M E T U
Department of Mathematics

Group	Field Extensions and Galois Theory Midterm 2		List No.
Code : *Math 368* Acad. Year : *2011* Semester : *Spring* Instructor : *Küçüksakallı* Date : *May 2, 2011* Time : *12:40* Duration : *90 minutes*	Last Name : Name : Department : Signature :	Student No. : Section :	
	6 QUESTIONS ON 4 PAGES 60 TOTAL POINTS		

1	2	3	4	5	6		

1. (12pts) True or false? Justify your answers.

- **(4pts)** Let K be a subfield of $\mathbb{C}$. If $f(t)$ is a non-constant polynomial which is irreducible over K, then $\gcd(f, Df)$ is constant.

It is given $\deg(f) \geq 1$. Since $K \subseteq \mathbb{C}$, we have $\deg(Df) \geq 0$. i.e. $Df \neq 0$.
$\gcd(f, Df) \mid f \Rightarrow \gcd(f, Df) = 1$ or $\gcd(f, Df) = f$
f does not divide Df. Thus $\gcd(f, Df) = 1$

TRUE

- **(4pts)** Let $K = \mathbb{Q}(\sqrt[3]{2})$. The extensions $N_1 = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ and $N_2 = \mathbb{Q}(\sqrt[6]{2}, \zeta_6)$ of K are both normal. Therefore the normal closure of $K/\mathbb{Q}$ may not be unique.

Note that $N_1 \subsetneq N_2$. Even though N_2 is a normal extension of K, it does not satisfy the minimality condition (N_1 is a "smaller" normal extension). Thus N_2 is not a normal closure.

FALSE ARGUMENT

- **(4pts)** A normal extension of a normal extension is a normal extension.

The extensions $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ and $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ are both normal but $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ is not.

FALSE

2. (8pts) Show that every extension in $\mathbb{C}$ of degree 2 is normal. Is this true if the degree is bigger than 2?

Let L/K be an extension of degree 2. Let $f(t) \in K[t]$ be an irreducible polynomial with $f(\alpha)=0$ for some $\alpha \in L$. Since $[L:K]=2$, $\deg(f)=1,2$. If $\deg(f)=2$, then

$$f(t) = (t-\alpha)(t-\beta) = 0$$

and $\alpha+\beta$, $\alpha\beta$ are both in K. If $\alpha+\beta=k$ for some $k \in K$ then $\beta = k-\alpha \in K$ as well. Thus L/K is normal.

3. (8pts) Show that any finite extension L/K in $\mathbb{C}$ has a normal closure N. Is $[N:K]$ finite? Is N unique?

Theorem 11.6 in the textbook.

4. (8pts) Suppose that L/K is an extension and M is an intermediate field. If τ is a K-automorphism of L, then show that $\tau(M)^* = \tau M^* \tau^{-1}$.

Lemma 12.2 in the textbook

5. (8pts) Let $n = 17m$ where m is a positive integer not divisible by 17 and let $K = \mathbb{Q}(\zeta_n)$ be the corresponding cyclotomic field. Find the number of intermediate fields M such that $\mathbb{Q}(\zeta_m) \subseteq M \subseteq K$ using the Galois correspondence.

We have $\mathrm{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/17\mathbb{Z})^\times \times (\mathbb{Z}/m\mathbb{Z})^\times$

$$\mathrm{Aut}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times$$

Galois correspondence $\Rightarrow$ $\left(\mathrm{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \Big/ \mathrm{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_m)) \right) \cong \mathrm{Aut}(\mathbb{Q}(\zeta_m)/\mathbb{Q}(\zeta_m))$

We see that $\underbrace{\mathrm{Aut}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_m))}_{G} \cong (\mathbb{Z}/17\mathbb{Z})^\times$

a cyclic group of order 16

For each divisor of 16 there is a subgroup of G.

1 2 4 8 16

$$\begin{array}{ccc} \mathbb{Q}(\zeta_n) & & \{id\} \\ | & & | \\ M & & M^* \\ | & & | \\ \mathbb{Q}(\zeta_m) & & G \end{array}$$

Galois correspondence $\Rightarrow$ There are 5 intermediate fields.

6. **(16pts)** Let $f(t) = t^5 - 2$ and let $L = \Sigma_{f,\mathbb{Q}}$ be the splitting field of f over $\mathbb{Q}$.

- **(5pts)** You are given that $|\mathrm{Aut}(L/\mathbb{Q})| = 20$. Show that $\mathrm{Aut}(L/\mathbb{Q})$ is isomorphic to a subgroup of S_5. Find generators for $\mathrm{Aut}(L/\mathbb{Q})$.

Define $\alpha_1 = \sqrt[5]{2}$, $\alpha_2 = \sqrt[5]{2}\zeta_5$, $\ldots$, $\alpha_5 = \sqrt[5]{2}\zeta_5^4$. Note that $\tau(\alpha_i) = \alpha_j$ for any $\tau \in \mathrm{Aut}(L/\mathbb{Q})$. Moreover $\tau_1(\alpha_i) = \alpha_i$ for each i implies $\tau = \text{identity}$. Thus $\mathrm{Aut}(L/\mathbb{Q}) \leq S_5$.

Define $\sigma: \begin{matrix} \sqrt[5]{2} \mapsto \sqrt[5]{2}\zeta_5 \\ \zeta_5 \mapsto \zeta_5 \end{matrix}$ $\quad \tau: \begin{matrix} \sqrt[5]{2} \longmapsto \sqrt[5]{2} \\ \zeta_5 \longmapsto \zeta_5^2 \end{matrix}$. Obviously $\sigma, \tau \in \mathrm{Aut}(L/\mathbb{Q})$

$$\left.\begin{matrix} \deg(\sigma) = 5 \\ \deg(\tau) = 4 \end{matrix}\right\} \ |\mathrm{Aut}(L/\mathbb{Q})| = 20 \Rightarrow \mathrm{Aut}(L/\mathbb{Q}) = \langle \sigma, \tau \rangle$$

- **(3pts)** Is $L/\mathbb{Q}$ normal? What is $[L/\mathbb{Q}]$? Find a basis for L over $\mathbb{Q}$.

Yes. It is a splitting field. $[L:\mathbb{Q}] = |\mathrm{Aut}(L/\mathbb{Q})| = 20$

$\{\sqrt[5]{2}^i \zeta_5^j : 0 \leq i \leq 4, 0 \leq j \leq 3\}$ is such a basis

- **(4pts)** Let σ be a $\mathbb{Q}$-automorphism of L such that $\sigma(\sqrt[5]{2}) = \sqrt[5]{2}\zeta_5$ and $\sigma(\zeta_5) = \zeta_5^2$. Let $H = \{\sigma^i : i \in \mathbb{Z}\}$ be the group generated by σ. What is $|H|$ and $[H^\dagger : \mathbb{Q}]$?

$\sigma \cong (1243)$. order of σ is 4. Thus $|H| = 4$

$$|H^+ : \mathbb{Q}| = \frac{[L:\mathbb{Q}]}{|H|} = 5$$

- **(4pts)** Consider $M_1 = \mathbb{Q}(\zeta_5)$ and $M_2 = \mathbb{Q}(\sqrt[5]{2})$. Answer the following questions for each $j \in \{1, 2\}$. Is M_j^* a normal subgroup of G? What is $|M_j^*|$?

$M_1/\mathbb{Q}$ is normal since it is the splitting field of $t^5 - 1$.

$$|M_1^*| = [L : M_1] = \frac{[L:\mathbb{Q}]}{\underbrace{[M_1:\mathbb{Q}]}_{4}} = 5$$

$M_2/\mathbb{Q}$ is not normal because $t^5 - 2$ has a root in M_2 but does not split in M_2

$$|M_2^*| = [L : M_2] = \frac{[L:\mathbb{Q}]}{\underbrace{[M_2:\mathbb{Q}]}_{5}} = 4$$