

Calculus 119
4th Week¹

1.

$$f(x) = \begin{cases} -x^2 + 1 & \text{if } -1 < x < 1 \\ x^2 - 1 & \text{otherwise} \end{cases}$$

$f(x)$ is differentiable everywhere except the points -1 and 1, and the derivative of $f(x)$ is given below;

$$f'(x) = -2x \quad \text{when } -1 < x < 1 \quad f'(x) = 2x \quad \text{when } x < -1 \text{ and } x > 1$$

Does $f'(x)$ exist at the points -1 and 1? If it exists, what?

At the point -1

$$\left. \begin{aligned} \lim_{x \rightarrow -1^+} \frac{f(x) - f(-1)}{x + 1} &= \lim_{x \rightarrow -1^+} \frac{-x^2 + 1 - 0}{x + 1} = \lim_{x \rightarrow -1^+} 1 - x = 2 \\ \lim_{x \rightarrow -1^-} \frac{f(x) - f(-1)}{x + 1} &= \lim_{x \rightarrow -1^-} \frac{x^2 - 1 - 0}{x + 1} = \lim_{x \rightarrow -1^-} x - 1 = -2 \end{aligned} \right\}$$

$$\Rightarrow \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x + 1} \quad \text{does not exist.}$$

At the point 1

$$\left. \begin{aligned} \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^+} \frac{x^2 - 1 - 0}{x - 1} = \lim_{x \rightarrow 1^+} x + 1 = 2 \\ \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} &= \lim_{x \rightarrow 1^-} \frac{-x^2 + 1 - 0}{x - 1} = \lim_{x \rightarrow 1^-} -(1 + x) = -2 \end{aligned} \right\}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} \quad \text{does not exist.}$$

Since $f'(x)$ does not exist at the points -1 and 1, The point -1 and 1 are critical points (or numbers).

Is there other critical point?

When $x < -1$ and $x > 1$, $f'(x) = 2x = 0 \Rightarrow x = 0$. Since $x \notin (-\infty, -1) \cup (1, \infty)$, $x = 0$ is not critical point. On the other hand,

¹Please email all corrections and suggestions to these solutions to htor@metu.edu.tr. All solutions are available on the web at the url <http://www.metu.edu.tr/~htor>.

When $-1 < x < 1$, $f'(x) = -2x = 0 \Rightarrow x = 0$. Then $x = 0$ is an other critical point of $f(x)$

Consequently, All critical points of $f(x)$ are -1, 0 and 1.

2. (a)

$$g(x) = \begin{cases} -2x - 3 & \text{if } -\frac{3}{2} < x \\ 2x + 3 & \text{if } -\frac{3}{2} > x \end{cases}$$

Assume $x \neq -\frac{3}{2}$

$$f'(x) = 2 \neq 0 \Rightarrow f \text{ has no critical point.}$$

Does $f'(x)$ exist?

$$\left. \begin{aligned} \lim_{x \rightarrow -\frac{3}{2}^+} \frac{f(x) - f(-\frac{3}{2})}{x + \frac{3}{2}} &= \lim_{x \rightarrow -\frac{3}{2}^+} \frac{2x + 3 - 0}{x + \frac{3}{2}} = 2 \\ \lim_{x \rightarrow -\frac{3}{2}^-} \frac{f(x) - f(-\frac{3}{2})}{x + \frac{3}{2}} &= \lim_{x \rightarrow -\frac{3}{2}^-} \frac{-2x - 3 - 0}{x + \frac{3}{2}} = -2 \end{aligned} \right\}$$

$$\Rightarrow \lim_{x \rightarrow -\frac{3}{2}^+} \frac{f(x) - f(-\frac{3}{2})}{x + \frac{3}{2}} \text{ does not exist.}$$

$\Rightarrow x = -\frac{3}{2}$ is critical point of $f'(x)$. Thus, $f(x)$ has only one critical point, namely $-\frac{3}{2}$.

(b)

$$f'(x) = 2(-\sin\theta) + 2\sin\theta\cos\theta$$

$$f'(x) = 0 \Rightarrow -\sin\theta + \sin\theta\cos\theta = 0$$

$$\Rightarrow \begin{cases} \sin\theta = 0 \\ \cos\theta = 1 \end{cases} \Rightarrow \begin{cases} \theta = k\pi \\ \theta = 2k\pi \end{cases} \text{ for any integer } k \text{ are critical point of } f(x)$$

3. First we have to find critical points.

$$f'(x) = \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow x = \frac{\pi}{4} \in (0, \frac{\pi}{3})$$

$\Rightarrow x = \frac{\pi}{4}$ is critical point.

$$f(\frac{\pi}{4}) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \sqrt{2} \text{ (For Critical Point } \frac{\pi}{4})$$

$$f(0) = \sin 0 + \cos 0 = 1 \text{ (For End Point 0)}$$

$$f(\frac{\pi}{3}) = \sin\frac{\pi}{3} + \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{\sqrt{3}+1}{2} \text{ (For End Point } \frac{\pi}{3})$$

Since $1 < \frac{\sqrt{3}+1}{2} < \sqrt{2}$, the absolute maximum value of $f(x)$ is $\sqrt{2}$ and the absolute minimum value of $f(x)$ is 1.

4. Consider $f(x) = 2x - 1 - \sin x$

Existence of a root

Since $f(x)$ is continuous on $(0, \frac{\pi}{2})$, we can apply the intermediate value theorem

$$\left. \begin{array}{l} f(0) = -1 < 0 \\ f(\frac{\pi}{2}) = \pi - 2 > 0 \end{array} \right\} \Rightarrow f \text{ has a root on } (0, \frac{\pi}{2}) \text{ by IVT}$$

Uniqueness of a root on $(0, \frac{\pi}{2})$.

Assume $f(x_1) = f(x_2) = 0$ where $x_1 \neq x_2 \in (0, \frac{\pi}{2})$ (i.e f has two different real root)

Since $f(x)$ is continuous on $[0, \frac{\pi}{2}]$, $f(x)$ is differentiable on $(0, \frac{\pi}{2})$ and $f(x_1) = f(x_2) = 0$, we can apply the Rolle's theorem.

Thus, there exists a root of $f'(x)$ on $(0, \frac{\pi}{2})$ by Rolle's Theorem.

On the other hand $f'(x) = 2 - \cos x > 0 \Rightarrow f'(x)$ can not have any root. It gives contradiction, so our assumption is wrong, that is f has exactly one real root.

5. f has local maximum at critical numbers 1, 5. f has local minimum at a critical number 3 and end points 0 and 6.

6. (a) $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x - 1)(x + 1) = 0 \Rightarrow x = 0, x = 1$ and $x = -1$ are critical point.

$$\begin{array}{l} f'(x) < 0 \text{ on } (-\infty, -1) \\ f'(x) > 0 \text{ on } (-1, 0) \\ f'(x) < 0 \text{ on } (0, 1) \\ f'(x) > 0 \text{ on } (0, \infty) \end{array} \Rightarrow \begin{array}{l} f(x) \text{ is increasing on } (-1, 0) \text{ and } (1, \infty) \\ f(x) \text{ is decreasing on } (-\infty, -1) \text{ and } (0, 1) \end{array}$$

(b) Local Maximum Values: $f(-1) = 2$ and $f(1) = 2$

Local Minimum Value: $f(0) = 3$

(c) In order to determine concavity and the inflection points, we have to find second derivative of $f(x)$

$$f''(x) = 12x^2 - 4$$

$$f''(x) = 0 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$$

$$\begin{array}{l} f(x) \text{ is concave upward on } (-\infty, -\frac{1}{\sqrt{3}}) \text{ since } f''(x) > 0 \text{ on } (-\infty, -\frac{1}{\sqrt{3}}) \\ \Rightarrow f(x) \text{ is concave downward on } (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \text{ since } f''(x) < 0 \text{ on } (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \\ f(x) \text{ is concave upward on } (\frac{1}{\sqrt{3}}, \infty) \text{ since } f''(x) > 0 \text{ on } (\frac{1}{\sqrt{3}}, \infty) \end{array}$$

Inflection Points:

Since concavity of $f(x)$ changes at the point $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$, all inflection points of $f(x)$ are $-\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}$.

7. (a)

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} = \lim_{x \rightarrow -\infty} \frac{|x^3| \sqrt{9 - \frac{1}{x^5}}}{x^3(1 + \frac{1}{x^3})} = -\frac{\sqrt{9}}{1} = -3$$

(b)

$$\begin{aligned} \lim_{x \rightarrow -\infty} (x + \sqrt{x^2 + 2x}) &= \lim_{x \rightarrow -\infty} \frac{(x + \sqrt{x^2 + 2x})(x - \sqrt{x^2 + 2x})}{x - \sqrt{x^2 + 2x}} = \lim_{x \rightarrow -\infty} \frac{x^2 - (x^2 + 2x)}{x - \sqrt{x^2 + 2x}} \\ &= \lim_{x \rightarrow -\infty} \frac{-2x}{x - |x|\sqrt{1 + \frac{2}{x}}} = \lim_{x \rightarrow -\infty} \frac{-2x}{x(1 + \sqrt{1 + \frac{2}{x}})} = \frac{-2}{1 + \sqrt{1}} = -1 \end{aligned}$$

(c)

$$\begin{aligned} \lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) &= \lim_{t \rightarrow 0^+} \frac{1}{t} \sin t \quad \text{where } t = \frac{1}{x} \text{ and } t \rightarrow 0^+ \text{ as } x \rightarrow \infty \\ &= 0 \end{aligned}$$

8. Recall: The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L.$$

Since $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$, $y = 0$ is a horizontal asymptote of the curve $f(x)$.

$y'(x) = \frac{-x^2+1}{(x^2+1)^2}$. If $f'(x) = 0$, then $x = \pm 1$ are critical numbers.

$y''(x) = \frac{2x^3-6x}{(x^2+1)^3} = 0 \Rightarrow x = 0, x = \pm\sqrt{3}$

Examining concave downward and upward intervals and increasing decreasing intervals, we obtain the following graph.

