M E T U
Department of Mathematics

	Discrete Mathematics MidTerm II		
Code : *Math 112* Acad. Year : *2010-2011* Semester : *Spring* Instructor : *Bhupal, Küçüksakallı, Okutmuştur, Seven.* Date : *05.05.2011* Time : *17.40* Duration : *90 minutes*	Last Name : Name : Student No : Department : Signature : 6 Questions on 4 Pages Total 60 Points		

1	2	3	4	5	6		

1. (4+3+3=10 pts.) Consider the non-homogenous recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2} + 3 \cdot 2^n$$

a. Find $a_n^{(h)}$, the general solution to the associated **homogenous** recurrence relation

$$a_n = 2a_{n-1} + 3a_{n-2}$$

The characteristic equation of the recurrence relation is $r^2-2r-3=0$ and therefore $r=3,-1$.
Thus

$$a_n^{(h)} = c_1 3^n + c_2 (-1)^n$$

b. Find a particular solution of the form $a_n^{(p)} = A \cdot 2^n$ for the non-homogenous recurrence relation.

$$a_n^{(p)} = A \cdot 2^n \Rightarrow A \cdot 2^n = 2 \cdot A \cdot 2^{n-1} + 3A \cdot 2^{n-2} + 3 \cdot 2^n$$

$$\Rightarrow 2^{n-2}(4A - 4A - 3A - 12) = 0$$

$$\Rightarrow A = -4$$

Thus $a_n^{(p)} = -4 \cdot 2^n$

c. Find the solution of the non-homogenous recurrence relation with the initial conditions $a_0 = 3$, $a_1 = 5$.

The solution is of the form

$$a_n = c_1 \cdot 3^n + c_2 (-1)^n - 4 \cdot 2^n$$

Using initial conditions, we get

$$\left.\begin{array}{l} a_0 = 3 = c_1 + c_2 - 4 \\ a_1 = 5 = 3c_1 - c_2 - 8 \end{array}\right\} \Rightarrow c_1 = 5,\ c_2 = 2$$

$$a_n = 5 \cdot 3^n + 2(-1)^n - 4 \cdot 2^n$$

2. (10 pts.) In how many ways can three A's, three B's and three C's be arranged so that no consecutive triple of the same letter appears (e.g. AABCBBCCA is allowed whereas ABBACCCBA is not)?

Set c_1 : permutation contains AAA
c_2 : // // BBB
c_3 : // // CCC

$\#$ of all permutations $= \frac{9!}{3!3!3!}$

Inclusion & Exclusion principle gives

$$N(\bar{c}_1\bar{c}_2\bar{c}_3) = \frac{9!}{3!3!3!} - 3\left(\frac{7!}{3!3!}\right) + 3\left(\frac{5!}{3!}\right) - 3!$$

3. (10 pts.) Find the number of solutions of $x + y + z = 200$ with $10 \le x \le 90$, $10 \le y \le 90$ and $10 \le z \le 70$.

Substitute $x = \bar{x}+10$, $y = \bar{y}+10$, $z = \bar{z}+10$.
Then $\bar{x}+\bar{y}+\bar{z} = 170$ and $0 \le \bar{x} \le 80$, $0 \le \bar{y} \le 80$, $0 \le \bar{z} \le 60$

Set $c_1 : \bar{x} \geq 81$
$c_2 : \bar{y} \geq 81$
$c_3 : \bar{z} \geq 61$

$N(c_1) = N(c_2) = \binom{170-81+2}{2}$

$N(c_3) = \binom{170-61+2}{2}$

$\#$ of solutions $= \binom{170+2}{2}$
$-(N(c_1)+N(c_2)+N(c_3))$
$+(N(c_1c_2)+N(c_1c_3)+N(c_2c_3))$
$-(N(c_1c_2c_3))$

$N(c_1c_3) = N(c_2c_3) = \binom{170-142+2}{2}$

$N(c_1c_2) = \binom{170-162+2}{2}$

$N(c_1c_2c_3) = 0$

4. (2+8=10 pts.) Let a_n be the number of strings of length n consisting of letters A, B, C that contain two consecutive A's.

a) Compute a_2 and a_3.

$a_2 = 1$ AA

$a_3 = 5$ AAA AA? ?AA

b) Find a recurrence relation for a_n.

Focus on the last two digits.

$\left.\begin{array}{l}\_\_\_\_\_\_ B \\ \_\_\_\_\_\_ C\end{array}\right\} \rightarrow 2a_{n-1}$

$\_\_\_\_\_\_ AA \rightarrow 3^{n-2}$

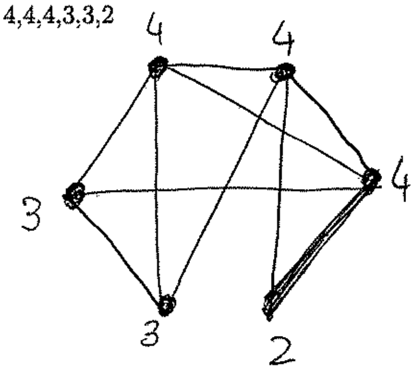
$\left.\begin{array}{l}\_\_\_\_\_\_ BA \\ \_\_\_\_\_\_ CA\end{array}\right\} \rightarrow 2 \cdot a_{n-2}$

$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$

for all $n \geq 4$

5. (4+4+2=10 pts.) Is there a simple graph whose vertices have the following degrees (if yes, please draw such a graph; if not, justify):

a) 4,4,4,3,3,2



Yes. There is such a graph.

b) 4,4,3,2,1,0

Say, these are degrees of $v_1, v_2, \ldots, v_6$ respectively.

v_1 is connected to v_2, v_3, v_4, v_5

v_2 " " " v_1, v_3, v_4, v_5 $\Rightarrow \deg(v_5) \geq 2$

Contradiction. Such a graph cannot exist.

c) 4,3,3,2,1,0

$4+3+3+2+1+0 = 13$ odd

Sum of degrees must be even. Such a graph cannot exist.

6. **(2+2+3+3=10 pts.)** Let G be the simple graph whose vertices are the integers i with $1 \leq i \leq 100$ such that two vertices i and j are connected if and only if $\gcd(i, j) = 1$.

a) Are the vertices 7 and 2 connected in G? What about 7 and 14?

$\gcd(7,2)=1$. Yes, connected.

$\gcd(7,14)=7$. No, they are not connected.

b) Find the degree of the vertex 7.

7 is not connected to $\underbrace{7, 14, \ldots, 98}_{14 \text{ vertices}}$ only.

$\deg(7) = 100 - 14 = 86$

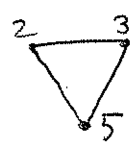
c) Find the vertices with smallest and largest degree. (one example for each is enough)

Largest possible degree is 99 and it is achived by 1
(also $\deg(p) = 99$ if $50 \leq p \leq 100$ is prime)

Note that the degree drops for each additional prime
For example $\deg(2) > \deg(2 \cdot 3)$.

A vertex with the smallest degree is 30. ($\deg(30) = 26$)
(also $\deg(60) = \deg(90) = 26$)

d) Is G bipartite? (Hint: Show there is a cycle of length 3 in G.)

No. Consider the subgraph 

which is a cycle of length 3 in G. If G is bipartite with $V = V_1 \cup V_2$ then $2 \in V_1$ without loss of genelity. Then $3, 5 \in V_2$ since they are adjacent with 2. Now $3, 5 \in V_2$ but there is an edge between them $\lightning$.