Calculus 119

10th Week[1]

Recall:

1. $\sin^{-1} x = \arcsin x = y \Leftrightarrow \sin y = x$ where $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
2. $\cos^{-1} x = \arccos x = y \Leftrightarrow \cos y = x$ where $0 \leq y \leq \pi$
3. $\tan^{-1} x = \arctan x = y \Leftrightarrow \tan y = x$ where $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
4. $\csc^{-1} x = \text{arccsc} x = y \quad (|x| \geq 1) \Leftrightarrow \csc y = x$ where $0 < y \leq \frac{\pi}{2}$ or $\pi < y \leq \frac{3\pi}{2}$
5. $\sec^{-1} x = \text{arcsec} x = y \quad (|x| \geq 1) \Leftrightarrow \sec y = x$ where $0 < y \leq \frac{\pi}{2}$ or $\pi < y \leq \frac{3\pi}{2}$
6. $\cot^{-1}(x \in \Re) \Leftrightarrow \cot y = x$ where $0 < y < \pi$

1. (a) $\tan^{-1}(\tan \dfrac{2\pi}{3}) = \tan^{-1}(\tan(\dfrac{2\pi}{3} + k\pi))$ where $k \in N$
   $= \tan^{-1}(\tan \frac{-\pi}{3}) = \frac{-\pi}{3}$

   (b) $\sin(\cos^{-1}(\frac{-1}{3}) =?$
   Let $\cos^{-1}(-\frac{1}{3}) = y$
   $\Rightarrow \cos y = -\frac{1}{3} \Rightarrow \sin(cos^{-1}(\frac{-1}{3}) \sin y = 1-(\cos y)^2 = 1-(\frac{-1}{3})^2 = \frac{8}{9}$

2. $f'(x) = \dfrac{1}{\sqrt{1 - (\frac{2x-1}{3})^2}} \left(\dfrac{2}{3}\right)$

3. By using implicit diferentiation
   $$\frac{1}{1 + \left(\frac{2x}{y}\right)^2} \left(\frac{d(\frac{2x}{y})}{dx}\right) = \frac{\pi y^2 + \pi x 2y \frac{dy}{dx}}{y^4}$$
   $\Rightarrow \frac{1}{1+\left(\frac{2x}{y}\right)^2} \frac{2y+2x\frac{dy}{dx}}{y^2} = \frac{\pi y^2 + \pi x 2y \frac{dy}{dx}}{y^4}$
   At $(1, 2)$
   $\frac{1}{1+\left(\frac{2}{2}\right)^2} \frac{2.2+2\frac{dy}{dx}}{2^2} = \frac{\pi 2^2 + \pi 1.2.2 \frac{dy}{dx}}{2^4} \Rightarrow y' = \frac{\pi-2}{1-\pi}$

4. The formulas are the following
   $$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$
   $$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
   Let's verify first equality
   Starting with Right hand side of the first equalitiy

[1] Please email all corrections and suggestions to these solutions to htor@metu.edu.tr. All solutions are available on the web at the url http://www.metu.edu.tr/∼htor.

$\cosh x \cosh y + \sinh x \sinh y = \frac{e^x + e^{-x}}{2} \frac{e^y + e^{-y}}{2} + \frac{e^x - e^{-x}}{2} \frac{e^y - e^{-y}}{2}$
$= \frac{e^x e^y + e^x e^{-y} + e^{-x} e^y + e^{-x} e^{-y}}{4} + \frac{e^x e^y - e^x e^{-y} - e^{-x} e^y + e^{-x} e^{-y}}{4} =$
$\frac{e^x e^y + e^{-x} e^{-y}}{2} = \frac{e^{x+y} + e^{-x-y}}{2} = \cosh(x + y)$
Let's verify second equality
Starting with right hand side of the second equalitiy
$\sinh x \cosh y + \cosh x \sinh y = \frac{e^x - e^{-x}}{2} \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \frac{e^y - e^{-y}}{2} =$
$... = \frac{e^{x+y} - e^{-x-y}}{2} = \sinh(x + y)$

5. Verify that $\int \text{sech} x dx = \tan^{-1}(\sinh x) + c$
   Diferentiating rihgt hand side of the above equality
   $\frac{d(\tan^{-1}(\sinh x) + c)}{dx} = \frac{1}{1 + (\sinh x)^2} \frac{d(\sinh x)}{dx} = \frac{1}{\cosh^2 x} \cosh x = \text{sech} x$
   $\Rightarrow \int \text{sech} x dx = \tan^{-1}(\sinh x) + c$

6. Verify $\int \tanh^{-1} x dx = x \tanh^{-1} x + \frac{1}{2} \ln(1 - x^2) + c$
   Diferentiating rihgt hand side of the above equality
   $\frac{d(x \tanh^{-1} x + \frac{1}{2} \ln(1 - x^2) + c)}{dx} = \tanh^{-1} x + x \frac{1}{1 - x^2} + \frac{1}{2} \frac{1}{1 - x^2}(-2x)$
   $= \tanh^{-1} x$