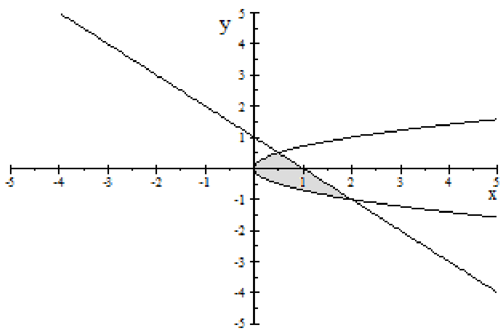
Calculus 119

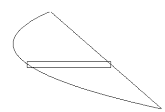
7th Week[1]

1.

$$\text{Area} = \int_0^3 2y - y^2 - (y^2 - 4y)dy = \int_0^3 6y - 2y^2 dy = [3y^2 - \frac{2y^3}{3}]\Big|_0^3 = 27 - 18 = 9$$

2. (a) $x = 2y^2$ and $x + y = 1$



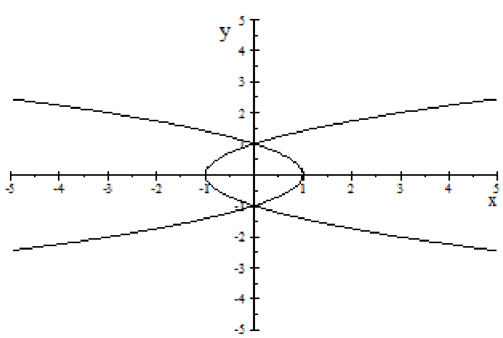


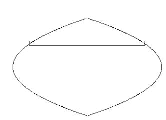
Let's determine the intersection points

$$\begin{cases} x = 2y^2 \\ x + y = 1 \end{cases} \Rightarrow 2y^2 + y - 1 = 0 \Rightarrow y = -1, y = 1/2 \Rightarrow (2, -1) \& (\frac{1}{2}, \frac{1}{2})$$

$$\begin{aligned} \text{Area} &= \int_{-1}^{\frac{1}{2}} 1 - y - (2y^2)dy = \left[y - \frac{y^2}{2} - \frac{2y^3}{3} \right]\Bigg|_{-1}^{\frac{1}{2}} \\ &= (\frac{1}{2} - \frac{1}{8} - \frac{1}{12}) - (-1 - \frac{1}{2} + \frac{2}{3}) = \frac{9}{8} \end{aligned}$$

(b) $x = 1 - y^2$ and $x = y^2 - 1$





[1] Please email all corrections and suggestions to these solutions to htor@metu.edu.tr. All solutions are available on the web at the url http://www.metu.edu.tr/~htor.

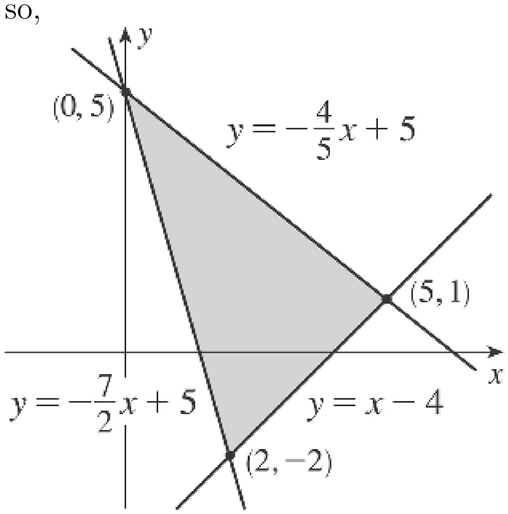
Let's determine the intersection points

$$\begin{cases} x = 1 - y^2 \\ x = y^2 - 1 \end{cases} \Rightarrow 1 - y^2 = y^2 - 1 \Rightarrow y^2 = 1 \Rightarrow y = \mp 1 \Rightarrow (0,-1) \& (0,1)$$

$$\text{Area } = \int_{-1}^{1} 1 - y^2 - (y^2 - 1)dy = \int_{-1}^{1} 2 - 2y^2 dy$$

$$= \left[2y - \frac{2y^3}{3}\right]\Big|_{-1}^{1} = 2[1 - \frac{1}{3} - (-1 + \frac{1}{3})] = \frac{8}{3}$$

3. First we should find the equation of the lines passes through the given three points.
   i)(0,5) and (5,1) $\Rightarrow y = -\frac{4}{5}x + 5$
   ii) (5,1) and (2,-2) $\Rightarrow y = x - 4$
   iii) (0,5) and (2,-2) $\Rightarrow y = -\frac{7}{2}x + 5$
   so,

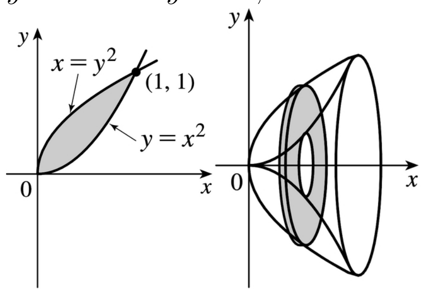


$$\text{Area } = \int_{0}^{2} (-\frac{4}{5}x + 5) - (-\frac{7}{2}x + 5)dx + \int_{2}^{5} (-\frac{4}{5}x + 5) - (-x - 4)dx$$

$$= \int_{0}^{2} \frac{27}{10}xdx + \int_{2}^{5} -\frac{9}{5}x + 9dx = \left[\frac{27}{20}x^2\right]\Big|_{0}^{2} + \left[-\frac{9}{10}x^2 + 9x\right]\Big|_{2}^{5}$$

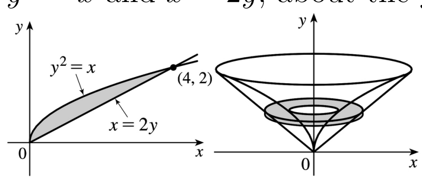
$$= \frac{27}{5} + (-\frac{45}{2} + 45) - (-\frac{18}{5} + 18) = \frac{27}{2}$$

4. (a) $y = x^2$ and $y^2 = x$; about x-axis



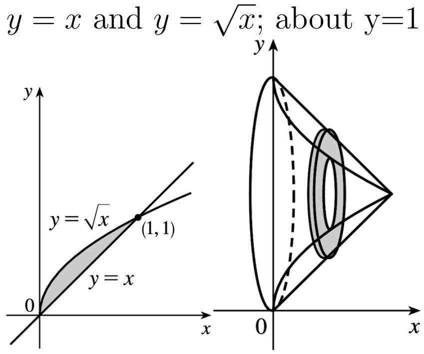
$x = y^2 \Rightarrow y = \sqrt{x}$
$A(x) = \pi(\sqrt{x})^2 - \pi(x^2)^2 = \pi x - \pi x^4$
Volume $= \int_0^1 A(x)dx = \int_0^1 \pi x - \pi x^4 dx = \left[\pi \frac{x^2}{2} - \pi \frac{x^5}{5}\right]\Big|_0^1$
$= \pi \frac{1}{2} - \pi \frac{1}{5} = \frac{3\pi}{10}$

(b) $y^2 = x$ and $x = 2y$; about the y-axis



$A(y) = \pi(2y)^2 - \pi(y^2)^2$
Volume $= \int_0^2 A(y)dy = \pi \int_0^2 4y^2 - y^4 dy = \pi \left[\frac{4}{3}y^3 - \frac{1}{5}y^5\right]\Big|_0^2$
$= \pi(\frac{32}{3} - \frac{32}{5}) = \frac{64\pi}{15}$

(c) $y = x$ and $y = \sqrt{x}$; about y=1



$A(x) = \pi(1-x)^2 - \pi(1-\sqrt{x})^2 = \pi(1 - 2x + x^2 - 1 + 2\sqrt{x} - x) =$
$\pi(x^2 - 3x + 2\sqrt{x}$

$$\text{Volume } = \int_0^1 A(x)dx = \int_0^1 \pi(x^2-3x+2\sqrt{x}dx = \pi \left[\frac{x^3}{3} - \frac{3}{2}x^2 + 2x\right]\Bigg|_0^1$$

$= \pi(\frac{5}{3} - \frac{3}{2}) = \frac{\pi}{6}$

5. (a)

$$V = \int_0^1 \pi(1-\sqrt[3]{y})^2 dy = \pi \int_0^1 (1 - 2y^{1/3} + y^{2/3})dy$$

$$= \pi \left(y - \frac{3}{2}y^{4/3} + \frac{3}{5}y^{5/3}\right)\Bigg|_0^1 = \pi(1 - \frac{3}{2} + \frac{3}{5}) = \frac{\pi}{10}$$

(b)

$$V = \int_0^1 \pi(y^2)^2 dy = \pi \int_0^1 y^4 = \pi \left(\frac{1}{5}y^5\right)\Bigg|_0^1 = \frac{\pi}{5}$$

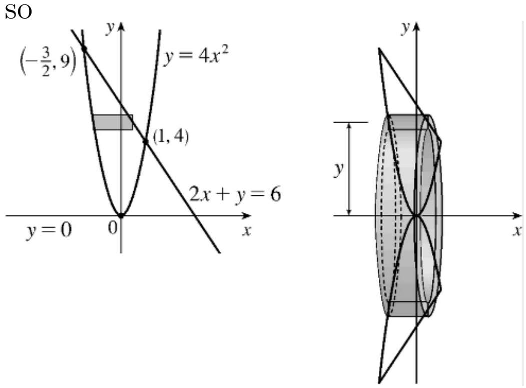
6. Hint: Consider the spheres $x^2+y^2+z^2=r^2$ & $(x-r)^2+y^2+z^2=r^2$. Obviously, the center of each sphere lies on the surface of the other spere. Use the formula from exercise 49 (or directly calculate).

7. $y = 4x^2$ and $2x+y=6$; about the x-axis
Let's find the intersection points of $y = 4x^2$ and $2x + y = 6$

$$\left.\begin{array}{l} y = 4x^2 \\ 2x + y = 6 \end{array}\right\} \Rightarrow 4x^2 = 6 - 2x \Rightarrow x = -\frac{3}{2} \text{ or } x = 1$$

$$\Rightarrow (-\frac{3}{2}, 9) \ \& \ (1,4) \text{ since } y = 6 - 2x$$

so

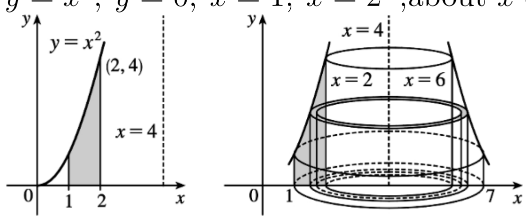


$V_{\text{clinder}} = 2\pi r h \Delta r$

If $y \in (0,4)$,then $V_{\text{cylinder}} = 2\pi y(\frac{1}{2}\sqrt{y} - (-\frac{1}{2}\sqrt{y})\Delta y.$
If $y \in (4,9)$, then $V_{\text{cylinder}} = 2\pi y(-\frac{1}{2}y + 3 - (-\frac{1}{2}\sqrt{y})\Delta y$
Thus,

$$\begin{aligned}
\text{Volume } &= \int_0^4 2\pi y(\frac{1}{2}\sqrt{y} - (-\frac{1}{2}\sqrt{y})dy + \int_4^9 2\pi y(-\frac{1}{2}y + 3 - (-\frac{1}{2}\sqrt{y})dy \\
&= 2\pi(\int_0^4 y\sqrt{y}dy + \int_4^9 (-\frac{1}{2}y^2 + 3y + \frac{1}{2}y^{\frac{3}{2}}dy \\
&= 2\pi[(\frac{2}{5}y^{\frac{5}{2}})\Big|_0^4 + (-\frac{1}{6}y^3 + \frac{3}{2}y^2 + \frac{1}{5}y^{\frac{1}{5}})\Big|_4^9 = \frac{250\pi}{3}
\end{aligned}$$

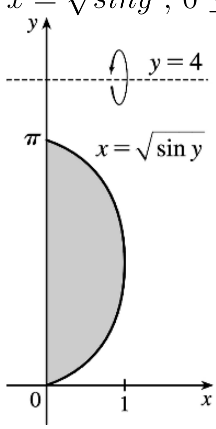
8. $y = x^2,\ y = 0,\ x = 1,\ x = 2$;about $x = 4$



$$V_{\text{clinder}} = 2\pi(4 - x)x^2\Delta x$$

$$\text{Volume } = \int_1^2 2\pi(4-x)x^2dx = 2\pi\left(\frac{4}{3}x^3 - \frac{x^4}{4}\right)\Big|_1^2 = \frac{67}{6}\pi$$

9. $x = \sqrt{siny}$, $0 \leq y \leq \pi$, $x = 0$; about $y = 4$



$$V_{\text{clinder}} = 2\pi(4-y)\sqrt{siny}\Delta y$$
$$\text{Volume } = \int_0^\pi 2\pi(4-y)\sqrt{siny}dy$$

10.

11.

$$f_{ave} = \frac{1}{\pi - 0}\int_0^{\pi} \cos^4 x \sin x dx = \frac{1}{\pi}\int_1^{-1} -u^4 du$$

usinge the subsution $u = \cos x \Rightarrow du = -\sin x dx$

$$\Rightarrow f_{ave} = -\frac{1}{\pi}(\frac{u^5}{5})_1^{-1} = \frac{2}{5\pi}$$

12. i) a)

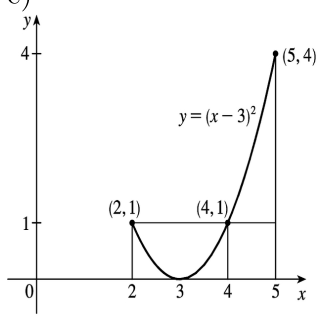
$$f_{ave} = \frac{1}{5-2}\int_2^5 (x-3)^2 dx = \frac{1}{3}\left.(\frac{(x-3)^3}{3})\right|_2^5$$

$$= \frac{1}{9}(8-(-1)) = 1$$

b)

$$f(c) = f_{ave} = 1 \Rightarrow (c-3)^2 = 1 \Rightarrow c = 4 or c = 2$$

c)



ii) a)

$$f_{ave} = \frac{1}{4-0}\int_0^4 \sqrt{x} dx = \left.\frac{1}{4}\frac{2}{3}x^{\frac{3}{2}}\right|_0^4 = \frac{4}{3}$$

b)

$$[f(c) = f_{ave} = \frac{4}{3} \Rightarrow \sqrt{c} = \frac{4}{3} \Rightarrow c = \frac{16}{9}$$

c)