Exercises for Calculus 119

#1. Evaluate the integral $\displaystyle\int_0^{\infty} \frac{x\tan^{-1}x}{(1+x^2)^2}\,dx$

Solution. $\displaystyle\int_0^{\infty} \frac{x\tan^{-1}x}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\int_0^{b} \frac{x\tan^{-1}x}{(1+x^2)^2}$

$$\int \frac{x\tan^{-1}x}{(1+x^2)^2}\,dx = \int u\,dv \ \underline{\text{by parts}}\ uv - \int v\,du$$

$$= -\frac{1}{2}\,\frac{\tan^{-1}x}{1+x^2} + \frac{1}{2}\int \frac{dx}{(1+x^2)^2}$$

$$= -\frac{1}{2}\,\frac{\tan^{-1}x}{1+x^2} + \frac{1}{2}\left[\frac{1}{2}\tan^{-1}x + \frac{1}{2}\,\frac{x}{1+x^2}\right] + C$$

$$\Rightarrow \int_0^{\infty} \frac{x\tan^{-1}x}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\int_0^{b} \frac{x\tan^{-1}x\,dx}{(1+x^2)^2}$$

$$= \lim_{b\to\infty}\left[-\frac{1}{2}\,\frac{\tan^{-1}x}{1+x^2} + \frac{1}{4}\tan^{-1}x + \frac{1}{4}\,\frac{x}{1+x^2}\right]_0^b$$

$$= \lim_{b\to\infty} -\frac{1}{2}\,\frac{\tan^{-1}b}{1+b^2} + \frac{1}{4}\tan^{-1}b + \frac{1}{4}\,\frac{b}{1+b^2} + \frac{1}{2}\,\frac{\tan^{-1}0}{1+0^2} - \frac{1}{4}\tan^{-1}0 - \frac{1}{4}\cdot\frac{0}{1+0^2}$$

$$= -\frac{1}{2}\cdot 0 + \frac{1}{4}\,\frac{\pi}{2} + 0 + \frac{1}{2}\cdot 0 - \frac{1}{4}\cdot 0 - \frac{1}{4}\cdot 0$$

$$= \pi/8$$

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let $\begin{cases} u = \tan^{-1}x \\ dv = \dfrac{x\,dx}{(1+x^2)^2} \end{cases}$

$\Rightarrow \begin{cases} du = \dfrac{dx}{1+x^2} \\ v = \dfrac{1}{2}\displaystyle\int \frac{2x\,dx}{(1+x^2)^2} = \dfrac{-1/2}{1+x^2} \end{cases}$

$\displaystyle\int \frac{dx}{(1+x^2)^2}$: Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$

$$= \int \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2} = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$

$$= \int \cos^2\theta\,d\theta$$

$$= \int\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)d\theta$$

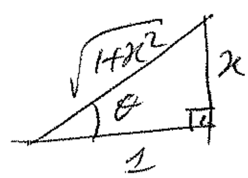
$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\cdot 2\sin\theta\cos\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\cdot\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}\tan^{-1}x + \frac{1}{2}\,\frac{x}{1+x^2}$$



#2. Evaluate $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{x^2-9}}$

Solution. $\displaystyle\int_3^\infty \frac{dx}{x\sqrt{x^2-9}} = \int_3^4 \frac{dx}{x\sqrt{x^2-9}} + \int_4^\infty \frac{dx}{x\sqrt{x^2-9}}$

$$= \lim_{a\to 3^+} \int_a^4 \frac{dx}{x\sqrt{x^2-9}} + \lim_{b\to\infty}\int_4^b \frac{dx}{x\sqrt{x^2-9}}$$

$$= \lim_{a\to 3^+}\left[\frac{1}{3}\sec^{-1}\frac{x}{3}\right]_a^4 + \lim_{b\to\infty}\left[\frac{1}{3}\sec^{-1}\frac{x}{3}\right]_4^b$$

$$= \frac{1}{3}\lim_{a\to 3^+}\left(\sec^{-1}\frac{4}{3} - \sec^{-1}\frac{a}{3}\right) + \frac{1}{3}\lim_{b\to\infty}\left(\sec^{-1}\frac{b}{3} - \sec^{-1}\frac{4}{3}\right)$$

$$= \frac{1}{3}\left[\sec^{-1}\frac{4}{3} - \sec^{-1}1\right] + \frac{1}{3}\left(\frac{\pi}{2} - \sec^{-1}\frac{4}{3}\right)$$

$$= \cancel{\frac{1}{3}\sec^{-1}\frac{4}{3}} - \frac{1}{3}\cdot 0 + \frac{1}{3}\frac{\pi}{2} - \cancel{\frac{1}{3}\sec^{-1}\frac{4}{3}}$$

$$= \frac{\pi}{6}$$

Let $x = 3\sec\theta$ where $0 \le \theta < \pi/2$ or $\pi \le \theta < \frac{3\pi}{2}$

then $dx = 3\sec\theta\tan\theta\, d\theta$ and

$$\int \frac{dx}{x\sqrt{x^2-9}} = \int \frac{\cancel{3\sec\theta}\tan\theta\, d\theta}{\cancel{3\sec\theta}\cdot\sqrt{9\sec^2\theta - 9}}$$

$$= \int \frac{\tan\theta\, d\theta}{3\sqrt{\sec^2\theta - 1}}$$

$$= \frac{1}{3}\int \frac{\tan\theta\, d\theta}{\tan\theta}$$

$$= \frac{1}{3}\int d\theta = \frac{1}{3}\theta$$

$$= \frac{1}{3}\cdot\sec^{-1}\left(\frac{x}{3}\right) + C$$

#3. Test the integral $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}+\sqrt{x^2}}$ for convergence

Solution. on $[0,1]$: $\displaystyle\frac{1}{\sqrt{x}+\sqrt{x^2}} \le \frac{1}{\sqrt{x}} \Rightarrow \int_0^1 \frac{dx}{\sqrt{x}+\sqrt{x^2}} \le \int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2$

on $[1,\infty)$: $\displaystyle\frac{1}{\sqrt{x}+x^2} \le \frac{1}{x^2} \Rightarrow \int_1^\infty \frac{dx}{\sqrt{x}+x^2} \le \int_1^\infty \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_1^\infty = 0+1 = 1$

Hence the given improper integral converges to a number < 3

#4. (a) Show that the integral $\int_0^\infty \frac{\cos x\,dx}{x^2+1}$ converges absolutely.

Solution: $\int_0^\infty \underbrace{\left|\frac{\cos x}{x^2+1}\right|}_{|\cos x| < 1} dx \le \int_0^\infty \frac{1 . dx}{x^2+1} = \left[\tan^{-1} x\right]_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

Hence by simple comparison, the given integral converges.

(b) Show that $\int_0^\infty \frac{\sin x}{x}\,dx$ converges.

Solution. $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$

- $\int_0^1 \frac{\sin x}{x}\,dx$ converges; because $\frac{\sin x}{x}$ is continuous in $(0,1]$ and $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$ (exists)

- $\int_1^\infty \frac{\sin x}{x}\,dx = \lim_{M\to\infty} \int_1^M \frac{\sin x}{x}\,dx$

$= \lim_{M\to\infty} \left[-\frac{1}{x}\cos x \Big|_1^M - \int_1^\infty \frac{\cos x\,dx}{x^2} \right]$

$\cancel{\left[-\frac{\cos x}{x}\right]}$

$= \lim_{M\to\infty}\left[-\frac{\cos M}{M} + \cos 1\right] - \int_1^\infty \frac{\cos x\,dx}{x^2}$

$= 0 + \cos 1 - \lambda$;

$\lambda = \int_1^\infty \frac{\cos x\,dx}{x^2} \le \int_1^\infty \frac{dx}{x^2}$ converges

$\int \frac{\sin x}{x}\,dx = ?$

Let $u = \frac{1}{x}$

$dv = \sin x\,dx$

$\Downarrow$

$du = -\cancel{dx}/x^2$

$v = \int \sin x\,dx = -\cos x$

$\Rightarrow \int \frac{\sin x}{x}\,dx = \int \frac{1}{x}.\sin x\,dx$

$= \int u\,dv \overset{\text{by parts}}{=} uv - \int v\,du$

$= -\frac{1}{x}\cos x + \int -\frac{\cos x\,dx}{x^2} + C$

#5. Find the length of the curve $y = \ln x$; $1 \le x \le \sqrt{3}$

Solution. $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{1}{x}\right)^2} = \sqrt{1+\frac{1}{x^2}}$

$= \frac{\sqrt{x^2+1}}{|x|} = \frac{\sqrt{x^2+1}}{x}$; $(x>0)$

Hence the length of arc is

$$\int_1^{\sqrt{3}} \sqrt{1+(y')^2}\,dx = \int_{x=1}^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}\,dx$$

; let $u = \sqrt{1+x^2}$

then $u^2 = 1+x^2$

and $u\,du = x\,dx$

$dx = \frac{u\,du}{x}$

x	u
1	$\sqrt{2}$
$\sqrt{3}$	$\sqrt{4}=2$

$x^2 = u^2-1$

$$= \int_{u=\sqrt{2}}^{2} \frac{u}{x}\cdot\frac{u\,du}{x} = \int_{\sqrt{2}}^{2} \frac{u^2\,du}{u^2-1}$$

$$= \int_{\sqrt{2}}^{2} \frac{(u^2-1)+1}{u^2-1}\,du = \int_{\sqrt{2}}^{2} 1\,du + \int_{\sqrt{2}}^{2} \frac{du}{u^2-1}$$

$$= u\Big|_{\sqrt{2}}^{2} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\Bigg|_{\sqrt{2}}^{2}$$

$$= \left(2-\sqrt{2}\right) + \frac{1}{2}\left[\ln\left|\frac{2-1}{2+1}\right| - \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right]$$

$$= 2-\sqrt{2} + \frac{1}{2}\left[\ln\frac{1}{3} - \ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right]$$

$$= 2-\sqrt{2} + \frac{1}{2}\ln\left|\frac{1/3}{(\sqrt{2}-1)/(\sqrt{2}+1)}\right| = 2-\sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{3(\sqrt{2}-1)}\right)$$

#6 Find the length of the curve $y=\ln\left(\frac{e^x+1}{e^x-1}\right)$; from $x=1$ to $x=2$.

Solution. $$y' = \frac{\frac{d}{dx}\left(\frac{e^x+1}{e^x-1}\right)}{\frac{e^x+1}{e^x-1}} = \left(\frac{e^x-1}{e^x+1}\right)\cdot\left[\frac{e^x(e^x-1)-e^x(e^x+1)}{(e^x-1)^2}\right]$$

$$= \left(\frac{e^x-1}{e^x+1}\right)\cdot\frac{\cancel{e^{2x}}-e^x-\cancel{e^{2x}}-e^x}{(e^x-1)^2} = \frac{-2e^x}{(e^x+1)(e^x-1)} = \frac{-2e^x}{e^{2x}-1}$$

length of the curve is $$= \int_1^2\sqrt{1+(y')^2}\,dx = \int_1^2\sqrt{1+\left(\frac{-2e^x}{e^{2x}-1}\right)^2}\,dx$$

$$= \int_1^2 \frac{\sqrt{(e^{2x}-1)^2+4e^{2x}}}{e^{2x}-1} = \int_1^2 \frac{\sqrt{e^{4x}-2e^{2x}+1+4e^{2x}}}{e^{2x}-1}\,dx$$

$$= \int_1^2 \frac{\sqrt{e^{4x}+2e^{2x}+1}}{e^{2x}-1}\,dx = \int_1^2 \frac{\sqrt{(e^{2x}+1)^2}}{e^{2x}-1}\,dx$$

$$= \int_1^2 \frac{e^{2x}+1}{e^{2x}-1}\,dx = \int_1^2 \frac{e^{-x}(e^{2x}+1)}{e^{-x}(e^{2x}-1)}\,dx$$

$$= \int_1^2 \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx = \ln|e^x-e^{-x}|\Big|_{x=1}^{2}$$

$$= \ln|e^2-e^{-2}| - \ln|e^1+e^{-1}|$$

$$= \ln\left|\frac{e^4-1}{e^2}\right| - \ln\left|\frac{e^2+1}{e}\right|$$

$$= \ln\left|\frac{e^4-1}{e^2}\cdot\frac{e}{e^2+1}\right| = \ln\left|\frac{e^2-1}{e}\right| \text{ units}$$

$$= \ln\left(\frac{e^2-1}{e}\right)$$

#7. Find the area of the surface obtained by rotating the curve $x = \frac{1}{3}(y^2+2)^{3/2}$ from $y=1$ to $y=2$ about the x-axis.

Solution. $x = \frac{1}{3}(y^2+2)^{3/2} \Rightarrow \frac{dx}{dy} = \frac{1}{3}\cdot\frac{3}{2}(y^2+2)^{1/2}(2y) = y\sqrt{y^2+2}$

$$\Rightarrow 1+\left(\frac{dx}{dy}\right)^2 = 1+\left[y\sqrt{y^2+2}\right]^2 = 1+y^2(y^2+2)$$

$$= 1+y^4+2y^2 = (y^2+1)^2$$

Hence, the surface area $= 2\pi\int_{y=1}^{2} y(y^2+1)\,dy$

$$= 2\pi\left[\frac{1}{4}y^4+\frac{1}{2}y^2\right]_1^2$$

$$= 2\pi\left[\frac{1}{4}2^4+\frac{1}{2}2^2-\frac{1}{4}-\frac{1}{2}\right]$$

$$= 2\pi\left[4+2-\frac{3}{4}\right]$$

$$= 2\pi\left(\frac{24-3}{4}\right)$$

$$= \frac{21}{2}\pi \text{ unit}^2$$