M E T U Department of Mathematics

Math 366, Spring 2020, Midterm I, 4 March 2020, 13:40-15:30						
FULL NAME					STUDENT ID	DURATION 110 MINUTES
5 QUESTIONS ON 4 PAGES						TOTAL 100 POINTS
1	2	3	4	5	Good Luck!	

Q1 (25 pts) Find all integer solutions of $15x + 21y + 35z = 11$.

Note that $15+21-35=1$. Thus $(11,11,-11)$ is a solution. Now consider the homogeneous equation

$$15x+21y+35z=0$$

We have $15x=-7(3y+5z)$ and $21y=-5(3x+7z)$. Thus x and y are divisible by 7 and 5, respectively. We put $x=7s$ and $y=5t$ for some integers s and t. Now

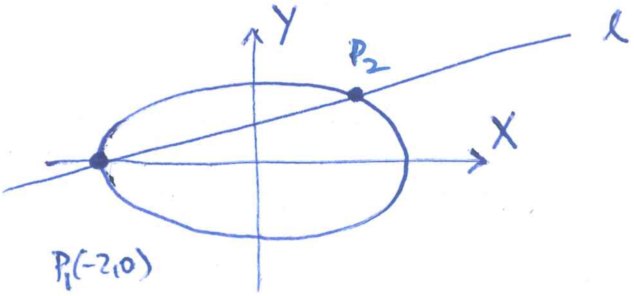
$$35z=-15x-21y=-105(s+t)$$

It follows that $z=-3s-3t$. We conclude that every solution is of the form

$$x=11+7s$$
$$y=11+5t$$
$$z=-11-3s-3t$$

for some integers s and t.

Q2 (25 pts) Find all integer solutions of $x^2 + 3y^2 = 4z^2$. Give an example of a solution that satisfies $x > 0$, $y > 0$ and $z = 301$.



Set $X = \frac{x}{z}$ and $Y = \frac{y}{z}$.

We have $X^2 + 3Y^2 = 4$

Consider the line

$$\ell : r(x+2)$$

The line ℓ intersects the ellipse at the points $P_1 = (X_1, Y_1) = (-2, 0)$ and $P_2 = (X_2, Y_2)$. In order to find x_2, we consider

$$\begin{aligned} X^2 + 3Y^2 - 4 &= X^2 + 3(r(X+2))^2 - 4 \\ &= (3r^2+1)X^2 + 12r^2 X + 12r^2 - 4 \\ &= (3r^2+1)\left(X^2 + \frac{12r^2}{3r^2+1} X + \text{constant term}\right) \end{aligned}$$

We must have $X_1 + X_2 = -\frac{12r^2}{3r^2+1}$ It follows that

$$X_2 = -\frac{12r^2}{3r^2+1} - (-2) = \frac{-6r^2+2}{3r^2+1}$$

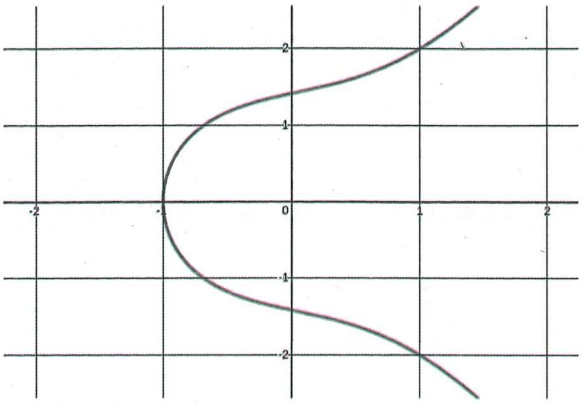
$$Y_2 = r(X_2+2) = \frac{4r}{3r^2+1}$$

Putting $r = \frac{m}{n}$ for some integers m and n, we obtain the triples $(-6m^2+2n^2, 4mn, 3m^2+n^2)$. All solutions are given by

$$\begin{aligned} x &= \mp d(6m^2 - 2n^2) \\ y &= \mp d(4mn) \\ z &= \mp d(3m^2 + n^2) \end{aligned}$$

In particular, choosing $m = 10$ and $n = 1$, we obtain the solution $(598, 40, 301)$.

Q3 (25 pts) Let E be the elliptic curve given by the equation $y^2 = x^3 + x + 2$.



(a) Consider the point $P = (1, 2)$ of E. Show that $P \oplus P = (-1, 0)$.

By using the implicit differentiation, we find that $\frac{dy}{dx} = \frac{3x^2+1}{2y}$.
It follows that $\frac{dy}{dx}\Big|_P = \frac{3 \cdot 1^2 + 1}{2 \cdot 2} = 1$ and the tangent line thru P is given by the equation $y = x + 1$. Putting $y = x+1$ in $y^2 = x^3 + x + 2$, the other x value is -1. We have $P * P = (-1, 0)$ and therefore $P \oplus P = (-1, 0)$.

(b) If $Q = (s, t)$ is an inflection point of E, then show that $3s^4 + 6s^2 + 24s - 1 = 0$.

We compute $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{6x \cdot 2y - 2\frac{dy}{dx}(3x^2+1)}{4y^2}$

$$= \frac{24xy^2 - 2(3x^2+1)^2}{8y^3}$$

Put $y^2 = x^3 + x + 2$

$$= \frac{3x^4 + 6x^2 + 24x - 1}{4y^3}$$

There is no inflection point with $y = 0$. We must have $3s^4 + 6s^2 + 24s - 1 = 0$ if $Q = (s, t)$ is an inflection point.

(c) Does E have a rational point of order 2, order 3 and order 4? For each case, either provide such a point or explain why such a point cannot exist.

The point $2P = (-1, 0)$ has order 2 since its y-coordinate is zero. It follows that $P = (1, 2)$ has order 4 since $4P = \infty$ but $2P \neq \infty$. To see that there is no rational point of order 3, we use (b) together with rational root theorem. A torsion point of E has integer points by Nagell-Lutz theorem. However, only possible integer roots 1 and -1 of $3s^4 + 6s^2 + 24s - 1 = 0$ are not roots at all.

Q4 (15 pts) Show that the equation $x^4 - 4y^4 = z^2$ has no solution in positive integers.

Assume that (x,y,z) is a solution in positive integers with $\gcd(x,y,z)=1$. Note that

$$(2y^2)^2 + z^2 = (x^2)^2$$

is a primitive Pythagorean triple. We must have

$$2y^2 = 2mn$$
$$z = m^2 - n^2$$
$$x^2 = m^2 + n^2$$

for some positive integers m and n with $\gcd(m,n)=1$. Now $y^2 = mn$ and $\gcd(m,n)=1$. We must have $m=r^2$ and $n=s^2$ for some positive integers r and s. It follows that

$$x^2 = m^2 + n^2 = r^4 + s^4$$

has a positive solution. This is a contradiction to the fact that $x^2 = r^4 + s^4$ has no solution in positive integers.

Q5 (10 pts) Show that 2 is not a congruent number, i.e. show that there is no rational right triangle with area 2. You may use the conclusion of **Q4** without proof.

Equivalently, let us consider a right triangle with integer sides (x,y,z) whose area is $2k^2$ for some positive integer k. We have $x^2+y^2=z^2$ and $2xy = 8k^2$. Observe that

$$(x+y)^2 = x^2+2xy+y^2 = z^2 + 8k^2$$
$$(x-y)^2 = x^2-2xy+y^2 = z^2 - 8k^2$$

Multiplying both sides, we obtain

$$(x^2-y^2)^2 = z^4 - 64k^2 = z^4 - 4(2k)^4$$

This contradicts to the conclusion of Q4.