

Math 119
Week 2
Solutions¹

1. Guessing δ

For given $\epsilon > 0$, we want to find $\delta > 0$ such that

$$|(2x + 3) - 5| < \epsilon \text{ whenever } |x - 1| < \delta$$

$$\Rightarrow |2(x - 1)| < \epsilon \text{ whenever } |x - 1| < \delta$$

$$\Rightarrow |x - 1| < \frac{\epsilon}{2} \text{ whenever } |x - 1| < \delta$$

Thus, we can choose $\delta = \frac{\epsilon}{2}$

Showing that this δ works

$$\forall \epsilon > 0 \exists \delta = \frac{\epsilon}{2} \text{ such that}$$

$$|(2x + 3) - 5| < \epsilon \text{ whenever } |x - 1| < \delta = \frac{\epsilon}{2}$$

$$|(2x + 3) - 5| = |2x - 2| = 2|x - 1| < 2 \cdot \frac{\epsilon}{2} = \epsilon$$

2. Guessing δ

For given M , we want to find $\delta > 0$ such that

$$\left| \frac{1}{(x + 3)^4} \right| > M \text{ whenever } |x - (-3)| < \delta$$

$$\Rightarrow |(x + 3)^4| < \frac{1}{M} \text{ whenever } |x + 3| < \delta$$

$$\Rightarrow |x + 3| < \sqrt[4]{\frac{1}{M}} \text{ whenever } |x + 3| < \delta$$

so, we can choose $\delta = \sqrt[4]{\frac{1}{M}}$

Showing that this δ works

$$\forall \epsilon > 0 \exists \delta = \sqrt[4]{\frac{1}{M}} \text{ such that}$$

$$\left| \frac{1}{(x + 3)^4} \right| > M \text{ whenever } |x - (-3)| < \delta = \sqrt[4]{\frac{1}{M}}$$

$$\left| \frac{1}{(x + 3)^4} \right| > \frac{1}{\delta^4} = \frac{1}{(\sqrt[4]{\frac{1}{M}})^4} = M$$

3. We want that $f(x)$ is continuous on $(-\infty, \infty)$. Obviously $f(x)$ is continuous on $(-\infty, 3)$ and $(3, \infty)$, because $f(x)$ is defined as a polynomial on the ray $(-\infty, 3)$ or $(3, \infty)$. Thus, it is enough that $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = f(3)$ to make it continuous.

$$\left. \begin{array}{l} \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (cx^2 - 1) = 9c - 1 \\ \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (cx + 1) = 3c + 1 \\ f(3) = 3c + 1 \end{array} \right\} \implies 9c - 1 = 3c + 1 \implies c = \frac{1}{3}$$

4. $[-4, -2), (-2, 2), [2, 4), (4, 6)$ and $(6, 8)$

¹Please email all corrections and suggestions to these solutions to htor@metu.edu.tr. All solutions are available on the web at the url <http://www.metu.edu.tr/~htor>.

5. **Recall(Intermediate value theorem):** Let $f(x)$ is continuous on close interval $[a,b]$ and N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$

Since the given function is continuous on $(1,2)$, we can use IVT. So,

$$\text{Let } f(x) = x^4 + x - 3$$

$$\left. \begin{array}{l} f(1) = -1 < 0 \\ f(2) = 15 > 0 \end{array} \right\} \implies \exists c \in (1, 2) \text{ s.t. } f(c) = 0 \text{ because } f(1) < 0 < f(2)$$

by IVT.

6. The slope m of the tangent line to the curve $y = \sqrt{2x+1}$ at the point

$$(4, 3) \text{ is } \lim_{x \rightarrow 4} \frac{\sqrt{2x+1} - \sqrt{9}}{x - 4}$$

$$m = \lim_{x \rightarrow 4} \frac{(\sqrt{2x+1} - 3)(\sqrt{2x+1} + 3)}{(x - 4)(\sqrt{2x+1} + 3)} = \lim_{x \rightarrow 4} \frac{(2x + 1) - 9}{(x - 4)(\sqrt{2x+1} + 3)} = \lim_{x \rightarrow 4} \frac{2}{\sqrt{2x+1} + 3}$$

$= \frac{1}{3}$ Thus, the equation of the tangent line to the curve at point $(4,3)$ is

$$y - 3 = \frac{1}{3}(x - 4).$$

7. sketch

$$\begin{aligned} 8. f'(a) &= \lim_{t \rightarrow a} \frac{\frac{2t+1}{t+3} - \frac{2a+1}{a+3}}{t - a} \\ &= \lim_{x \rightarrow a} \frac{2at + 6t + a + 3 - 2at - 6a - t - 3}{t - a} \\ &= \lim_{x \rightarrow a} \frac{5t - 5a}{(t - a)(t + 3)(a + 3)} \\ &= \lim_{x \rightarrow a} \frac{5}{(t + 3)(a + 3)} \\ &= \frac{5}{(a + 3)^2} \end{aligned}$$

9. sketch

$$\begin{aligned} 10. f'(a) &= \lim_{x \rightarrow a} \frac{1 - 3x^2 - 1 + 3a^2}{x - a} \\ &= \lim_{x \rightarrow a} -3(a + x) = -6a \end{aligned}$$

Since both $f(x)$ and $f'(x)$ are polynomials, their domain are \mathfrak{R} .