an ordinary double point of C', and so $G\cdot C \geq m_j q_j$ is equivalent to the coefficients of G satisfying m_j linear equations. Hence, $G\cdot C' \geq D'+\Delta$ is equivalent to the coefficients of G satisfying the following number of linear equations:

$$\sum_{i=1}^{\delta}(n_i'+n_i''+1)+\sum_{j=1}^{\gamma} m_j = \left(\sum_{i=1}^{\delta}(n_i'+n_i'')+\sum_{j=1}^{\gamma} m_j\right)+\delta$$

$$= \deg(D')+\delta = d'+\delta,$$ which proves the claim. ∎

Now we'll show that if n is large enough such polynomials G exist. We've seen above that all such polynomials G which satisfy condition (b) form a subspace A of S^n, and condition (b) is equivalent to the coefficients of G satisfying $d'+\delta$ linear equations. Therefore,

$$\dim A \geq \dim S^n - (d'+\delta) = \frac{1}{2}(n+1)(n+2) - d' - \delta.$$

However, the dimension of the set of such G which do not satisfy condition (a) is

$$\dim\{G \in S^n \mid F \mid G\} = \dim S^{n-m} = \frac{1}{2}(n-m+1)(n-m+2).$$

Clearly, for n sufficiently large, $\dim A > S^{n-m}$, and hence there exist $G \in S^n$ which simultaneously satisfy condition (a) and (b).

Fix a G which satisfy conditions (a) and (b), and let $E = G \cdot C - D' - \Delta$. Let

$$S = \{H \in S^n \mid H \cdot C \geq \Delta + E + D''\}.$$

Let us now calculate $\dim S$. By Bezout's Theorem

$\deg(G \cdot C) = \deg(G)\deg(F) = nm$, and thus

$\deg E = nm - d' - 2\delta$. As in our previous discussion of G satisfying condition (b), the condition

$H \cdot C \geq \Delta + E + D''$ is equivalent to the coefficients of H satisfying $(\delta + \deg E + \deg D'')$ linear equations. Moreover,

$$\begin{aligned} \delta + \deg E + \deg D'' &= \delta + (nm - d' - 2\delta) + d'' \\ &= -\delta + nm - d, \end{aligned}$$

hence, $\dim S \geq \frac{1}{2}(n+1)(n+2) + \delta - nm + d$.

If now for every $H \in S$, we let $f_H = \pi^*\left(H/G\big|_{C'}\right)$, then

$$\begin{aligned} (f_H) + D &= (H \cdot C) - (G \cdot C) + D \\ &\geq (\Delta + E + D'') - (E + D' + \Delta) + D = 0. \end{aligned}$$

Therefore, $f_H \in \mathcal{L}(D)$, and we have a mapping

$$\alpha : S \longrightarrow \mathcal{L}(D), \quad H \longmapsto f_H.$$

It is easy to see that α is a vector space homomorphism and induces and injective map

$\alpha : S/_{F\cdot S^{n-m}} \longrightarrow \mathcal{L}(D)$, because $\ker(\alpha) = F\cdot S^{n-m}$.

To see this note that $\alpha(H) = f_H = 0$ if and $H=0$ on C', that is to say $H = F\cdot Q$ where $Q \in S^{n-m}$.

Hence, $\ell(D) \geq \dim S - \dim S^{n-m}$

$$\geq \frac{1}{2}(n+1)(n+2) + \delta - nm + d - \frac{1}{2}(n-m+1)(n-m+2)$$

$$= d - \frac{1}{2}(m^2-3m) + \delta = d - g + 1, \text{ by the genus}$$

formula. ∎

<u>Corollary 4.2.</u> If $d \geq g$, then $\ell(D) \neq 0$. In other words, if $\ell(D) = 0$, then $d \leq g-1$.

<u>Proposition 4.3.</u> Assume the hypothesis of Proposition 4.1. Then $i(D) \geq g-d-1$.

<u>Proof</u>: We'll continue the notation used in the proof of Proposition 4.1.

As in the proof of Prop. 4.1 we may choose a homogeneous polynomial $G(\xi^0, \xi^1, \xi^2)$ of degree n such that $F \nmid G$ and $G\cdot C \geq D''$.

Also let $S_1 = \{ H \in S^{n+m-3} \mid H\cdot C \geq D' + \Delta + (G\cdot C - D'')\}$.

Then $\dim S_1 \geq 1/2\,(n+m-1)(n+m-2) - d'' - \delta - mn + d''$.

For any $H \in S_1$, let $\varphi_H = \pi^*\left(\frac{H(1,x,y)\,dx}{G(1,x,y)\, F_y(1,x,y)} \Big|_{C'} \right)$.

In the proof of Prop. 2.3, we've already proved that for $p \in C'$,

$$\frac{dx}{F_y(1,x,y)}\Big|_p = \infty \text{ and } \frac{dy}{F_x(1,x,y)}\Big|_p = \infty \Rightarrow \pi^{-1}(p) \in \Delta,$$

and therefore

$$(\varphi_H) = H\cdot C - G\cdot C + \pi^{-1}\left(\left(\frac{dx}{F_y(1,x,y)}\Big|_{C'} \right)\right), \text{ so that } \varphi_H \in K'(D).$$

Thus, we obtain a homomorphism $\beta: S_1 \to K'(D)$, $H \mapsto \varphi_H$. Clearly, $\ker\beta = F\cdot S^{n+m-3-m} = F\cdot S^{n-3}$, and hence

$$\tau(D) = \dim K'(D) \geq \dim S_1 - \dim(F\cdot S^{n-3})$$
$$\geq g-d-1. \quad \blacksquare$$

Note that the Riemann inequality and Prop. 1.4. greatly simplified the above proof. Indeed, if $\omega \in K'(C)$, then $\tau(D) = l((\omega)-D) \geq \deg((\omega)-D) - g+1$

$$= 2g-2-d-g+1$$
$$= g-d-1.$$

<u>Corollary 4.4.</u> If $d \leq g-2$ then $\tau(D) \neq 0$. In other words, if $\tau(D)=0$, then $d \geq g-1$.

§ <u>5. The Riemann-Roch Theorem.</u>

<u>Theorem 5.1.</u> (R-R. Theorem)

Suppose C is a compact Riemann surface of genus g and

$$D = \sum_{i=1}^{k} n_i p_i \in \mathrm{Div}(C).$$

Then $l(D) = d-g+\tau(D)+1$, where $d=\deg D$, and $l(D)$ and $\tau(D)$ are as before.

<u>Proof</u> We'll first prove that if $D \geq 0$, then

$$l(D) \leq d-g+\tau(D)+1.$$

Let $\eta = (\eta_1, \dots, \eta_k)$, where $\eta_i = \frac{a_{in_i}}{z_i^{n_i}} + \dots + \frac{a_{i1}}{z_i}$ is a Laurent principal part near p_i. The linear space of all such η is isomorphic to $\mathbb{C}^d$. Regarding η as a vector in $\mathbb{C}^d$ consider the bilinear mapping

$$\mathrm{Res} : \mathbb{C}^d \times \Omega^1(C) \to \mathbb{C}, \quad (\eta, \omega) \mapsto \sum_{i=1}^{k} \mathrm{Res}_{p_i}(\eta_i \omega).$$

It is easy to see that

$(*)$ $\mathrm{Res}(\eta, \omega) = 0 \ \forall \eta \in \mathbb{C}^d \iff \omega \in \Omega^1(D)$:

Indeed, if $\omega \in \Omega^1(D)$ then $\eta_i \omega$ does not have p_i as pole for all $i = 1, 2, \ldots, k$. Thus, $R_{p_i}(\eta_i \omega) = 0$ and hence $\sum_{i=1}^{k} \mathrm{Res}_{p_i}(\eta_i \omega) = 0$. This proves the "$\Leftarrow$" part of $(*)$.

For the other part, assume the contrary: let $\omega \in \Omega^1(C)$ such that $R(\eta, \omega) = 0$, $\forall \eta \in \mathbb{C}^d$ and near p_i $(i = 1, 2, \ldots, k)$ let

$\omega = b_{i0} z_i^{m_i} + \cdots$ such that $b_{i0} \neq 0$ and $m_i < n_i$ for some i.

Then choose a Laurent principal part near p_i $\xi_i = (\xi_{i1}, \ldots, \xi_{ik})$, where $\xi_{ii} = 1/z_i^{m_i+1}$, $\xi_{ij} = 0$ $(j \neq i)$.

Since $m_i + 1 \leq n_i$, our definition of $\mathbb{C}^d$ as Laurent polynomial parts at p_i shows that $\xi_i \in \mathbb{C}^d$. However,

$$0 = \mathrm{Res}(\xi_i, \omega) = \mathrm{Res}_{p_i}\left(\frac{b_{i0}}{z_i}\right) = b_{i0} \neq 0.$$

This contradiction shows that $m_i \geq n_i$ $(i = 1, 2, \ldots, k)$. Hence, $\omega \in \Omega^1(D)$.

For the rest of the proof we'll need the following linear algebra fact:

Let $B : V \times W \to \mathbb{C}$ be a bilinear mapping so that for all $w \in W$, if $B(v, w) = 0$, $\forall v \in V$, then $\omega = 0$. We say in this case that B is nondegenerate with respect to W. Now let there is some subspace $L \subseteq V$ so that $B(L \times W) = 0$. Then $\dim L \leq m - n$, where $m = \dim V$ and $n = \dim W$.

The proof of this fact will be left as an exercise.

Note that the property (*) which we've discussed in the previous page implies that the bilinear form, induced by Res,

$$\mathrm{Res}_1 : \mathbb{C}^d \times \Omega^1(C)/\Omega^1(D) \longrightarrow \mathbb{C}$$

is nondegenerate with respect to $\Omega^1(C)/\Omega^1(D)$.

By the results of §1 we know that $\mathcal{L}(D)/\mathbb{C}$ can be considered as a subspace of $\mathbb{C}^d$, and by the residue theorem

$$\mathrm{Res}_1(\mathcal{L}(D)/\mathbb{C} \times \Omega^1(C)/\Omega^1(D)) = 0.$$

Hence, by the linear algebra fact we mentioned above

$$\dim(\mathcal{L}(D)/\mathbb{C}) \leq \dim \mathbb{C}^d - \dim(\Omega^1(C)/\Omega^1(D))$$

$$\Rightarrow l(D) - 1 \leq d - \Omega^1(C) + i(D).$$

On the other hand, by Proposition 2.1 of §2,

$\dim \Omega^1(C) = g$ and hence, $l(D) \leq d - g + i(D) + 1$, so that our assertion is proved.

The rest of the proved is divided into four cases.

a) $l(D) \neq 0$ and $i(D) = 0$.

$l(D) \neq 0$ implies that there is some $f \in K(C)$ such that $(f) + D \geq 0$. Let $E = (f) + D$, then $(f) = E - D$ i.e. $E \sim D$.

By Prop. 1.3. we have $l(D) = l(E)$, $i(D) = i(E)$ and $\deg D = \deg E$.

Thus, in the proof of R.R. we may replace D by E. In particular, we may assume that $D \geq 0$.

We've proved already the first half: $l(D) \leq d - g + i(D) + 1$.

On the other hand, by the Riemann inequality

$d-g+i(D)+1 = d-g+1 \leq l(D)$ so that the theorem is proven in this case.

b) $l(D) \neq 0$ and $i(D) \neq 0$.

As in (a), since $l(D) \neq 0$ we may assume that $D \geq 0$. Hence, again we have the first half:

$$l(D) \leq d-g+i(D)+1.$$

Also, since $i(D) \neq 0$ there is some $\omega \in \Omega^1(C)$ such that $(\omega) \geq D$. Let $E = (\omega) - D$, then $E \geq 0$. So by Prop 1.4 we get

$i(D) = l(E) \leq \deg E - g + i(E) + 1$. By the Poincaré-Hopf formula $\deg(\omega) = 2g-2$ and hence

$\deg E = \deg(\omega) - \deg D = 2g-2-d$. Finally, since $i(E) = l(D)$ the above line becomes

$i(D) \leq 2g-2-d-g+1+l(D)$ or equivalently,

$$l(D) \geq d-g+i(D)+1.$$

Hence, we are done in this case also.

c) $l(D) = 0$ and $i(D) \neq 0$.

As in Case (b), since $i(D) \neq 0$, there is some $\omega \in \Omega^1(C)$ such that

$$(\omega) = D + E, \text{ where } E \geq 0.$$

By Prop 1.4. $l(E) = i(D) \neq 0$ and $i(E) = l(D) = 0$.

So we can assume Case (a) to E to conclude

$$l(E) = \deg E - g + i(E) + 1.$$

On the other hand, since
$$\deg E = \deg(\omega) - \deg D = 2g-2-d \quad \text{we get}$$

$\bar{\imath}(D) = (2g-2-d) - g + \ell(D) + 1$ and hence
$$\ell(D) = d - g + 1 + \bar{\imath}(D).$$

d) $\ell(D) = 0$ and $\bar{\imath}(D) = 0$.

By Corollary 4.2 and Corollary 4.4 we have
$d = g-1$ or $\ell(D) = 0 = d - g + 1 = d - g + 1 + \bar{\imath}(D)$.

Exercise 5.1. For $C = \mathbb{P}^1$, $D > 0$ give a direct proof of the Riemann Roch Theorem.

Exercise 5.2. For $C = \mathbb{C}/\Lambda$, where $\Lambda = \{m_1\omega_1 + m_2\omega_2 \mid m_i \in \mathbb{Z}, i = 1, 2\}$ and $D \geq 0$ give a direct proof of the Riemann-Roch Theorem.

Exercise 5.3. Show that if $\deg D < 0$, then $\ell(D) = 0$, and if $\deg D > 2g-2$, then $\bar{\imath}(D) = 0$.

Exercise 5.4. Suppose C is a Riemann surface of genus g, $D \in \mathrm{Div}(C)$, and $d = \deg D \geq 2g+1$ (from Exercise 5.3, $\bar{\imath}(D) = 0$, and thus $\ell(D) = \deg D - g + 1 = d - g + 1$).

Suppose $\{f_0, f_1, \dots, f_{d-g}\}$ is a basis for $L(D)$. Define the mapping,
$$\varphi : C \longrightarrow \mathbb{CP}^{d-g}, \quad p \longmapsto [f_0(p), \dots, f_{d-g}(p)].$$

Show that φ is well defined, injective and its derivative $(\varphi_*)_p \neq 0$, for all $p \in C$.

CHAPTER IV. Applications of the Riemann-Roch Theorem

§ The Case of Genus Zero.

Proposition 1.1. Suppose C is a compact Riemann surface of genus 0. Then $C \cong \mathbb{CP}^1$.

Proof: Choose any point p on C and let $D = p \in \mathrm{Div}(C)$. Then, since $K^1(D) = \Omega^1(D) \subseteq \Omega^1(C)$ is a subspace, $0 \leq i(D) = \dim K^1(D) \leq \dim \Omega^1(C) = g = 0$ and hence $i(D) = 0$. Now by the R-R. Theorem

$$\ell(D) = d - g + i(D) + 1 = 1 - 0 + 0 + 2 = 0.$$

So, $\ell(D)$ has dimension two. Since constant functions are in $\mathcal{L}(D)$, $\mathcal{L}(D)$ must have a meromorphic function f which has a pole of order 1 at p. So $f^{-1}(\infty) = p$ and thus the holomorphic mapping $f: C \to \mathbb{CP}^1$ has $\deg f = \#\{f^{-1}(\infty)\} = 1$. In other words, f is a biholomorphic mapping from C to $\mathbb{CP}^1$. This finishes the proof. ∎

§ 2. The Case of Genus 1.

Proposition 2.1. Suppose C is a compact Riemann Surface of genus 1. Then C can be represented by a smooth algebraic curve of degree 3 in $\mathbb{CP}^2$.

Proof: Our aim is to construct a holomorphic injection map $f: C \to \mathbb{CP}^2$ so that its image is a smooth algebraic curve of degree 3.

Let $\omega \in \Omega^1(C)$ be any nontrivial element. Then for any $p \in C$ $\omega(p) \neq 0$. Indeed, if $\omega(p) = 0$ then by the Poincaré-Hopf, we have
$\deg(\omega) = 2g - 2 = 2 \cdot 1 - 2 = 0$ and hence ω would have a pole too. However, this is not possible since ω is holomorphic.

If $D \geq 0$ is effective then $\Omega^1(D) \subseteq \Omega^1(C)$, where $\dim \Omega^1(C) = g = 0$ so that $i(D) = \dim \Omega^1(D) = 0$.

Case 1. We first construct two meromorphic functions x and y on C (which will correspond $\wp$ and $\wp'$, the Weierstrass function and its derivative) which will embedd C into $\mathbb{CP}^2$.

Choose any $p \in C$ and let $D = 2p \in \mathrm{Div}(C)$. Since $2p > 0$ $i(D) = 0$ and thus by R-R Theorem, we have

$$\ell(D) = d - g + i(D) + 1 = 2 - 1 + 0 + 1 = 2.$$

Thus $\mathcal{L}(D)$ contains a meromorphic function which has p as singular point (of order ≤ 2), and clearly this type of meromorphic function must take p as a second-order pole. In fact, by the Riemann-Roch theorem we know that

$\ell(p) = 1 - 1 + 0 + 1 = 1$ and hence, $\mathcal{L}(p)$ has only constant functions. Thus the order of any meromorphic function with pole is at least two.

Suppose $x \in \mathcal{L}(2p)$. Then near the point $p \in C$, we have

$$x = \frac{c_2}{t^2} + \frac{c_1}{t} + h_1(t), \text{ where } h_1(t) \text{ is holomorphic,}$$

t is a local coordinate, $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$. If the coordinate is centered at p then we must have $c_1 = 0$: To see this, first note that $\dim \Omega^1(C) = g = 1$ and let $\omega \in \Omega^1(C)$ be a nontrivial element. Let us write ω as $\omega = g(t)\, dt$. By the arguments at the beginning of the proof $g(0) \neq 0$. Finally, since the meromorphic differential $x\omega$ has no poles other than p, by the residue theorem $0 = \mathrm{Res}_p(x\omega) = c_1$.

So, after a change of coordinates near p, we may assume that $x = 1/z^2 + h(z)$.

Fix some $\omega \in \Omega^1(C)$ and let $y = dx/\omega$. Define the mapping

$$f : C \to \mathbb{CP}^2, \quad f(q) = [1, x(q), y(q)],$$

$p \mapsto [0, 0, 1]$. Then, (clearing the denominators) we see that f is a holomorphic mapping.

<u>Step 2</u>. Now we'll see that $f(C)$ lies in $\mathbb{CP}^2$ as a curve of degree 3.

Let us consider x as a holomorphic mapping from C to $\mathbb{P}^1$. Then $x^{-1}(\infty)=2p$. Therefore $\deg x = 2$ and hence the ramification index of x at its ramification points never exceeds 2. Suppose R is the ramification divisor of x. Then the coefficient of every point in R must be 1. By the Riemann-Hurwitz formula,

$$\deg R = 2(\deg x + g - 1) = 2(2+1-1) = 4.$$

Hence, in addition to the point p, x has three more ramification points, say $R = p_1 + p_2 + p_3 + p$, where they are all distinct points. Suppose $x(p_i) = a_i$, $i = 1,2,3$.

For a meromorphic function or meromorphic differential α, let $(\alpha)_0$ and $(\alpha)_\infty$ denote, respectively, its set of zeros and poles with multiplicity. Since ω has neither zeros or poles and $y = dx/\omega$, we see that

$$(y)_0 = (dx)_0 \quad \text{and} \quad (y)_\infty = (dx)_\infty.$$

Hence, (y_0) and $(y)_\infty$ can only contain the points of x with ramification index > 1, i.e., p_1, p_2, p_3, p. Now let's compute $(y)_0$ and $(y)_\infty$.

Suppose z_i $(i=1,2,3)$ be a local coordinate near p_i, with $z_i(p_i) = 0$. Since the multiplicity of x at p_i is 2, we see that

$x - a_i = z_i^2 h_i(z_i)$, where h_i is a holomorphic function and $h_i(0) \neq 0$. Thus

$$dx = 2z_i h_i(z_i)\,dz_i + z_i^2 h_i'(z_i)\,dz_i = z_i\left(2h_i(z_i) + z_i h_i'(z_i)\right)dz_i$$

and $\left.\left(2h_i(z_i) + z_i h_i'(z_i)\right)\right|_{z_i=0} = 2h_i(0) \neq 0$, p_i is a simple zero of dx.

Choosing a local coordinate function z near p as before, we have

$x = 1/z^2 + h(z)$. So, $dx = \frac{-2}{z^3} + h'(z)$ and thus

dx has a pole of order 3 at p. Hence, we get

$(y)_0 = p_1 + p_2 + p_3$ and $(y)_\infty = 3p$ and thus

$$(y) = p_1 + p_2 + p_3 - 3p.$$

Consider the meromorphic function on $\mathbb{P}^1$ given by
$$g(x) = (x-a_1)(x-a_2)(x-a_3).$$

Consider g as a function on C i.e. $(x^* g = g \circ x)$, then

$$(g)_0 = 2p_1 + 2p_2 + 2p_3 \text{ and } (g)_\infty = 6p.$$

This is because near each p_i $(i=1,2,3)$, $x - a_i = z_i^2 h_i(z_i)$ so that g takes p_i as a zero of order 2; and near the point p, $x = 1/z^2 + h(z)$ and $x - a_i$ takes p as a pole of order 2.

In particular, the divisor of g is equal to that of y^2 and thus the divisor $(y^2/g) = 0$. In other words, y^2/g is a holomorphic function on C. Since C is compact this is possible only if $y^2/g = a \neq 0$ is a constant map.

Replacing y with $y/\sqrt{a}$ we obtain $y^2 = g(x)$. Hence, we've shown that $f(C)$, the image of C under f lies on a curve of degree 3:

$$C' \doteq \{[1,x,y] \in \mathbb{P}^2\mathbb{C} \mid y^2 - g(x) = 0\} \cup \{[0,0,1]\}.$$

<u>Step 3.</u> We'll prove now that f is injective.

Consider the involution $J : C \to C$, $q_1 \mapsto q_2$, where $x(q_1) = x(q_2)$. Since x is a local biholomorphic map on C, except at finitely many points, J is a holomorphic map. Note that $J(p_i) = p_i$, $i = 1,2,3$.

<u>Exercise 2.1.</u> Prove the above claim that $J \in \mathrm{Aut}(C)$.

J induces homomorphisms, which we all denote by J^* on $K(C)$, $K^1(C)$ and $\Omega^1(C)$: $J^* a = a \circ J$.

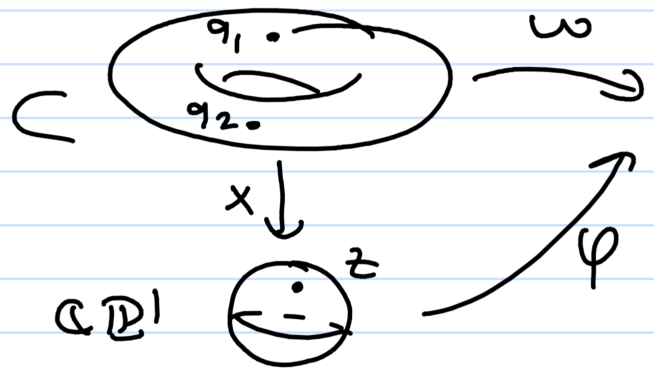
Clearly, $J^* x = x$, $J^*(dx) = dx$. For $y = dx/\omega$, we have

$$J^* y = \frac{J^*(dx)}{J^*\omega} = \frac{dx}{J^*\omega}.$$ Now since $\dim \Omega^1(C) = g = 1$

we have $\Omega^1(C) \cong \mathbb{C}$. Also $J^2 = 1$ and hence $J^{*2} = 1$ and hence $J^* = 1$ or $J^* = -1$ on $\mathbb{C}$.

If $J^* = 1$ then $J^*\omega = \omega$.

<u>Exercise 2.2.</u> If $J^*\omega = \omega$ there is a holomorphic 1-form φ on $\mathbb{CP}^1$ so that $\omega = \varphi \circ x = x^*\varphi$.



$\varphi(z) \doteq \omega(q_1) = \omega(J(q_1)) = \omega(q_2)$

$x(q_1) = z$

Now, this clear contradiction shows that we cannot have $J^*\omega = \omega$. Hence, $J^*\omega = -\omega$ so that

$$J^* y = -y.$$

Now suppose $q_1, q_2 \in C$ are such that $f(q_1) = f(q_2)$. If $q_i = p$ for some i, then $f(q_1) = f(q_2) = \infty$ and $q_1 = q_2 = p$. Thus we may assume that $q_1 \neq p \neq q_2$.

Then $f(q_i) = [1, x(q_i), y(q_i)]$, $i = 1, 2$. By assumption

$x(q_1) = x(q_2)$ and $y(q_1) = y(q_2)$. If $q_1 \neq q_2$ then $x(q_1) = x(q_2) \Rightarrow q_2 = J(q_1)$ so that $y(q_1) = y(q_2) = y(J(q_1)) = -y(q_1)$.

Hence, $y(q_1) = 0$. But $(y)_0 = p_1 + p_2 + p_3$ so that $q_1 = p_1$ or p_2 or p_3. Hence, $q_2 = J(q_1) = q_1$, which contradicts to our assumption that $q_1 \neq q_2$.
Thus, f must be injective.

Step 4. $f(C)=C'$, where C' is the degree 3 curve in Step 2.

The equation $y^2-g(x)$ of C' is irreducible and thus C' is connected. Since C is compact $f(C)$ is a closed subset of C'. However, f is a holomorphic map and thus $f(C)$ is an open subset of C'. Hence, the closed and open subset $f(C)\neq\emptyset$ of C' is equal to C'.

Step 5. We prove C' is smooth by direct computation.

The projective equation for C' in $[z_0,z_1,z_2]$ is

$$F=z_0z_2^2-(z_1-a_1z_0)(z_1-a_2z_0)(z_1-a_3z_0)=0.$$

$$\partial F/\partial z_0=z_2^2+a_1(z_1-a_2z_0)(z_1-a_3z_0)+a_2(z_1-a_1z_0)(z_1-a_3z_0)$$
$$+a_3(z_1-a_1z_0)(z_1-a_2z_0).$$

$$\partial F/\partial z_1=-(z_1-a_2z_0)(z_1-a_3z_0)-(z_1-a_1z_0)(z_1-a_3z_0)$$
$$-(z_1-a_1z_0)(z_1-a_2z_0).$$

$$\partial F/\partial z_2=2z_0z_2.$$ (Recall that $\partial F/\partial z_i=0, i=1,2,3\Rightarrow F=0$).

Now $2z_0z_2=0\Rightarrow z_0z_2^2=0$ so that since $F=0$ also we get

$$(z_1-a_1z_0)(z_1-a_2z_0)(z_1-a_3z_0)=0.$$

So one of the factors above must be zero. On the other hand if only one of them is zero then $\partial F/\partial z_1\neq 0$, which is a contradiction. Hence, at least two of them must be zero, say w.l.o.g. $z_1=a_1z_0$ and $z_1=a_2z_0$. Since a_i's are all distinct we must have then $z_0=z_1=0$. Plug these in $\partial F/\partial z_0=0$ we obtain $z_2=0$.

This contradiction finishes the proof. ∎

§3. Canonical Maps.

Definition 3.1. Suppose C is a compact Riemann surface of genus $g \geq 2$ and $\{\omega_1, \dots, \omega_g\}$ is a basis for $\Omega^1(C)$. Then the mapping
$$\varphi_K : C \longrightarrow \mathbb{P}^{g-1}, \; p \longmapsto [\omega_1(p), \dots, \omega_g(p)],$$
is called the canonical mapping.

Remark 3.2. Note that if z is a local coordinate around a point $p \in C$ then we may write $\omega_\alpha = f_\alpha(z)\,dz$, $\alpha = 1, \dots, g$, so that
$$\varphi_K(p) = [f_1(z(p)), \dots, f_g(z(p))].$$

Exercise 3.1. Prove that φ_K is independent of the choice of local coordinate z near p.

Remark 3.3. $\varphi_K(p)$ is well defined because $\omega_1(p), \dots, \omega_g(p)$ are not all zero. To prove this assume on the contrary that $\omega_\alpha(p) = 0$ $\forall \alpha$ at some $p \in C$. Let $D = p \in \mathrm{Div}(C)$. Then $\Omega^1(D) = \Omega^1(C)$ and hence $i(D) = \dim \Omega^1(C) = g$. So by the R-R. Theorem $\ell(p) = 1 - g + g + 1 = 2$, which implies that there is a nonconstant holomorphic map $f \in \mathcal{L}(p)$, with $f^{-1}(\infty) = p$. Therefore, $\deg f = 1$ so that the map $f : C \longrightarrow \mathbb{P}^1$ is an isomorphism, which is a contradiction. This finishes the proof of the assertion.

Proposition 3.4. φ_K is non degenerate.

Proof: $\{\omega_1, \dots, \omega_g\}$ is a basis for $\Omega^1(C)$ and this finishes the proof. ∎

Proposition 3.5. If $\varphi_K(p) = \varphi_K(q)$, where $p, q \in C$, $p \neq q$, then there exists a degree 2 mapping from C to $\mathbb{P}^1$.

Proof: If $\varphi_K(p) = \varphi_K(q)$ then there is some $\lambda \neq 0$ such that $\omega_\alpha(p) = \lambda\, \omega_\alpha(q)$ for all $\alpha = 1, \dots, g$. Choose $D = p + q \in \mathrm{Div}(C)$. Let's compute $i(D)$.

First we claim $\Omega^1(D) = \Omega^1(p)$: let $\omega \in \Omega^1(C)$. Then $\omega = \sum \mu_\alpha \omega_\alpha$, $\mu_\alpha \in \mathbb{C}$. Then $\omega(p) = 0 \Leftrightarrow \sum_\alpha \mu_\alpha \omega_\alpha(p) = 0$ $\Leftrightarrow \sum_\alpha \mu_\alpha \lambda \omega_\alpha(q) = 0 \Leftrightarrow \omega(q) = 0$.

So, $\Omega^1(p) = \Omega^1(p+q) = \Omega^1(D)$ and thus $i(D) = i(p)$. Let's compute $i(p)$: Consider the equation below in the variables $\lambda_1 \omega_1(p) + \cdots + \lambda_g \omega_g(p) = 0$.

From Remark 3.3 we know that not all $\omega_\alpha(p) = 0$. Hence, this equation has $g-1$ linearly independent solutions in $\mathbb{C}^g$, which gives $(g-1)$ linearly independent elements in $\Omega^1(p)$. Thus, we have

$i(D) = i(p) = g-1$. Now by the R.R. theorem for $D = p+q$ gives

$$\ell(D) = 2 - g + (g-1) + 1 = 2,$$

and hence there is a non constant meromorphic function $f \in L(p+q)$. This gives a holomorphic mapping $f: C \to \mathbb{P}^1$ with $f^{-1}(\infty) = p+q$ (it cannot be p or q, since in that case $f: C \to \mathbb{P}^1$ would be a biholomorphic map). Hence, $\deg f = 2$. ∎

By the definition below we divide the set of all compact Riemann surfaces into two classes.

<u>Definition 3.6.</u> A compact Riemann surface C of genus $g > 1$ is called hyperelliptic if there is a degree two holomorphic mapping from C to $\mathbb{P}^1$. Otherwise, it is called nonhyperelliptic.

<u>Definition 3.7.</u> Suppose C is a nonhyperelliptic compact Riemann surface of genus $g \geq 2$, and $\varphi_K : C \to \mathbb{P}^1$ is its canonical map. Then $\varphi_K(C) \subseteq \mathbb{P}^{g-1}$ is called a canonical curve.

We'll prove in section 5 that any compact Riemann surface of genus 2 is hyperelliptic. Hence, we may assume in the definition above that $g \geq 3$.

Proposition 3.8. If C is a non hyperelliptic then φ_K is injective and $\varphi_K(C)$ is smooth.

Proof: Proposition 3.5 already showed that φ_K is injective. To show that $\varphi_K(C)$ is smooth it is enough to show that the derivative map $(\varphi_K)_*$ is never zero.

Let $p \in C$. In the proof of Prop. 3.5. we've seen that $\dim \Omega^1(p) = g-1$.

Claim: $\dim \Omega^1(2p) = g-2$.

Proof: Let z be a local coordinate on C centered at p, i.e, $z(p)=0$. Then near p, $\omega_\alpha(p) = f_\alpha(z)dz$, $\alpha = 1, \dots, g$, where $\{\omega_1, \dots, \omega_g\}$ is a basis for $\Omega^1(C)$. For any $\omega \in \Omega^1(2p)$, suppose

$$\omega = \sum \lambda_\alpha \omega_\alpha, \quad \lambda_\alpha \in \mathbb{C}. \text{ Then}$$

we have

$$(*) \quad \begin{cases} \sum_{\alpha=1}^{g} \lambda_\alpha f_\alpha(0) = 0 \\ \sum_{\alpha=1}^{g} \lambda_\alpha f_\alpha'(0) = 0. \end{cases}$$

To finish the proof we need to show that these two equations are linearly independent. Suppose not, then $\Omega^1(p) = \Omega^1(2p)$. Then by the R.R. Theorem

$\ell(2p) = 2-g+(g-1)+1 = 2$ and hence there is a nonconstant meromorphic function f with second order pole p in $\mathcal{L}(2p)$ (as before, f cannot p as a first order pole, because then $f: C \to \mathbb{P}^1$ would be an isomorphism).

Now $f: C \to \mathbb{P}^1$ has degree two. However, this is a contradiction since C is nonhyperelliptic.

Thus, we must have $\dim \Omega^1(2p) < \dim \Omega^1(p)$. Now, by $(*)$ $\dim(\Omega^1(C)/\Omega^1(p)) = 1$ and $\dim(\Omega^1(p)/\Omega^1(2p)) = 1$.

In particular, $\dim \Omega^1(2p) = g-2$.

So we can choose a basis $\omega_1, \ldots, \omega_g$ of $\Omega^1(C)$ such that near the point p, we have

$$\omega_1 = h_1(z)\,dz, \quad \omega_2 = z h_2(z)\,dz, \quad \omega_\alpha = z^2 h_\alpha(z)\,dz, \quad \alpha = 3, \ldots, g,$$

where the h_α are all holomorphic and $h_1(0) \neq 0 \neq h_2(0)$.

Clearly, we may choose $h_1(z) = 1$ so that near p,

$$\varphi_K(z) = [1, z h_2(z), z^2 h_3(z), \ldots, z^2 h_g(z)].$$

Since $\left.\dfrac{d(z h_2(z))}{dz}\right|_{z=0} = h_2(0) \neq 0$, $(\varphi_K)_* \neq 0$ at p.

However, $p \in C$ was arbitrary. Thus $(\varphi_K)_* \neq 0$ at all points. This finishes the proof. ∎

§4. Hyperelliptic Compact Riemann Surfaces.

We'll see that hyperelliptic Riemann surfaces are similar to elliptic ones in many respects.

__Proposition 4.1.__ Any hyperelliptic Riemann surface can be represented as the normalization of a plane algebraic curve of degree $(2g+2)$, where g is the genus of this Riemann surface. By a suitable linear projections this plane curve will be a two sheeted covering of a straight line.

__Proof:__ Our aim is to construct, as in $g=1$ case, a holomorphic mapping $f: C \to \mathbb{CP}^2$ such that $f(C)$ is a very special type of algebraic curve of degree $2g+2$.

Since C is given to be hyperelliptic there is a degree two map $x: C \to \mathbb{P}^1$. Let R be its ramification divisor. By the Riemann-Hurwitz formula

$$\deg R = 2(2+g-1) = 2g+2.$$

However, $\deg x = 2$ and thus the coefficients of every point in R must be as

$$R = p_1 + p_2 + \cdots + p_{2g+2}, \text{ for some points } p_i \in C,$$

$i = 1, \ldots, 2g+2$.

Now let $x^{-1}(\infty) = p + q$ and $x(p_i) = a_i$ $(i = 1, \ldots, 2g+2)$.

As in case $g=1$, define an involution $J : C \to C$, $J(q_1) = q_2$, where $x(q_1) = x(q_2)$. As before $J \in Aut(C)$ and $J^2 = 1$. Again J^* on $\Omega^1(C) \cong \mathbb{C}^g$ is an involution with eigenvalues ± 1. Since $+1$ eigenvectors give holomorphic differentials on $\mathbb{P}^1$, which is impossible, every efunc of J^* is -1. In other words,

$$J^* : \Omega^1(C) \to \Omega^1(C), \quad J^*(\omega) = -\omega.$$

Let $D = (g+1)p + (g+1)q \in Div(C)$, when $x^{-1}(\infty) = p + q$.

For any $\omega \in K^1(C)$, by the Poincaré-Hopf theorem

$\deg(\omega) = 2g - 2 < \deg D$. Thus $(\omega) - D \geq 0$ is not possible and so $i(D) = 0$. From the R.-R. Theorem

$$\ell(D) = (2g+2) - g + 1 = g + 3 \text{ and hence, } \mathcal{L}(D) \cong \mathbb{C}^{g+3}.$$

Since $x(p) = x(q) = \infty$, $J(D) = D$ so that J^* can be also regarded as a linear transformation on $\mathcal{L}(D)$. $J^{*2} = 1$ as usual and hence it has ± 1 eigenspaces on $\mathcal{L}(D)$. So we may write

$$\mathcal{L}(D) = \mathcal{L}^+(D) \oplus \mathcal{L}^-(D).$$

Note that any meromorphic function in $\mathcal{L}(D)^+$ can be factored as the composition of x and a meromorphic function on $\mathbb{P}^1$ (since it belongs to $+1$ eigenspace) with poles only on p and q, and the order of pole at each point not exceeding $g+1$. Thus, regarded as a meromorphic function on C, $1, x, x^2, \ldots, x^{g+1}$ for a basis for $\mathcal{L}^+(D)$.

Hence, $\dim \mathcal{L}^+(D) = g + 2$.

However, $\ell(D) = g + 3$ and thus $\dim$ $\dim \mathcal{L}^-(D) = 1$.

So there is some $y \in \bar{L}(D) \subseteq L(D)$ with $J^* y = -y$.

<u>Claim:</u> $y^2 = c\,g(x)$, for some $c \in \mathbb{C}^*$ and
$g(x) = (x-a_1)\cdots(x-a_{2g+2})$.

To prove the claim we just need to compare the divisors (y^2) and $(g(x))$.

Since $J^*(y) = -y$ and $J(p_i) = p_i$ ($i = 1, \ldots, 2g+2$), we see that $y(p_i) = -y(p_i)$ so that $y(p_i) = 0$.

Therefore, $\deg(y)_0 \geq \deg \sum_{i=1}^{2g+2} p_i = 2g+2$.

However, $\deg(y)_0 = \deg(y)_\infty \leq \deg D = 2g+2$ and hence $\deg(y)_0 = \deg(y)_\infty = 2g+2$. So, $(y)_0 = \sum_{i=1}^{2g+2} p_i$ and $(y)_\infty = D$.

Let's now compute $(g(x))$. By our assumptions $(x)_\infty = p+q$ and $(x-a_i)_\infty = p+q$ for all $i = 1, \ldots, 2g+2$.

So $(g(x)) = (2g+2)p + (2g+2)q = 2D$. Let z_i be a local coordinate near p_i with $z(p_i) = 0$. Then

$x - a_i = z_i^2 h_i(z_i)$, where h_i is holomorphic and $h_i(0) \neq 0$. Thus,

$$(g(x))_0 = \sum_{i=1}^{2g+2} 2p_i = 2 \sum_{i=1}^{2g+2} p_i.$$

Hence, the results,

$(y)_0 = \sum_{i=1}^{2g+2} p_i$, $(y)_\infty = D$, $(g(x))_\infty = 2D$ and $(g(x))_0 = 2\sum_{i=1}^{2g+2} p_i$

imply $(y^2) = (g(x))$. So $y^2/g(x)$ is a holomorphic function on C and thus it is a constant, say C.

$y^2/g(x) = C$. Replace y with y/c, $c = \sqrt{C}$ so that $y^2 = g(x)$.

Now define the mapping

$f: C \longrightarrow \mathbb{CP}^2$; $t \mapsto [1, x(t), y(t)]$, $p, q \mapsto [0,0,1]$.

Also let $C' = \{y^2 - g(x) = 0\} \cup \{[0,0,1]\}$.

As in the case $g=1$, we see that f is injective on $C \setminus \{p, q\}$ and $f(C) = C'$.

It is not difficult to see that a_i's are all distinct and consequently C' is smooth in $\mathbb{C}^2$, but singular at $[0,0,1]$. Linear projection from this point to the x-axis, $(x,y) \mapsto x$, realizes $C' \to \mathbb{P}^1$ as a 2-sheeted covering. ∎

<u>Remark 4.2.</u> As a converse statement, given any $(2g+2)$ mutually distinct complex numbers $a_1, \ldots, a_{2g+2}$,

$$C' = \{y^2 - \prod_{i=1}^{2g+2} (x - a_i) = 0\} \cup \{[0,0,1]\} \subseteq \mathbb{P}^2$$ has normalization C, a compact Riemann surface of genus g.

In the next proposition we'll produce an explicit basis for the canonical (bundle) map for Riemann surfaces of genus $g \geq 2$.

<u>Proposition 4.3.</u> $\omega_1 = dx/y$, $\omega_2 = (x\,dx)/y, \ldots, \omega_g = (x^{g-1}dx)/y$ form a basis of $\Omega^1(C)$, where the precise meaning of C, x and y are given in Prop 4.1 and its proof.

<u>Proof</u>: By their definition the forms are clearly linearly independent. So, we just need to show that they are holomorphic. To see this consider their divisors (ω_α), $\alpha = 1, \ldots, g$,

$$(\omega_\alpha) = (x^{\alpha-1}) + (dx) - (y).$$

<u>$(x^{\alpha-1})$</u>: Suppose that $x^{-1}(0) = p' + q' = D'$, where the case $p' = q'$ is <u>not</u> excluded. Then

$$(x^{\alpha-1})_0 = (\alpha-1)D'.$$

Let $D^* = p+q$, then $(x^{\alpha-1})_\infty = (\alpha-1)D^*$. So

$$(x^{\alpha-1}) = (\alpha-1)D' - (\alpha-1)D^*.$$

(dx): Let z_i be a local coordinate near p_i ($i=1,\dots,2g+2$) with $z_i(p_i)=0$. Then $x-a_i = z_i^2 h_i(z_i)$, where h_i is holomorphic and $h_i(0)\neq 0$. Thus

$$\begin{aligned} dx &= 2z_i h_i(z_i)dz_i + z_i^2 h_i'(z_i)dz_i \\ &= z_i\,(2h_i(z_i)+z_i h_i'(z_i))\,dz_i. \end{aligned}$$

Thus, dx has p_i as a zero of order 1.

Now take a coordinate near p so that $z(p)=0$. Near the point $\infty\in\mathbb{P}^1$ choose the usual local coordinate $u = 1/x$. Then we have

$u = z h(z)$, where $h(z)$ is holomorphic and $h(0)\neq 0$. So

$$dx = -\frac{du}{u^2} = -\frac{(h(z)+zh'(z))dz}{z^2 h^2(z)}.$$

Since $(h(z)+zh'(z))|_{z=0}\neq 0$ and $h^2(z)|_{z=0}\neq 0$, dx has p as a pole of order two.

The case of the point q is the same as that of p, and so

$$(dx) = \sum_{i=1}^{2g+2} p_i - 2D^*.$$

$(y) = \sum_{i=1}^{2g+2} p_i - 2D^*$ was computed in the proof of Prop. 4.1.

Now combining all these we conclude that

$$\begin{aligned} (\omega_\alpha) &= (\alpha-1)D' - (\alpha-1)D^* + \Big(\sum_{i=1}^{2g+2} p_i - 2D^*\Big) - \Big(\sum_{i=1}^{2g+2} p_i - (g-1)D^*\Big) \\ &= (\alpha-1)D' + (g-\alpha)D^*. \end{aligned}$$

Since $1\le\alpha\le g$, $\alpha-1\geq 0$ and $g-\alpha\geq 0$. Hence, all ω_α's are holomorphic. This finishes the proof. ■

Remark: This basis makes the canonical map of C has the expression $\varphi_K : C\to\mathbb{P}^{g-1}$, $t\mapsto[1, x(t),\dots, x(t)^{g-1}]$.

Moreover, φ_K has a factorization as shown below

$$\begin{tikzcd} C \arrow[r, "\varphi_K"] \arrow[d, "x"'] & \mathbb{P}^{g-1} \\ \mathbb{P}^1 \arrow[ur, "r"'] & \end{tikzcd}$$

$r: \mathbb{P}^1 \to \mathbb{P}^{g-1}$
$x \longmapsto [1, x, \ldots, x^{g-1}]$, which is a rational curve.

§5. The Case of genus 2.

Proposition 5.1. All compact Riemann surfaces of genus 2 are hyperelliptic.

Proof: Suppose C is a compact Riemann surface of genus 2. According to Proposition 3.5, it suffices to prove that the canonical map φ_K on C is not injective. In fact, if $\varphi_K: C \to \mathbb{P}^1$ is injective, then C has genus zero. ∎

Using the notation of the previous sections, let x represent a holomorphic mapping of degree 2 from C to $\mathbb{P}^1$, we get

$$y^2 = \prod_{i=1}^{6} (x - a_i), \text{ where } a_1, \ldots, a_6 \text{ are}$$

mutually distinct complex numbers; since $\omega_1 = dx/y$, $\omega_2 = x\,dx/dy$, we then have

$$\varphi_K: C \to \mathbb{P}^1, \ t \mapsto [\omega_1, \omega_2] = [1, x(t)].$$

§6. The Case of genus 3.

Definition 6.1. Suppose C is a smooth algebraic curve in $\mathbb{P}^n$, H is a hyperplane in $\mathbb{P}^n$ defined by the equation

$$L(\xi) = \sum_{i=0}^{n} l_i \xi^i = 0, \text{ and } H \text{ does not contain any}$$

irreducible branches of C. Then the hyperplane section of C is the element of $\mathrm{Div}(C)$ obtained through the following procedure; choose a generic linear function

$$L'(\xi) = \sum_{i=0}^{n} l_i' \xi^i \ \ (\text{In other words, a linear function,}$$

where $\{L' = 0\} \cap C \cap \{L = 0\} = \phi$.)

Then $L/L' \in K(\mathbb{P}^n)$, $L/L'\big|_C \in K(C)$.

The hyperspace section of C (denoted $H \cdot C$), is then just all the zeros of $L/L'\big|_C$ on C (counting multiplicity);

that is $H\cdot C = \left(L/L'|_C\right)_0$, or $H\cdot C = C\cap (L(\xi)=0)$.

Note that $H\cdot C$ is independent of the choice of L'.

<u>Definition</u>: Suppose C is a smooth curve in $\mathbb{P}^n$. Then we call $\deg(H\cdot C)$ the degree of C, and denote it by $\deg C$.

<u>Remark</u>: This is a well-defined notion, i.e., independent of H. For, if $H_1 = \{L_1(\xi)=0\}$, then

$$\deg(H_1\cdot C) = \deg\left(L_1/L'|_C\right)_0 = \deg\left(L_1/L'|_C\right)_\infty = \deg\left(L/L'|_C\right)_\infty$$
$$= \deg\left(L/L'|_C\right)_0 = \deg(H\cdot C).$$

<u>Remark 6.4.</u> If $n=2$ then H is just a line in $\mathbb{P}^2$ and thus $H\cdot C$ is independent of H by the Bezout's Theorem.

For any nonhyperelliptic compact Riemann surface of genus 1, the canonical map is injective and thus we'll identify the Riemann surface with its image, which is a smooth curve in $\mathbb{P}^{g-1}$.

<u>Proposition 6.5.</u> Suppose C is a canonical curve of genus g. Then $\deg C = 2g-2$.

<u>Proof</u>: Suppose $\omega_1, \omega_2, \dots, \omega_g$ form a basis of $\Omega^1(C)$. Then

$C = \{[\omega_1(p), \dots, \omega_g(p)] \mid p\in C\}$, and suppose

$$H = \left\{\sum_{i=0}^{g-1} l_i\xi^i = 0\right\},\quad H' = \left\{\sum_{i=0}^{g-1} l_i'\xi^i = 0\right\}.$$

Now by the definition of hyperplane section, we have

$$H\cdot C = \left\{p\in C \,\middle|\, \frac{\sum l_i\xi^i}{\sum l_i'\xi^i}\Big|_p = 0\right\} = \left\{p\in C \,\middle|\, \frac{\sum l_i\omega_i(p)}{\sum l_i'\omega_i(p)} = 0\right\} = (\omega)_0,$$

where $\omega = \sum_{i=0}^{g-1} l_i\omega_i$. Since $\omega\in\Omega^1(C)$, $(\omega)_0 = (\omega)$, and consequently, $H\cdot C = (\omega)$. Thus

$\deg C = \deg(H\cdot C) = \deg(\omega) = 2g-2.$ ∎

Now we can state and prove

Proposition 6.6. A canonical curve of genus 3 is a smooth plane algebraic curve of degree 4.

Proof: By definition, a canonical curve of genus 3 must be in $\mathbb{P}^{g-1} = \mathbb{P}^{3-1} = \mathbb{P}^2$. In Prop. 3.8 we have seen that this curve is smooth and hence, by the previous result, we have
$$\deg C = 2g-2 = 2\times 3 - 2 = 4.$$

Remark 6.7. We claim that almost all compact Riemann surfaces of genus 3 are canonical curves. We'll argue this below:

First note that almost all algebraic curves defined by degree four homogeneous equations $F(\xi^0, \xi^1, \xi^2) = 0$ are smooth. Indeed, the coefficients of a fourth-degree homogeneous equation in ξ^0, ξ^1, ξ^2 form a complex projective space, whose dimension is $\binom{4+3-1}{3-2} - 1 = 14$ (-1 comes from the fact that $f=0$ and $\lambda f = 0$ defines the same curve).

Such a curve is singular if and only if the system below has non trivial solutions:

$\partial F/\partial\xi^0 = 0$, $\partial F/\partial \xi^1 = 0$, $\partial F/\partial\xi^2 = 0$. The coefficients of these F's must satisfy a non-trivial set of algebraic equations. These coefficients form a proper subvariety of $\mathbb{P}^{14}$, and hence we have proved our assertion.

Next, since the automorphism group of $\mathbb{P}^2$, $PGL(3,\mathbb{C})$ has dimension 8, we may say that the number of projectively inequivalent smooth fourth degree plane algebraic curves is $\infty^{14-8} = \infty^6$.

Moreover, by the genus formula $g = \frac{1}{2}(d-1)(d-2) - \delta = 3$. Finally, among the ∞^6 compact Riemann surfaces of genus 3, there are only $\infty^{2g-1} = \infty^{2\cdot 3-1} = \infty^5$ hyperelliptic compact Riemann surfaces whereas the number of canonical curves of genus 3 is ∞^6. In other words, almost all compact Riemann surfaces of genus 3 are canonical curves.

§7. The Case of genus 4:

First we need to give a general discussion: Let C be a smooth algebraic curve in $\mathbb{P}^n$, and $H = \{\xi^0 = 0\}$. Let $D = H\cdot C \in \mathrm{Div}(C)$.

Both $\bigoplus_{k=0} \mathcal{L}(kD)$ and $\mathbb{C}[\xi^0, \xi^1, \ldots, \xi^n]$ are both graded rings. Let $F(\xi)$ be a homogeneous polynomial of degree k in $\mathbb{C}[\xi^0, \xi^1, \ldots, \xi^n]$. Define

$$r(F(\xi)) = F(\xi)/(\xi^0)^k\big|_C \in \mathcal{L}(kD)$$

Extend r to all of $\mathbb{C}[\xi^0, \ldots, \xi^n]$ linearly:

$r: \mathbb{C}[\xi^0, \xi^1, \ldots, \xi^n] \to \bigoplus_{k=0}^{\infty} \mathcal{L}(kD)$. It is a theorem of Noether that if C is a canonical curve then r is a surjective mapping.

Suppose now that C is a canonical curve of genus g and S^k denote the vector space of homogeneous polynomials of degree k in $\mathbb{C}[\xi^0, \xi^1, \ldots, \xi^n]$. Set

$$r_k = r|_{S^k} : S^k \to \mathcal{L}\{kD\},$$ which is a vector space homomorphism.

Then we have, $r_k(F(\xi)) = \dfrac{F(\xi)}{(\xi^0)^k}\Big|_C = \dfrac{F(\omega_1, \ldots, \omega_g)}{\omega_1^k}$, where

$\omega_1, \ldots, \omega_g$ form a basis of $\Omega^1(C)$ and $F(\omega_1, \ldots, \omega_g)$ means

$$F(\omega_1, \ldots, \omega_g) = \sum_{i_1+\cdots+i_g=k} a_{i_1\ldots i_g}\, \omega_1^{i_1}\cdots\omega_g^{i_g}.$$

So $$r_k(F(\xi)) = \sum_{i_1+\cdots+i_g=k} a_{i_1\ldots i_g} \left(\frac{\omega_2}{\omega_1}\right)^{i_2}\cdots\left(\frac{\omega_g}{\omega_1}\right)^{i_g}.$$

Thus $r_k(F(\xi)) \in \mathcal{L}(kD)$, where $D = (\omega_1)$.

Let $I_k = \ker r_k$, then $\dim I_k = \dim S^k - \dim \mathcal{L}(kD)$.

Claim: $\ell(D) = g$ and $\ell(kD) = (2k-1)(g-1)$, $k \geq 2$.

Proof: First note that $i(D) = 1$. This is because if $\omega \in \Omega^1(D)$ and $\omega \neq 0$, then $(\omega) \geq D$. Also, since $(\omega) = D = (\omega_1)$ so that ω/ω_1 is a holomorphic function on C and thus is a constant function. Hence, ω and ω_1 are linearly dependent.

When $k \geq 2$, $\ell(kD) = 0$ because $\deg kD = k(2g-2)$. However, the degree of the divisor of any holomorphic differential is $2g-2$. Thus when $k \geq 2$, there are no holomorphic differentials in $\Omega^1(kD)$.

Now applying R-R to both cases finishes the proof of the claim. ■

Note that $\dim S^k = \binom{k+g-1}{g-1}$. Hence from the computation of the previous page that

$\dim I_k = \dim S^k - \dim \mathcal{L}(kD) = \dim S^k - \ell(kD)$ we see that

$$\dim I_k = \binom{k+g-1}{g-1} - (2k-1)(g-\delta), \text{ where } \delta = \begin{cases} 0, & k=1 \\ 1, & k>1. \end{cases}$$

Now, we can prove the following

Proposition 7.1. A canonical curve of genus 4 is the intersection of a quadric and a cubic hypersurface in $\mathbb{P}^3$.

Proof: Suppose C is a canonical curve of genus $g = 4$. Then $C \subseteq \mathbb{P}^{g-1} = \mathbb{P}^3$.

Using the formulas above Proposition 7.1 we form the table below:

k	$\dim I_k$	$\dim S^k$	$\dim \mathcal{L}(kD)$
1	0	4	4
2	1	10	9
3	5	20	15

If $F(\xi^0, \dots, \xi^3)$ is a degree k homogeneous polynomial then $F \in I_k$ if and only if

$$\frac{F_k(\omega_1, -, \omega_4)}{\omega_1^k} = 0; \quad \text{that is } F(\xi^0, \dots, \xi^3)$$

vanishes identically on C. In the above table $\dim I_2 = 1$ and $\dim I_3 = 5$, so there exists a homogeneous polynomial of degree 2 and five linearly independent homogeneous polynomials of degree 3 which vanish identically on C. Letting $F(\xi^0, \dots, \xi^3)$ denote the quartic homogeneous polynomial which vanishes identically on C, $\xi^0 F, \xi^1, \xi^2 F$ and $\xi^3 F$ are then four linearly independent cubic homogeneous polynomials which also vanishes identically on C. So there must a fifth degree 3 homogeneous polynomial, linearly independent of $\{\xi^i F \mid i = 0, 1, 2, 3\}$ which vanishes identically on C. Call that polynomial $G(\xi^0, \dots, \xi^3)$.

Let $Q = \{F(\xi^0, \dots, \xi^3) = 0\}$ and $V = \{G(\xi^0, \dots, \xi^3) = 0\}$.

Then clearly, $C \subseteq Q \cap V$.

Our aim is to show that $C = Q \cap V$.

First we claim that Q is irreducible. Assume on the contrary that Q is reducible so that $F(\xi) = L_1(\xi) L_2(\xi_2)$. Since $\deg F = 2$ we must have $\deg L_1 = \deg L_2 = 1$.

Since F vanishes on C, one of L_1 and L_2 should have infinitely many zeros on C, say L_1. Then

$$1/\omega_1 \, L_1(\omega_1, -, \omega_4) \text{ is a meromorphic}$$

function on C with an infinite number of zeros. Since C is compact we see that $L_1(\omega_1, -, \omega_4) = 0$. Hence $L_1 \in I_1$. However, by the above table $\dim I_1 = 0$, which is a contradiction to the fact that $L_1 \in I_1$.

Next we will show that Q and V have no common hypersurface components. Assume the contrary. However, since Q is irreducible the common hypersurface must be Q itself. Then $V = Q \cup H$, where H is a hypersurface of degree 1, i.e., a hyperplane.

Let $H = \{L(\xi) = 0\}$, where degree $L = 1$.

Hence, we get $G(\xi) = F(\xi)H(\xi)$, which implies that G, $\xi^0 F, \dots, \xi^3 F$ are linearly dependent, contradicting the choice of G.

Now, we will show that there is a hyperplane $H_0 \subset \mathbb{P}^3$ such that $H_0 \cap Q$ and $H_0 \cap V$ have no common curve components. Note that it is enough to prove that there is some H_0 such that $H_0 \cap Q$ is irreducible, and $H_0 \cap Q \not\subset H_0 \cap V$.

To prove this, first note that since $Q \not\subset V$, there is some $p \in Q$ so that $p \notin V$. Let Φ denote the set of all hyperplanes in $\mathbb{P}^3$ which pass through the point p, then

$$H \in \Phi \Rightarrow H \cap Q \not\subset H \cap V.$$

Since the equation of a hyperplane in Φ is of the form $\sum_{i=0}^{3} l_i \xi^i = 0$ and these hyperplanes passes through p, the hyperplanes in Φ depends on two independent parameters, denoted by λ and μ.

For $H = H_{\lambda\mu} \in \Phi$, let $C_H = Q \cap H$, which is a quadric curve in the plane H. Let $T_{\lambda H}$ (fix $\mu = \mu_0$ and let λ vary) be the matrix of the quadric curve C_H.

We know that C_H is reducible if and only if $T_{\lambda H}$ has rank ≤ 1, i.e. $\det(T_{\lambda H}) = 0$. So we just need to choose λ so that $\det(T_{\lambda H}) \neq 0$ in order that the hyperplane H_0 in Φ corresponding to λ_0, μ_0 satisfies our requirement.

Now we'll finish the proof: Fix $H_0 \in \mathcal{P}$ hyperplane in $\mathbb{P}^3$ such that $H_0 \cap Q$ and $H_0 \cap V$ have no common curve components. By a linear change of coordinates we may assume that

$$H_0 = \{ \xi^3 = 0 \} = \mathbb{P}^2.$$

Then $H_0 \cap Q$ is a quadric and $H_0 \cap V$ a cubic curves in $H_0 = \mathbb{P}^2$. By Bezout's Theorem, we get

$$\# \left((H_0 \cap Q) \cap (H_0 \cap V) \right) = \deg (H_0 \cap Q) \deg (H_0 \cap V)$$
$$= 2 \times 3 = 6.$$

By the definition of curve degree, $\deg(Q \cap V) = 6$, i.e, $Q \cap V$ is a degree 6 curve in $\mathbb{P}^3$.

Finally assume that $Q \cap V = C + C'$. Noting that

$\deg C = 2g - 2 = 2 \times 2 - 2 = 6$ we deduce that $\deg C' = 0$, and so $Q \cap V = C$. This finishes the proof. ■