Partial fractions, improper integrals

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Problem 1. Compute the following indefinite integrals.

(a) $\displaystyle\int \frac{x^3 - x}{x^2 - x - 6}\,\mathrm{d}\,x$

(b) $\displaystyle\int \frac{x}{x^2 + x + 1}\,\mathrm{d}\,x$

(c) $\displaystyle\int \frac{x}{(x^2 + x + 1)^2}\,\mathrm{d}\,x$

(d) $\displaystyle\int \frac{x^4 - 3x^3 + 4x^2 - 3x + 1}{x^4 + 3x^3 + 4x^2 + 3x + 1}\,\mathrm{d}\,x$

(e) $\displaystyle\int \frac{4x^3 + 9x^2 + 8x + 3}{x^4 + 3x^3 + 4x^2 + 3x + 1}\,\mathrm{d}\,x$

Problem 2. Assuming $\deg f < 8$, analyse

$$\frac{f(x)}{(x^2 + 4x - 5)^2 \cdot (x^2 + 4x + 5)^2}$$

as a sum of partial fractions. (No need to compute the numerators.)

Problem 3. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^{32} \frac{\mathrm{d}\,x}{1 + x^{1/5}}$

(b) $\displaystyle\int_0^{\pi/2} \frac{\mathrm{d}\,x}{2 + \sin x}$

Problem 4. Evaluate, if possible, the following improper integrals.

(a) $\displaystyle\int_{-1}^{1} \frac{\mathrm{d}\,x}{x^3}$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}\,x}{x^3 \cdot \mathrm{e}^{1/x^2}}$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}\,x}{|x^3| \cdot \mathrm{e}^{1/x^2}}$

Problem 5. Determine whether the following improper integrals converge.

(a) $\displaystyle\int_0^{\infty} \frac{\sin(x^3)}{x + \mathrm{e}^x}\,\mathrm{d}\,x$

(b) $\displaystyle\int_2^{\infty} \frac{\mathrm{d}\,x}{x^2 - x^{-2}}$

(c) $\displaystyle\int_1^{\infty} \frac{\mathrm{d}\,x}{x^2 - x^{-2}}$

(d) $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x + x^3}}$

e) $\displaystyle\int_2^{\infty} \frac{dx}{\ln x}$

Rec 12

① a) $\displaystyle\int \frac{x^3 - x}{x^2 - x - 6}\,dx = ?$

$$\begin{array}{r|l} x^3 - x & x^2 - x - 6 \\ \cline{2-2} -\,x^3 - x^2 - 6x & x + 1 \\ \hline x^2 + 5x & \\ -\,x^2 - x - 6 & \\ \hline 6x + 6 & \end{array}$$

$$\Rightarrow \frac{x^3 - x}{x^2 - x - 6} = x + 1 + \frac{6x + 6}{x^2 - x - 6}$$

$$= x + 1 + 6\,\frac{(x+1)}{x^2 - x - 6}$$

Now $\displaystyle\frac{x+1}{x^2 - x - 6} = \frac{x+1}{(x-3).(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)}$

$$\Rightarrow A = \frac{4}{5} \qquad B = \frac{1}{5}$$

So $\displaystyle\int \frac{x^3 - x}{x^2 - x - 6}\,dx = \int (x+1)\,dx + 6\int \frac{4/5}{x-3}\,dx + 6\int \frac{1/5}{x+2}\,dx$

$$= x^2 + x + \frac{24}{5}\ln|x-3| + \frac{6}{5}\ln|x+2| + C$$

b) $\displaystyle\int \frac{x}{x^2 + x + 1}\,dx = \frac{1}{2}\int \frac{2x}{x^2 + x + 1}\,dx = \frac{1}{2}\int \frac{2x + 1 - 1}{x^2 + x + 1}\,dx$

$$= \frac{1}{2}\left[\int \frac{2x + 1}{x^2 + x + 1}\,dx - \int \frac{1}{x^2 + x + 1}\,dx\right]$$

$$= \frac{1}{2}\left[\ln|x^2 + x + 1| - \int \frac{1}{(x + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\,dx\right.$$

$$= \frac{1}{2}\left[\ln|x^2 + x + 1| - \frac{2}{\sqrt{3}}\arctan\frac{2(x + \frac{1}{2})}{\sqrt{3}}\right] + C$$

Recall that $\displaystyle\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C.$

c) $\int \frac{x}{(x^2+x+1)^2}\,dx = ?$ Note that

$$x^2+x+1 = (x+\tfrac{1}{2})^2 + (\tfrac{\sqrt{3}}{2})^2$$

$$\int \frac{x}{[(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2]^2}\,dx$$

Recall that $\int \frac{1}{x^2+a^2}\,dx \to$ we use $x = a\tan u$ transformation

then.

$$\int \frac{x}{[(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2]^2}\,dx = \int \frac{\frac{\sqrt{3}}{2}\tan u - \frac{1}{2}}{[(\frac{\sqrt{3}}{2}\tan u)^2+(\frac{\sqrt{3}}{2})^2]^2}\cdot\frac{\sqrt{3}}{2}\sec^2 u\,du$$

$$\left(\begin{array}{l} x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan u \to x = \frac{\sqrt{3}}{2}\tan u - \frac{1}{2} \\ dx = \frac{\sqrt{3}}{2}\sec^2 u\,du \end{array} \right)$$

$$= \int \frac{\frac{\sqrt{3}}{2}\tan u}{[\frac{3}{4}\tan^2 u + \frac{3}{4}]^2}\,\frac{\sqrt{3}}{2}\sec^2 u\,du - \frac{1}{2}\int \frac{\sqrt{3}/2\,\sec^2 u\,du}{[\frac{3}{4}\tan^2 u+\frac{3}{4}]^2}$$

$$= \frac{3}{4}\int \frac{\tan u}{[\frac{3}{4}(\underbrace{\tan^2 u+1}_{\sec^2 u})]^2}\sec^2 u\,du - \frac{\sqrt{3}}{4}\int \frac{\sec^2 u}{[\frac{3}{4}(\underbrace{\tan^2 u+1}_{\sec^2 u})]^2}\,du$$

$$= \frac{3}{4}\int \frac{\tan u}{\frac{9}{16}\sec^{\cancel{4}} u}\cancel{\sec^2 u}\,du - \frac{\sqrt{3}}{4}\int \frac{\cancel{\sec^2 u}}{\frac{9}{16}\sec^{\cancel{4}} u}\,du$$

$$= \frac{4}{3}\int \frac{\tan u}{\sec^2 u}\,du - \frac{4}{3\sqrt{3}}\int \frac{1}{\sec^2 u}\,du$$

$$= \frac{4}{3}\int \frac{\frac{\sin u}{\cancel{\cos u}}}{\frac{1}{\cos^{\cancel{2}} u}}\,du - \frac{4}{3\sqrt{3}}\int \frac{1}{\frac{1}{\cos^2 u}}\,du$$

$$\boxed{\begin{array}{l} \int \sin u\cos u\,du = \int v\,dv = \frac{v^2}{2}+C = \frac{\sin^2 u}{2}+C \\ \sin u = v \\ \cos u\,du = dv \\ \int \cos^2 u\,du = \int \frac{\cos 2u+1}{2}\,du \\ = \int \frac{\cos 2u}{2}+\frac{1}{2}\,du \\ = \frac{\sin 2u}{4}+\frac{u}{2}+C \end{array}}$$

$$= \frac{4}{3}\int \sin u\cos u\,du - \frac{4}{3\sqrt{3}}\int \cos^2 u\,du = \frac{{}^2\cancel{4}}{3}\left[\frac{\sin^2 u}{\cancel{2}}\right] - \frac{4}{3\sqrt{3}}\left[\frac{\sin 2u}{4}+\frac{u}{2}\right] + C$$

$$\boxed{u = \arctan\frac{2}{\sqrt{3}}(x+\tfrac{1}{2})}$$

$$= \frac{2}{3}\left[\sin^2(\arctan\tfrac{2}{\sqrt{3}}(x+\tfrac{1}{2})\right] - \frac{1}{3\sqrt{3}}\sin 2\arctan\tfrac{2}{\sqrt{3}}(x+\tfrac{1}{2}) - \frac{2}{3\sqrt{3}}\arctan\tfrac{2}{\sqrt{3}}(x+\tfrac{1}{2}) + C.$$

e) $\int \frac{4x^3+9x^2+8x+3}{x^4+3x^3+4x^2+3x+1}\,dx = ?$

Recall that $\int \frac{1}{u}\,du = \ln|u| + c$

then $\int \frac{4x^3+9x^2+8x+3}{x^4+3x^3+4x^2+3x+1}\,dx = \int \frac{du}{u} = \ln|u| + c$

$= \ln|x^4+3x^3+4x^2+3x+1| + c$

Let $u = x^4+3x^3+4x^2+3x+1$

$du = (4x^3+9x^2+8x+3)\,dx$

② $\deg f < 8$

$$\frac{f(x)}{(x^2+4x-5)^2\cdot(x^2+4x+5)^2} = \frac{f(x)}{(x+5)^2(x-1)^2\cdot(x^2+4x+5)^2}$$

$(x^2+4x-5) \to (x+5)\cdot(x-1) \to$ product of linear factor (repeated)

$(x^2+4x+5)^2 \to$ repeated irreducible quadratic factor

$$= \frac{A}{(x+5)} + \frac{B}{(x+5)^2} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2} + \frac{Ex+F}{x^2+4x+5}$$

$$+ \frac{Gx+H}{(x^2+4x+5)^2}$$

③ a) $\displaystyle\int_0^{32} \frac{dx}{1+x^{1/5}} = ?$

Let $x^{\frac{1}{5}} = u \Rightarrow x = u^5$ boundary: when $x=0 \Rightarrow u=0$

$dx = 5u^4\,du.$ when $x=32 \Rightarrow u=2.$

Now $\displaystyle\int_0^2 \frac{5u^4\,du}{1+u} = 5\int_0^2 \frac{u^4}{1+u}\,du.$

$$= 5\left[\int_0^2 (u^3-u^2+u-1) + \frac{1}{u+1}\,du\right]$$

$$= 5\left[\frac{u^4}{4} - \frac{u^3}{3} + \frac{u^2}{2} - u + \ln|u+1| \Big|_0^2\right]$$

$$= 5\left[\ln 3 + \frac{4}{3}\right]$$

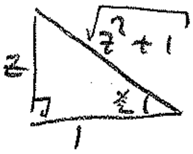
$$\begin{array}{r|l} u^4 & u+1 \\ \hline -u^4+u^3 & u^3-u^2+u-1 \\ -u^3 & \\ -u^3-u^2 & \\ +u^2 & \\ -u^2+u & \\ -u & \\ -u-1 & \\ 1 & \end{array}$$

$$\Rightarrow \boxed{\frac{u^4}{u+1} = u^3-u^2+u-1+\frac{1}{u+1}}$$

3 b) $$\int_0^{\pi/2} \frac{dx}{2+\sin x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{2+2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$z = \tan\frac{x}{2}$$



$$= \int_0^{\frac{\pi}{2}} \frac{dx}{2+2\cdot\tan\frac{x}{2}\cdot\cos^2\frac{x}{2}}$$

$$\cos\frac{x}{2} = \frac{1}{\sqrt{z^2+1}}$$

$$\cos^2\frac{x}{2} = \frac{1}{z^2+1}$$

$$= \int_0^1 \frac{\frac{\cancel{2}\,dz}{1+z^2}}{\cancel{2}+\cancel{2}\cdot z\cdot\frac{1}{z^2+1}}$$

$$dz = \left(1+\tan^2\frac{x}{2}\right)\cdot\frac{1}{2}\,dx$$

$$dz = \frac{(1+z^2)}{2}\,dx$$

$$dx = \frac{2\,dz}{1+z^2}$$

$$= \int_0^1 \frac{dz}{z^2+z+1}$$

$$= \int_0^1 \frac{dz}{(z+\frac{1}{2})^2+\frac{3}{4}} = \int_0^1 \frac{dz}{(z+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\arctan\frac{2(z+\frac{1}{2})}{\sqrt{3}}\Bigg|_0^1 = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

(Recall that $\int_0^1 \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + c$)

④ a) $\displaystyle\int_{-1}^{1} \frac{dx}{x^3}$ → undefined at $x=0$ so improper integral.

$$\Rightarrow \int_{-1}^{0} \frac{dx}{x^3} + \int_{0}^{1} \frac{dx}{x^3}$$

$$= \lim_{A \to 0^-} \int_{-1}^{A} \frac{dx}{x^3} + \lim_{B \to 0^+} \int_{B}^{1} \frac{dx}{x^3}$$

$$= \lim_{A \to 0^-} \int_{-1}^{A} x^{-3}\, dx + \lim_{B \to 0^+} \int_{B}^{1} x^{-3}\, dx$$

$$= \lim_{A \to 0^-} \left[\frac{x^{-2}}{-2} \Big|_{-1}^{A} \right] + \lim_{B \to 0^+} \left[\frac{x^{-2}}{-2} \Big|_{B}^{1} \right]$$

$$= \lim_{A \to 0^-} \left[\frac{1}{-2A^2} + \frac{1}{2} \right] + \lim_{B \to 0^+} \left[-\frac{1}{2} + \frac{1}{2B^2} \right]$$

Note that $\displaystyle\lim_{A \to 0^-} -\frac{1}{2(A)^2} + \frac{1}{2} = -\infty$ so

$\displaystyle\int_{-1}^{0} \frac{dx}{x^3}$ → divergent $\Rightarrow$ $\displaystyle\int_{-1}^{1} \frac{dx}{x^3}$ also divergent.

b) $I = \int_{-\infty}^{\infty} \frac{dx}{x^3 \cdot e^{1/x^2}}$ → improper on boundary and at $x=0$

$$I = \int_{-\infty}^{-1} \frac{dx}{x^3 \cdot e^{1/x^2}} + \int_{-1}^{0} \frac{dx}{x^3 \cdot e^{1/x^2}} + \int_{0}^{1} \frac{dx}{x^3 \cdot e^{1/x^2}} + \int_{1}^{\infty} \frac{dx}{x^3 \cdot e^{1/x^2}}$$

↓ improper ↓ improper ↓ improper ↓ improper

Note that $\int \frac{dx}{x^3 \cdot e^{1/x^2}} = -\frac{1}{2} \int \frac{du}{e^u} = -\frac{1}{2} \int e^{-u}\, du$

$\frac{1}{x^2} = u$ $\qquad = \frac{1}{2} e^{-u} + c$

$-\frac{2}{x^3} dx = du$ $\qquad = \frac{1}{2} e^{-1/x^2} + c$

i) $\int_{-\infty}^{-1} \frac{dx}{x^3 \cdot e^{1/x^2}} = \lim_{R \to -\infty} \int_{R}^{-1} \frac{dx}{x^3 e^{1/x^2}} = \lim_{R \to -\infty} \left(\frac{1}{2} e^{-1/x^2} \Big|_{R}^{-1} \right)$

$= \lim_{R \to -\infty} \left(\frac{1}{2} e^{-1} - \frac{1}{2} e^{-1/R^2} \right) = \frac{1}{2} e^{-1} - \frac{1}{2}$

ii) $\int_{-1}^{0} \frac{dx}{x^3 e^{1/x^2}} = \lim_{R \to 0^-} \int_{-1}^{R} \frac{dx}{x^3 e^{1/x^2}} = \lim_{R \to 0^-} \left(\frac{1}{2} e^{-1/x^2} \Big|_{-1}^{R} \right) = -\frac{1}{2} e^{-1}$

$\frac{1}{2}(e^{-1/R^2} - e^{-1})$, $e^{-1/R^2} \to 0$

iii) $\int_{0}^{1} \frac{dx}{x^3 e^{1/x^2}} = \lim_{R \to 0^+} \int_{R}^{1} \frac{dx}{x^3 e^{1/x^2}} = \lim_{R \to 0^+} \left(\frac{1}{2} e^{-1/x^2} \Big|_{R}^{1} \right) = \frac{1}{2} e^{-1}$

$\frac{1}{2}(e^{-1} - e^{-1/R^2})$, $e^{-1/R^2} \to 0$

iv) $\int_{1}^{\infty} \frac{dx}{x^3 e^{1/x^2}} = \lim_{R \to \infty} \int_{1}^{R} \frac{dx}{x^3 e^{1/x^2}} = \lim_{R \to \infty} \left(\frac{1}{2} e^{-1/x^2} \Big|_{1}^{R} \right) = \frac{1}{2} - \frac{1}{2} e^{-1}$

$\frac{1}{2}(e^{-1/R^2} - e^{-1})$

So $I = \cancel{\frac{1}{2} e^{-1}} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2} e^{-1}} + \cancel{\frac{1}{2} e^{-1}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2} e^{-1}} = 0$

c) $$\int_{-\infty}^{\infty} \frac{dx}{|x^3| e^{1/x^2}} = \int_{-\infty}^{-1} \frac{dx}{-x^3 e^{1/x^2}} + \int_{-1}^{0} \frac{dx}{-x^3 e^{1/x^2}} + \int_{0}^{1} \frac{dx}{x^3 e^{1/x^2}} + \int_{1}^{\infty} \frac{dx}{x^3 \cdot e^{1/x^2}}$$

$$= -\underbrace{\int_{-\infty}^{-1} \frac{dx}{x^3 e^{1/x^2}}}_{\substack{\downarrow \text{ from part (b)} \\ \frac{1}{2}e^{-1} - \frac{1}{2}}} - \underbrace{\int_{-1}^{0} \frac{dx}{x^3 e^{1/x^2}}}_{\substack{\downarrow \text{ from part (b)} \\ -\frac{1}{2}e^{-1}}} + \underbrace{\int_{0}^{1} \frac{dx}{x^3 e^{1/x^2}}}_{\substack{\downarrow \text{ from part (b)} \\ \frac{1}{2}e^{-1}}} + \underbrace{\int_{1}^{\infty} \frac{dx}{x^3 \cdot e^{1/x^2}}}_{\substack{\downarrow \text{ from (b)} \\ \frac{1}{2} - \frac{1}{2}e^{-1}}}$$

$$= \cancel{-\frac{1}{2}e^{-1}} + \frac{1}{2} + \cancel{\frac{1}{2}e^{-1}} + \cancel{\frac{1}{2}e^{-1}} + \frac{1}{2} - \cancel{\frac{1}{2}e^{-1}}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

5) a) $$\int_0^\infty \frac{\sin(x^3)}{x+e^x}\, dx \le \int_0^\infty \left| \frac{\sin x^3}{x+e^x} \right| dx \le \int_0^\infty \frac{dx}{1+e^x}$$

then $$\le \int_0^\infty \frac{1}{e^x}\, dx = 1$$

Now $$\int_0^\infty \frac{1}{e^x}\, dx = \lim_{R\to\infty} \int_0^R e^{-x}\, dx = \lim_{R\to\infty} -e^{-x}\Big|_0^R$$

$$= \lim_{R\to\infty} (-\cancel{e^{-R}} + e^0) = 1$$

$\therefore$ $\int_0^\infty \frac{1}{e^x}\, dx$ is convergent $\Rightarrow$ $\int_0^\infty \frac{\sin x^3}{x+e^x}$ is also convergent by absolutely convergence test, 5 b) $$\int_2^{\infty} \frac{dx}{x^2 - x^{-2}}$$

limit comparison test.

$$\boxed{\lim_{x\to\infty} \frac{\dfrac{1}{x^2 - x^{-2}}}{\dfrac{1}{x^2}} = \lim_{x\to\infty} \frac{x^4}{x^4 - 1} = 1}$$

$$\int_2^{\infty} \frac{1}{x^2}\, dx = \lim_{R\to\infty} \int_2^{R} x^{-2}\, dx = \lim_{R\to\infty} \frac{x^{-1}}{-1} \Big|_2^R$$

$$= \lim_{R\to\infty} - \overset{0}{\cancel{x^{-R}}} + 2^{-1}$$

$$= 2^{-1}$$

so $\int_2^{\infty} \frac{1}{x^2}\, dx$ is convergent then

by limit comparison test.

$\int_2^{\infty} \frac{dx}{x^2 - x^{-2}}$ is convergent.

<u>limit comparison test !</u>

c) $\displaystyle\int_1^{\infty} \frac{dx}{x^2 - x^{-2}} = \underbrace{\int_1^{2} \frac{dx}{x^2 - x^{-2}}}_{\text{improper at } 1} + \underbrace{\int_2^{\infty} \frac{dx}{x^2 - x^{-2}}}_{\text{improper at } \infty}$

$\displaystyle\int_2^{\infty} \frac{dx}{x^2 - x^{-2}}$ is convergent from part (b).

Now $\displaystyle\int_1^{2} \frac{dx}{x^2 - x^{-2}}$ dx. check whether conv. or div.

$$\int_1^2 \frac{x^2}{x^4 - 1}\,dx = \lim_{R \to 1^-} \int_R^2 \frac{x^2}{x^4-1}\,dx$$

↳ use partial fractions

$$= \lim_{R \to 1^-} \int_R^2 \left(\frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{1/2}{x^2+1} \right) dx$$

$$= \lim_{R \to 1^-} \left[\frac{1}{4} \ln|x-1| - \frac{1}{4} \ln|x+1| + \frac{1}{2} \arctan x \right] \Bigg|_R^2 =$$

$$= \lim_{R \to 1^-} -\frac{1}{4}\ln 3 + \frac{1}{2}\arctan 2 - \frac{1}{4}\underbrace{\ln|R-1|}_{-\infty} + \frac{1}{4}\underbrace{\ln|R+1|}_{\ln 2} - \frac{1}{2}\underbrace{\arctan R]}_{\arctan 1 = \frac{\pi}{4}}$$

(note that when $R \to 1^-$, $\ln|R-1| \to -\infty$)

$= +\infty$ so this integral divergent then

$\displaystyle\int_1^{\infty} \frac{dx}{x^2 - x^{-2}}$ is also divergent.

$$\frac{x^2}{x^4-1} = \frac{x^2}{(x-1)\cdot(x+1)\cdot(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1} \Rightarrow A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$C = 0 \quad D = \frac{1}{2}$$

d) $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x+x^3}}$

The integral is improper of both types, so we write

$$\int_0^{\infty} \frac{dx}{\sqrt{x+x^3}} = \underbrace{\int_0^{1} \frac{dx}{\sqrt{x+x^3}}}_{I_1} + \underbrace{\int_1^{\infty} \frac{dx}{\sqrt{x+x^3}}}_{I_2} = I_1 + I_2$$

i) $I_1 = \displaystyle\int_0^1 \frac{dx}{\sqrt{x+x^3}}$ On $(0,1]$ we have

$$\sqrt{x} < \sqrt{x+x^3}$$

$$\Rightarrow \frac{1}{\sqrt{x+x^3}} < \frac{1}{\sqrt{x}}$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x+x^3}} < \underbrace{\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0} \int_R^1 x^{-\frac{1}{2}}\,dx = 2}_{\text{convergent}}$$

then $\displaystyle\int_0^1 \frac{dx}{\sqrt{x+x^3}}$ is also <u>convergent.</u>

ii) $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x+x^3}}$ On $[1,\infty)$ we hve $\sqrt{x^3} < \sqrt{x+x^3}$

$$\Rightarrow \frac{1}{\sqrt{x+x^3}} < \frac{1}{\sqrt{x^3}}$$

$$\Rightarrow \int_1^{\infty} \frac{dx}{\sqrt{x+x^3}} < \underbrace{\int_1^{\infty} x^{-3/2}\,dx = \lim_{R\to\infty} \int_1^R x^{-3/2}\,dx = 2}_{\text{convergent}}$$

then $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x+x^3}}$ is also <u>convergent</u>

Hence the given integral converges.

e) Recall that $\ln x \leq x-1$ for all $x > 0$

In particular $\ln x < x$ for $x \geq 2$.

so $\int_2^\infty \frac{dx}{\ln x}$ then $\ln x < x$.

$$\Rightarrow \frac{1}{x} < \frac{1}{\ln x}.$$

$$\Rightarrow \int_2^\infty \frac{1}{x}\,dx < \int_2^\infty \frac{1}{\ln x}\,dx$$

$$\| $$

$$\lim_{R \to \infty} \int_2^R \frac{1}{x}\,dx$$

$$= \lim_{R\to\infty} \left(\ln x \Big|_2^R \right) = \lim_{R \to \infty} \ln R - \ln 2 = \infty. \rightarrow$$

so $\int_2^\infty \frac{1}{x}\,dx$ diverges to infinity

hence the given integral diverges to infinity.