M E T U
Department of Mathematics

Group	Field Extensions and Galois Theory Midterm 1	List No.

Code : *Math 368* Acad. Year : *2011* Semester : *Spring* Instructor : *Küçüksakallı* Date : *March 28, 2011* Time : *12:40* Duration : *90 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 5 QUESTIONS ON 4 PAGES 60 TOTAL POINTS

1	2	3	4	5	6		

1. **(15pts)** True or false? Justify your answers.

- **(4pts)** There exists integers $a, b \in \mathbb{Z}$ such that $368a + 11b = 1$.

True!

$$368 = 33 \cdot 11 + 5 \quad \Rightarrow \quad 1 = 11 - 2 \cdot 5$$
$$11 = 2 \cdot 5 + 1 \qquad\qquad = 11 - 2(368 - 33 \cdot 11)$$
$$5 = 5 \cdot 1 + 0 \qquad\qquad = -2 \cdot 368 + 11 \cdot 67$$

- **(5pts)** There exists polynomials $f, g \in \mathbb{Q}[t]$ such that $(t^2-1)f + (t^3+t^2+t+1)g = 1$.

False! The left hand side is divisible by $t+1$ whereas the right hand side is not!

- **(6pts)** There exists an **irrational** number α such that $\alpha = f(\sqrt{3}) = g(\sqrt[3]{2})$ for some $f, g \in \mathbb{Q}[t]$.

False! Assume $\alpha = f(\sqrt{3}) = g(\sqrt[3]{2})$ with $f, g \in \mathbb{Q}[t]$.

Then $\alpha \in \underbrace{\mathbb{Q}(\sqrt{3}) \cap \mathbb{Q}(\sqrt[3]{2})}_{K}$. Note that

$[K:\mathbb{Q}]$ divides both 2 and 3. So $[K:\mathbb{Q}] = 1$

Thus $\alpha \in \mathbb{Q}$. i.e. must be rational.

2. (15pts) Let θ be a complex number which satisfies the equation $\theta^3 - 5\theta - 10 = 0$.

- **(3pts)** Show that $f(t) = t^3 - 5t - 10$ is irreducible over $\mathbb{Q}$.

Eisenstein's Lemma with $q=5 \Rightarrow f(t)$ is irreducible over $\mathbb{Z}$. Gauss' Lemma $\Rightarrow f(t)$ is irreducible over $\mathbb{Q}$.

- **(4pts)** Give a basis for the field $K = \mathbb{Q}(\theta)$ over $\mathbb{Q}$. Why is this a basis?

θ is algebraic. Thus $\mathbb{Q}(\theta) \cong \mathbb{Q}[t]/(t^3-5t-10)$

we have a natural basis $\{1, \theta, \theta^2\}$.

- **(8pts)** Express elements $\beta = (1+\theta)(1+\theta+\theta^2)$ and $\gamma = (1+\theta)/(1+\theta+\theta^2)$ in the basis you have given.

$$\beta = 1 + 2\theta + 2\theta^2 + \underbrace{\theta^3}_{5\theta+10} = 2\theta^2 + 7\theta + 11$$

Euclidean Algorithm with $t^3-5t-10$ & t^2+t+1 gives

$$\left(-\frac{5}{61}t^2 + \frac{9}{61}t + \frac{21}{61}\right)(t^2+t+1) + \left(\frac{5}{61}t - \frac{4}{61}\right)(t^3-5t-10) = 1$$

Thus

$$\gamma = (1+\theta)\cdot\left(-\frac{5}{61}\theta^2 + \frac{9}{61}\theta + \frac{21}{61}\right)$$

$$= \frac{4}{61}\theta^2 + \frac{5}{61}\theta - \frac{29}{61}$$

3. **(8pts)** Let a, b, c, d, e be odd integers. Show that

$$f(t) = at^4 + bt^3 + ct^2 + dt + e$$

is irreducible over $\mathbb{Q}$.

Reducing $f(t)$ modulo 2, we obtain $\bar{f}(t) = t^4+t^3+t^2+t+1$.
Note that $\bar{f}(t)$ has no roots since $\bar{f}(0)=\bar{f}(1)=1$.
Assume

$$\begin{aligned}\bar{f}(t) &= (t^2+at+b)(t^2+ct+d) \\ &= t^4 + \underbrace{(a+c)}_{1}t^3 + (\underbrace{ac+b+d}_{1})t^2 + (\underbrace{ad+bc}_{1})t + \underbrace{bd}_{1}\end{aligned}$$

$\Rightarrow b=d=1 \Rightarrow ac=1 \Rightarrow a=c=1 \Rightarrow a+c=0$ ↯

Thus $\bar{f}(t)$ is irreducible. Therefore $f(t)$ is irreducible over $\mathbb{Z}$. Result follows from Gauss' Lemma.

4. **(10pts)** Let K be a subfield of $\mathbb{C}$. Prove that if $[K(\alpha) : K]$ is odd then $K(\alpha) = K(\alpha^2)$.

It is easy to see that $K(\alpha^2) \subseteq K(\alpha)$. The minimal polynomial of α over $K(\alpha^2)$ divides

$$f(t) = t^2 - \alpha^2$$

Thus $[K(\alpha):K(\alpha^2)] = 1$ or 2. On the other hand

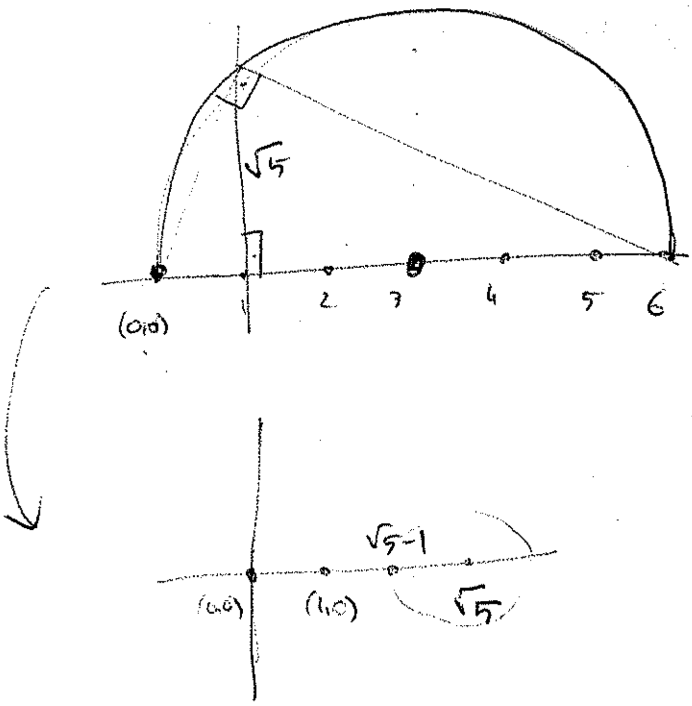
$$[K(\alpha):K(\alpha^2)][K(\alpha^2):K] = \text{odd}$$

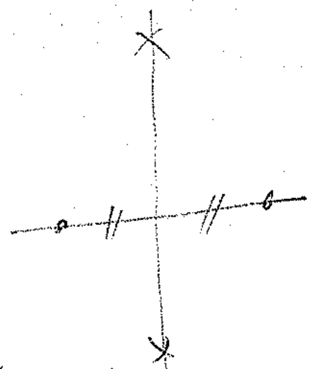
Thus $[K(\alpha):K(\alpha^2)] = 1$ and therefore $K(\alpha^2) = K(\alpha)$.

5. **(12pts)** You are given that

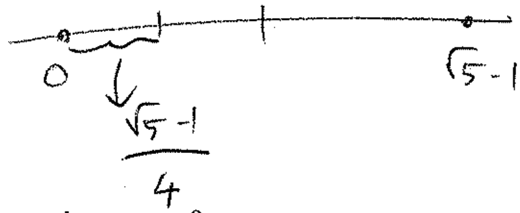
$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}.$$

- **(7pts)** Starting from $P_0 = \{(0,0),(1,0)\}$, can you construct the point $r = (\cos(\frac{2\pi}{5}), 0)$ using a ruler and a compass?

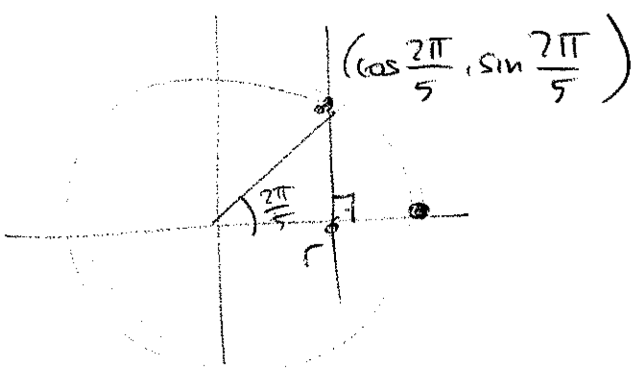




Then use bisection twice!



- **(5pts)** Can you construct a regular pentagon ⬠ using a ruler and a compass?



Consider the line $x = \cos\frac{2\pi}{5}$. It intersects the unit circle at the point $\left(\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5}\right)$

We proceed in a similar fashion to obtain the remaining 3 corners.