METU, Spring 2012, Math 515.

Homework 3

(due April 2)

1. A nonzero A-module M is called irreducible if 0 and M are the only submodules of M.
   - Determine all the irreducible $\mathbf{Z}$-modules.
   - Show that M is irreducible if and only if $M \cong A/\mathfrak{m}$ as A-modules where $\mathfrak{m}$ is a maximal ideal of A.
   - If M is irreducible then prove that $\mathrm{Hom}_A(M, M)$ is a field. What about the converse, is M irreducible if $\mathrm{Hom}_A(M, M)$ is a field?

2. Let f', f, f'' be A-module homomorphisms such that the following diagram commutes and suppose that the rows are exact. If f' and f'' are surjective then show that f is surjective.

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \to & N' & \to & N & \to & N'' & \to & 0 \end{array}$$

3. Show the exactness at $\mathrm{Coker}(f')$ in the sequence obtained by the snake lemma when it is applied to the commutative diagram above.

$$0 \to \mathrm{Ker}(f') \to \mathrm{Ker}(f) \to \mathrm{Ker}(f'') \to \mathrm{Coker}(f') \to \mathrm{Coker}(f) \to \mathrm{Coker}(f'') \to 0.$$

4. Give an exact sequence of A-modules

$$0 \to M' \to M \to M'' \to 0$$

and an A-module N such that

$$0 \to \mathrm{Hom}(N, M') \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M'') \to 0$$

is not exact.

5. Let $\{e_1, e_2\}$ be a basis of $\mathbf{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $\mathbf{R}^2 \otimes_{\mathbf{R}} \mathbf{R}^2$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbf{R}^2$.

6. Is $\mathbf{Q}$ a flat $\mathbf{Z}$-module? Prove or disprove.