METU, Spring 2020, Math 366.

Exercise Set 4

1. Prove that an integer of the form $4^n(8m+7)$ cannot be represented as the sum of three squares.

2. Prove that every integer $n \geq 170$ is a sum of five squares, none of which are equal to zero. (Hint: Write $n-169 = a^2+b^2+c^2+d^2$ for some integers). Represent $10169, 10170$ and 10171 as a sum of five squares, none of which are equal to zero.

3. Express 366 and 2015 as a sum four squares using the Hamiltonian product.

4. A combination of three squares:

   (a) Show that a positive integer n can be represented as the difference of two squares if and only if n is not of the form $4k+2$.

   (b) Show that every positive integer is of the form $x^2 + y^2 - z^2$.

   (c) Represent 366 and 2015 in the form $x^2 + y^2 - z^2$.

5. Show that the number of positive cubes needed to represent every positive integer n is at least 9. (Hint $n = 23$). Show that the number of positive cubes needed to represent every positive integer $n > N$ for some N is at least 4. (Hint: $n = 9k \pm 4$).

6. Determine all solutions of $x^2 - a^2y^2 = n$ for fixed integers a and n.

7. Set $(x_0, y_0) = (10, 1)$ and define $(x_n, y_n) = (10x_{n-1} + 99y_{n-1}, x_{n-1} + 10y_{n-1})$ for $n \geq 1$.

   - Show that (x_n, y_n) is a solution of the Diophantine equation $x^2 - 99y^2 = 1$ for all $n \geq 0$.
   - Show that the Diophantine equation $x^2 - 99y^2 = 1$ has infinitely many solutions.

8. Define $a_n = 6a_{n-1} - a_{n-2} + 2$ with $a_0 = 0$ and $a_1 = 3$. We have

$$a_0 = 0, a_1 = 3, a_2 = 20, a_3 = 119, a_4 = 696, \ldots.$$

   Show that $a_n^2 + (a_n + 1)^2$ is a square.