**METU, Spring 2012, Math 515.**

# Homework 1

(due March 5)

1. In this question, use only the basic definitions.

   - Construct a field $\mathbf{F} = \{0, 1, a, b\}$ with 4 elements and give its addition and multiplication tables. Verify that multiplication is distributive over addition.
   - Show that there is no field $\mathbf{F} = \{0, 1, a, b, c, d\}$ with 6 elements.

2. Let $A$ be a ring and let $B = A[[x]]$ be the ring of formal power series

$$f = \sum_{n=0}^{\infty} a_n x^n$$

   with coefficients in $A$. Show that $f \in B^\times \Leftrightarrow a_0 \in A^\times$. Is $B$ a local ring?

3. Let $A$ be a finite ring. Prove that every prime ideal of $A$ is maximal.

4. Let $A$ be the ring of all continuous functions from the open interval $(0, 1)$ to $\mathbf{R}$ and set $M_c = \{f \in A : f(c) = 0\}$ for any $c \in (0, 1)$.

   - Show that $M_c$ is a maximal ideal of $A$.
   - Prove that $M_c \neq (x - c)$.
   - Find a maximal ideal $\mathfrak{m}$ of $A$ such that $\mathfrak{m} \neq M_c$ for any $c \in (0, 1)$.

5. Let $\mathfrak{a} \subset A$ be an ideal and define

$$\operatorname{rad}(\mathfrak{a}) = \{x \in A : x^n \in \mathfrak{a} \text{ for some } n \in \mathbf{Z}^+\}$$

   called the radical of $\mathfrak{a}$.

   - Give two distinct examples such that $\operatorname{rad}(\mathfrak{a}) \neq \mathfrak{a}$.
   - Prove that $\operatorname{rad}(\mathfrak{a})$ is an ideal containing $\mathfrak{a}$.
   - Prove that $\operatorname{rad}(\mathfrak{a})/\mathfrak{a} = \mathfrak{N}(A/\mathfrak{a})$.