Calculus 119

7th Week[1]

1.

$$
\begin{aligned}
LowerEslimate &= L_5 = \textstyle\sum_{i=1}^{5} \Delta x f(x_{i-1}) = \Delta x(f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4)) \\
&= 5(f(0) + f(5) + f(10) + f(15) + f(20)) \\
&= 5(-42 - 37 - 25 - 6 + 15) = -475 \\
UpperEstimate &= R_5 = \textstyle\sum_{i=1}^{5} \Delta x f(x_i) = \Delta x(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)) \\
&= 5(f(5) + f(10) + f(15) + f(20) f(25)) \\
&= 5(-37 - 25 - 6 + 15 + 36) = -85
\end{aligned}
$$

2.

$$
\begin{aligned}
g(x) &= \textstyle\int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}} dt = \int_{\tan x}^{0} \frac{1}{\sqrt{2+t^4}} dt + \int_{0}^{x^2} \frac{1}{\sqrt{2+t^4}} dt \\
&= -\textstyle\int_{0}^{\tan x} \frac{1}{\sqrt{2+t^4}} dt + \int_{0}^{x^2} \frac{1}{\sqrt{2+t^4}}
\end{aligned}
$$

Let $u = u(x) = \tan x$ and $v = v(x) = x^2$

$$
\begin{aligned}
\Rightarrow g(x) &= -\textstyle\int_{0}^{u} \frac{1}{\sqrt{2+t^4}} dt + \int_{0}^{v} \frac{1}{\sqrt{2+t^4}} \\
\Rightarrow g'(x) &= -\tfrac{1}{\sqrt{2+u^4}} u'(x) + \tfrac{1}{\sqrt{2+v^4}} v'(x) \\
&= -\tfrac{1}{\sqrt{2+(\tan x)^4}} \sec^2 x + \tfrac{1}{\sqrt{2+(x^2)^4}} 2x \\
&= -\tfrac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \tfrac{2x}{\sqrt{2+(x)^8}}
\end{aligned}
$$

3.

$$
I = \lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + ... + \sqrt{\frac{n}{n}}\right) =?
$$

[1] Please email all corrections and suggestions to these solutions to htor@metu.edu.tr. All solutions are available on the web at the url http://www.metu.edu.tr/∼htor.

Consider $\Delta x = \frac{1}{n}, x_i = 0 + i\Delta x = \frac{i}{n}$ and $f(x) = \sqrt{x}$

$$I = \int_0^1 \sqrt{x}dx = \left.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 = \frac{2}{3}(1^{\frac{3}{2}} - 0) = \frac{2}{3}$$

4. <u>First Way</u>
Let $u = 2\theta \Rightarrow du = 2d\theta$

$$\int \sec 2\theta \tan 2\theta d\theta = \int \sec u \tan u \frac{du}{2} = \frac{1}{2}\int \sec u \tan u d\theta = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2\theta + C$$

<u>Second Way</u>

$$I = \int \sec 2\theta \tan 2\theta d\theta = \int \frac{1}{\cos 2\theta}\frac{\sin 2\theta}{\cos 2\theta}d\theta$$

Let $\cos 2\theta = u \Rightarrow -2\sin 2\theta d\theta = du$

$$\Rightarrow I = \int \frac{-\frac{du}{2}}{u^2} = -\frac{1}{2}\frac{u^{-1}}{-1} + C = \frac{1}{2u} + C = \frac{1}{2\cos 2\theta} + C = \frac{1}{2}\sec 2\theta + C$$

5. Let $u = x^3 + 1 \Rightarrow du = 3x^2dx$ and $x^3 = u - 1$

$$\int_0^2 \sqrt[3]{x^3+1}\, x^5 dx = \int_0^2 \sqrt[3]{x^3+1}\, x^3x^2dx = \int_1^9 \sqrt[3]{u}\,(u-1)\frac{du}{3}$$

$$= \frac{1}{3}\int_1^9 u^{\frac{4}{3}} - u^{\frac{1}{3}}du = \left.\frac{1}{3}\left(\frac{u^{\frac{7}{3}}}{\frac{7}{3}} - \frac{u^{\frac{4}{3}}}{\frac{4}{3}}\right)\right|_1^9 = \left.\frac{u^{\frac{7}{3}}}{7} - \frac{u^{\frac{4}{3}}}{4}\right|_1^9$$

$$= \frac{9^{\frac{7}{3}}}{7} - \frac{9^{\frac{4}{3}}}{4} - \left(\frac{1^{\frac{7}{3}}}{7} - \frac{1^{\frac{4}{3}}}{4}\right)$$

6. <u>First Way</u>
Let $f(x) = \frac{x^2 \sin x}{1+x^6}$.

$$I = \int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}dx = \int_{-\pi/2}^{0} \frac{x^2\sin x}{1+x^6} + \int_0^{\pi/2}\frac{x^2\sin x}{1+x^6}dx = -\int_0^{-\pi/2}\frac{x^2\sin x}{1+x^6} + \int_0^{\pi/2}\frac{x^2\sin x}{1+x^6}dx$$

Let's use the subsitution $t = -x$ for first integral($\Rightarrow dt = -dx$)

$$I = -\int_0^{\pi/2}\frac{(-t)^2\sin(-t)}{1+(-t)^6}(-dt) + \int_0^{\pi/2}\frac{x^2\sin x}{1+x^6}dx = -\int_0^{\pi/2}\frac{t^2\sin t}{1+t^6}dt + \int_0^{\pi/2}\frac{x^2\sin x}{1+x^6}dx = 0$$

Second Way

Observe $f(-x) = -f(x)$(i.e. f is odd.)

$$f(-x) = \frac{(-x)^2 \sin(-x)}{1 + (-x)^6} = \frac{x^2(-\sin x)}{1 + x^6} = -\frac{x^2 \sin x}{1 + x^6} = -f(x)$$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6} dx = 0 \quad \text{by Theorem 6 part (b) on the page 364}$$