M E T U
Department of Mathematics

Group	Commutative Algebra Midterm	List No.

Code : *Math 515* Acad. Year : *2012* Semester : *Spring* Instructor : *Küçüksakallı* Date : *16/04/2012* Time : *13:40* Duration : *110 minutes*	Name : Last Name : Signature : 6 QUESTIONS ON 4 PAGES 60 TOTAL POINTS

1	2	3	4	5	6	7		

1. (12pts) Let A be the ring of all continuous functions from $D = [-1,1] \times [-1,1]$ to $\mathbf{R}$ and set $M_P = \{f \in A : f(x_0, y_0) = 0\}$ for any point $P = (x_0, y_0) \in D$.

- Show that M_P is a maximal ideal of A.

- Prove that $M_{(0,0)} \neq (x, y)$.

- Is there a prime ideal $\mathfrak{p}$ of A which is not maximal?

2. (9pts) Let p be a prime number. Let $\mathbf{F}$ be a finite field with p elements and let G be a group with p elements. Consider the group ring $A = \mathbf{F}[G]$. Find the nilradical $\mathfrak{N}$ and Jacobson radical $\mathfrak{R}$ of A. (Hint: Construct an ideal $\mathfrak{m} = \ker(\varphi)$ such that $\mathfrak{R} \subseteq \mathfrak{m} \subseteq \mathfrak{N} \subseteq \mathfrak{R}$.)

3. (9pts) Let A be an integral domain and let u be a nonzero element of A. Show that $B = A[x]/(xu-1)$ is a finitely generated A-module if and only if u is a unit in A. (Hint: If B is a finitely generated A-module, then show that $x^{k+1} = \sum_{n=1}^{k} a_n x^n$ for some $k > 0$.)

4. (9pts) Let f', f, f'' be A-module homomorphisms such that the following diagram commutes and suppose that the rows are exact. If f' and f'' are injective then show that f is injective.

$$\begin{array}{ccccccccc} 0 & \to & M' & \to & M & \to & M'' & \to & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \to & N' & \to & N & \to & N'' & \to & 0 \end{array}$$

5. (9pts) Let $\mathfrak{p}$ be a prime ideal of A. Show that $A_{\mathfrak{p}}$ is a local ring.

6. (12pts) Let $A_m = \mathbf{Z}/m\mathbf{Z}$ and $A_n = \mathbf{Z}/n\mathbf{Z}$. Set $d = \gcd(m, n)$.

- Construct a non-zero bilinear map $\varphi : A_m \times A_n \to A_d$ if $d \neq 1$.

- Show that $A_m \otimes_{\mathbf{Z}} A_n \cong A_d$.

- Let $\psi : A_m \times A_n \to A_d$ be a bilinear map and let $\overline{\psi} : A_m \otimes_{\mathbf{Z}} A_n \to A_d$ be the corresponding map induced by ψ. When is it possible to recover ψ if you are given $\overline{\psi}(x)$ for some $x \in A_m \otimes_{\mathbf{Z}} A_n$?