**Local equivalences between finite Lie groups**

Bob Oliver

Fix a prime $p$. Two finite groups $G$ and $H$ will be called *$p$-locally equivalent* if there is an isomorphism from a Sylow $p$-subgroup $S$ of $G$ to a Sylow $p$-subgroup $T$ of $H$ which preserves all conjugacy relations between elements and subgroups of $S$ and $T$.

Martino and Priddy proved that if the $p$-completed classifying spaces $BG^\wedge_p$ and $BH^\wedge_p$ are homotopy equivalent, then $G$ and $H$ are $p$-locally equivalent. They also conjectured the converse, a result which has since been proven, but only by using the classification theorem of finite simple groups.

Anyone who works much with finite groups of Lie type (such as linear, symplectic, or orthogonal groups over finite fields) notices that there are many cases of $p$-local equivalences between them. For example, if $q$ and $q'$ are two prime powers such that $q^2-1$ and $(q')^2-1$ have the same 2-adic valuation, then $SL_2(q)$ and $SL_2(q')$ are 2-locally equivalent.

In joint work with Carles Broto and Jesper Møller, we proved, among other results, the following very general theorem about such $p$-local equivalences between finite Lie groups.

**Theorem** *Fix a prime $p$, a connected, reductive, integral group scheme $\mathbb{G}$, and a pair of prime powers $q$ and $q'$ both prime to $p$. Then $\mathbb{G}(q)$ and $\mathbb{G}(q')$ are $p$-locally equivalent if $\overline{\langle q\rangle} = \overline{\langle q'\rangle}$ as subgroups of $\mathbb{Z}_p^\times$.*

Our proof of this theorem is topological: we show that the $p$-completed classifying spaces have the same homotopy type, and then apply the theorem of Martino and Priddy mentioned above. The starting point is a theorem of Friedlander, which describes the space $B\mathbb{G}(q)^\wedge_p$ as a "homotopy fixed space" of a some self map of $B\mathbb{G}(\mathbb{C})^\wedge_p$ of a certain type (an "unstable Adams operation"). This is combined with a theorem of Jackowski, McClure, and Oliver that classifies more precisely the self maps of $B\mathbb{G}(\mathbb{C})^\wedge_p$; and with a result of Broto, Møller, and Oliver which says that under certain hypotheses on a space $X$, the homotopy fixed space of a self equivalence $f$ of $X$ depends (up to homotopy type) only on the closed subgroup $\overline{\langle f\rangle}$ in the group $\mathrm{Out}(X)$ of all homotopy classes of self equivalences of $X$.

This theorem is not surprising to group theorists. But currently, no other proof seems to be known of this purely algebraic theorem.