Dynamical systems and CA

- Outline
  - Dynamical systems
  - Cellular automata (CA) as models of nonlinear dynamical systems
  - Complexity

Dynamical systems and CA

- References

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  - Foley, D.K. (1998), "Introduction", in P.S. Albin, *Barriers and Bounds to Rationality,* Princeton, NJ: Princeton University Press, Chapter 1.
  - Albin, B.S. (1998), *Barriers and Bounds to Rationality,* Princeton, NJ: Princeton University Press, Chapters 3, 4 and 5.
  - Day, R.H. (1982), "Irregular Growth Cycles", *American Economic Review* (72): 406-414.

Dynamical systems

- A dynamical system can be defined by the relation

  $x_{t+1} = F_a(x_t)$

  where x_t is a vector in a state space X, Fa an operator on the state space, a the vector of potentially changeable parameters of the system.

Linear dynamical systems

- In a linear dynamical system, F_a is a matrix, A

  $x_{t+1} = A(x_t)$

  A matrix determines the trajectories of the system that can be decomposed into three types of independent motion.

Linear dynamical systems

- Geometrix expansion or contraction
  - Eigenvalues of A are real, positive
- Osilating geometric expansion or contraction
  - Eigenvalues of A are real, negative
- Inward or outward spiraling
  - Eigenvalues of A are complex

Linear dynamical systems

- Stability of linear dynamical systems
  - The magnitude of the eigenvalues determines the stability of the system.
  - If an eigenvalue (or pair of complex eigenvalues) has a magnitude smaller than 1, the corresponding component of the motion will be stable, moving to the origin (the "equilibrium" point).
  - If an eigenvalue (or pair of complex eigenvalues) has a magnitude greater than 1, the corresponding component of the motion will be unstable, moving away to the origin indefinetly.
  - Eigenvalues with magnitude just equal to 1 are neutral, neither stable nor unstable.

Linear dynamical systems

- Example

  $x_{t+1} = \lambda \cos\omega \; x_t - \lambda \sin\omega \; y_t$

  $y_{t+1} = \lambda \sin\omega \; x_t + \lambda \cos\omega \; y_t$

- The eigenvalues are

  $\lambda (\cos\omega \pm i \sin\omega)$

If $\lambda < 1$, stable; if $\lambda > 1$, unstable.

ω determines the speed and direction of spiraling.

Linear dynamical systems

Example

Rewrite the system in polar coordinates

$r(x, y) = (x^2 + y^2)^{1/2}$

$\Theta = \arctan(y/x)$

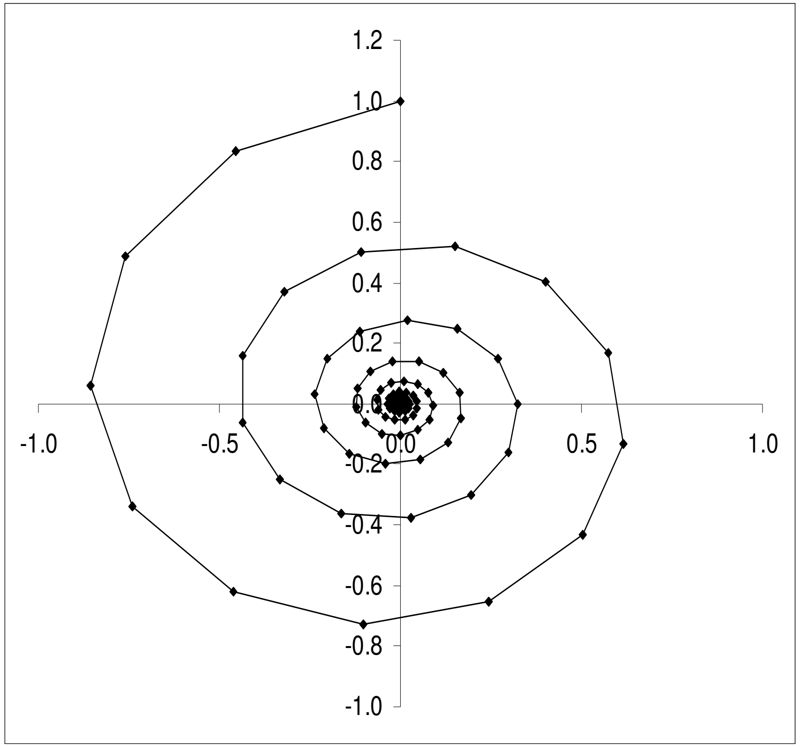
Then,

$r_{t+1} = \lambda r_t$

$\Theta_{t+1} = \Theta_t + \omega$

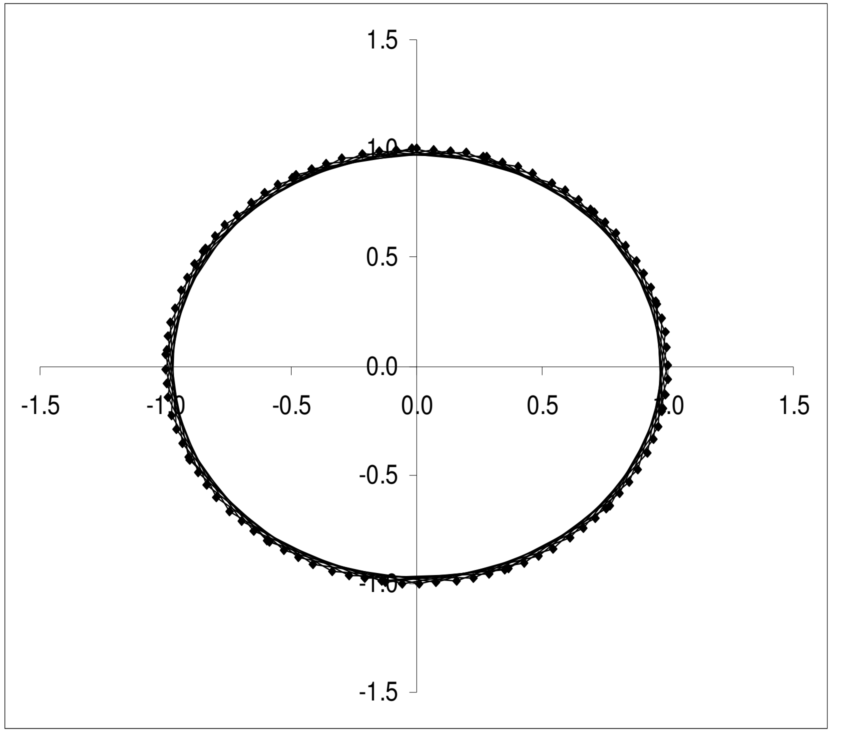
Linear dynamical systems

- $\lambda = 0.95$, $\omega = 0.5$, $x_0 = 0$, $y_0 = 1$



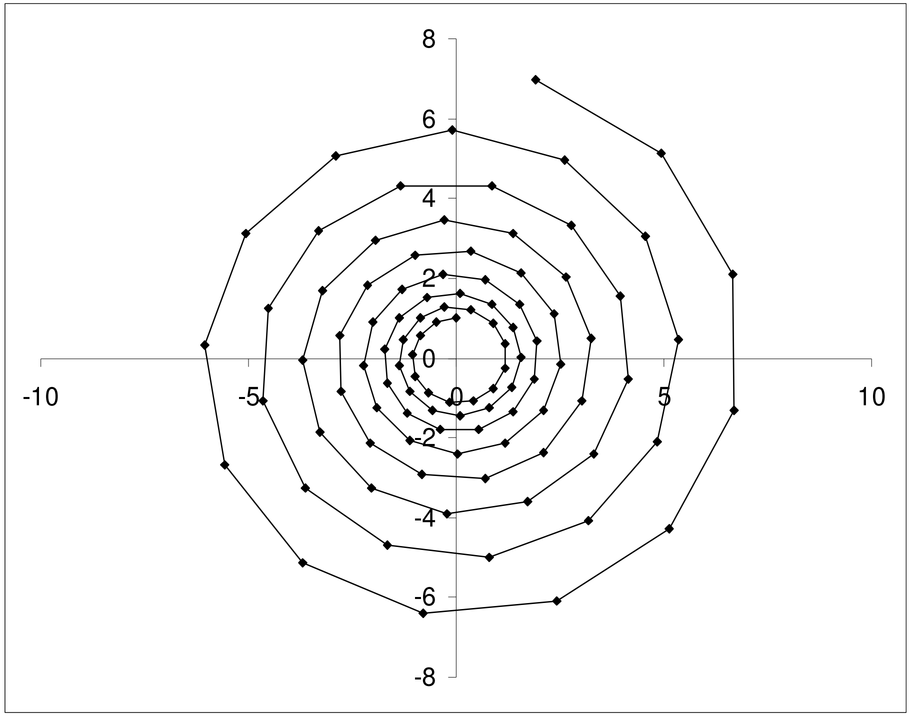
Linear dynamical systems

- $\lambda = 1$, $\omega = 0.5$, $x_0 = 0$, $y_0 = 1$



Linear dynamical systems

- $\lambda = 1.02$, $\omega = 0.5$, $x_0 = 0$, $y_0 = 1$



Linear dynamical systems

Characteristics of linear systems

- It is possible to predict the motion of linear systems by observing their past trajectories (for example, estimate the A matrix).
- Their behavior in all regions of the state space is proportional to their behavior in a small neighborhood of the origin (i.e., the observation of a linear system in a small part of state space is potentially sufficient to understand its behavior everywhere).
- Smooth changes in parameter values a (the elements of the matrix A) lead to smooth changes in the behavior of the trajectories (i.e., estimation errors in parameters/initial conditions do not create a major problem).

Nonlinear dynamical systems

- If the operator $F_a(.)$ is not linear, then the system is nonlinear.
- Assume that the origin of the state space is an equilibrium of the system, i.e.,

  $F_a(0) = 0$
- A nonlinear system can be approximated by linear systems (a power series expansion, for example, Taylor series).

Nonlinear dynamical systems

Types of motion in nonlinear systems

- Moves toward the origin
- Developes a stable limit cycles
- Multiple cycles
- Develop a "chaotic" behavior

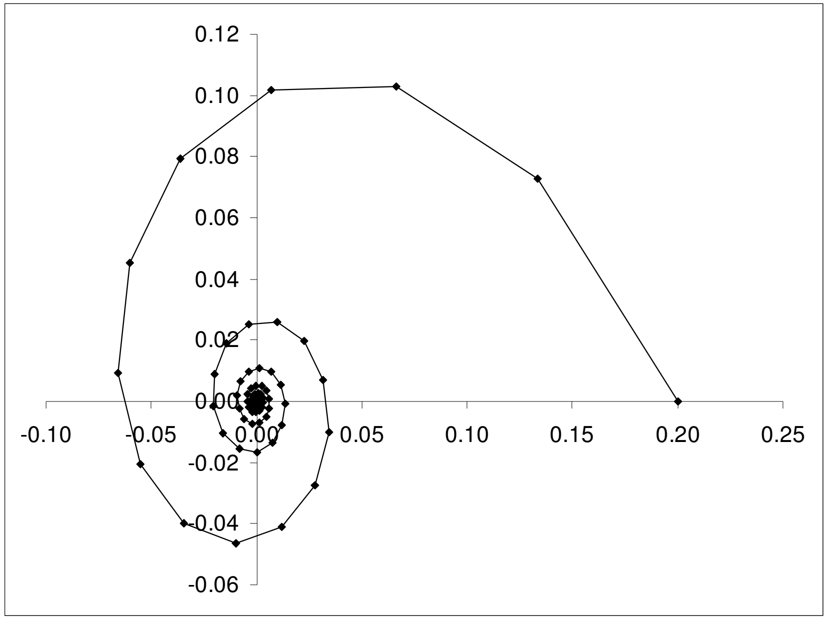
Consider the following nonlinear system

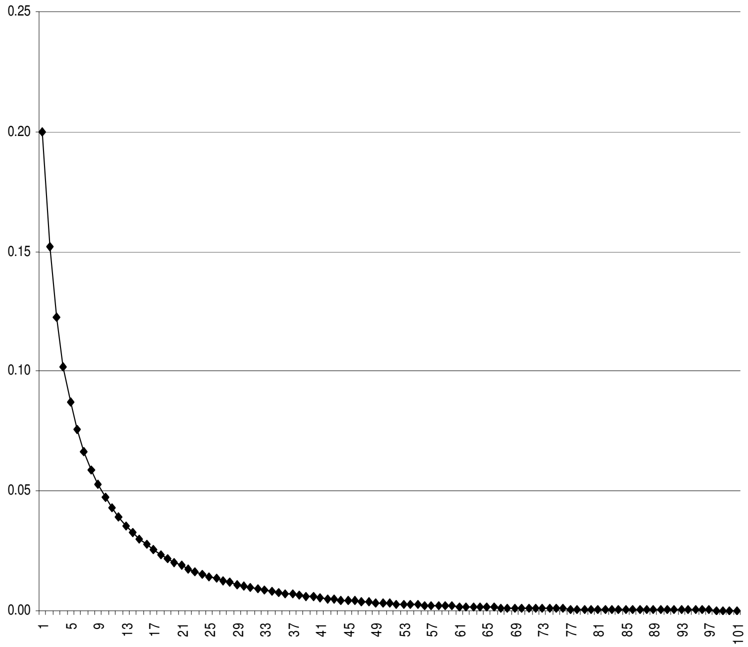
$r_{t+1} = \lambda r_t - \lambda r_t^2 = \lambda r_t(1 - r_t)$

$\Theta_{t+1} = \Theta_t + \omega$

Nonlinear dynamical systems

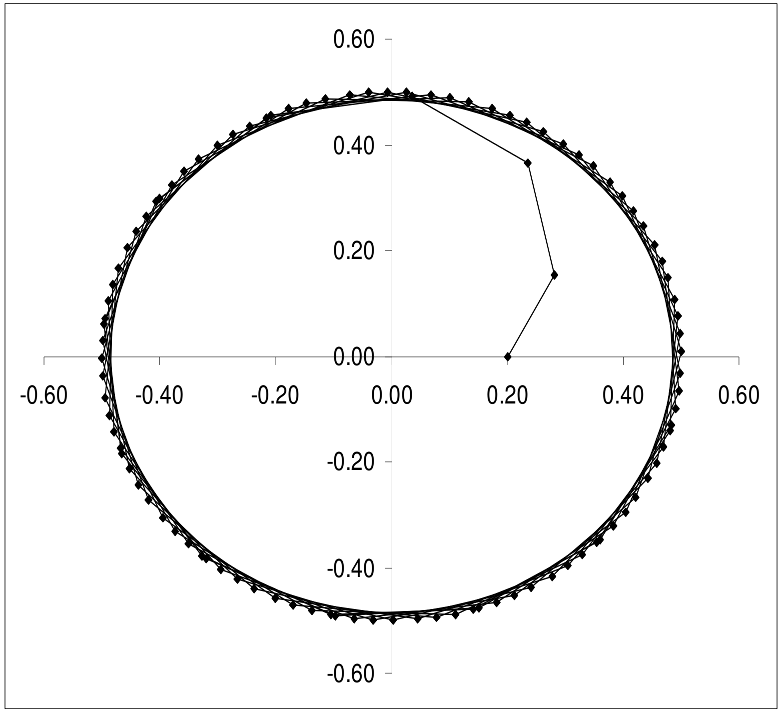
- $\lambda = 0.95$, $\omega = 0.5$, $x_0 = 0.2$, $y_0 = 0$

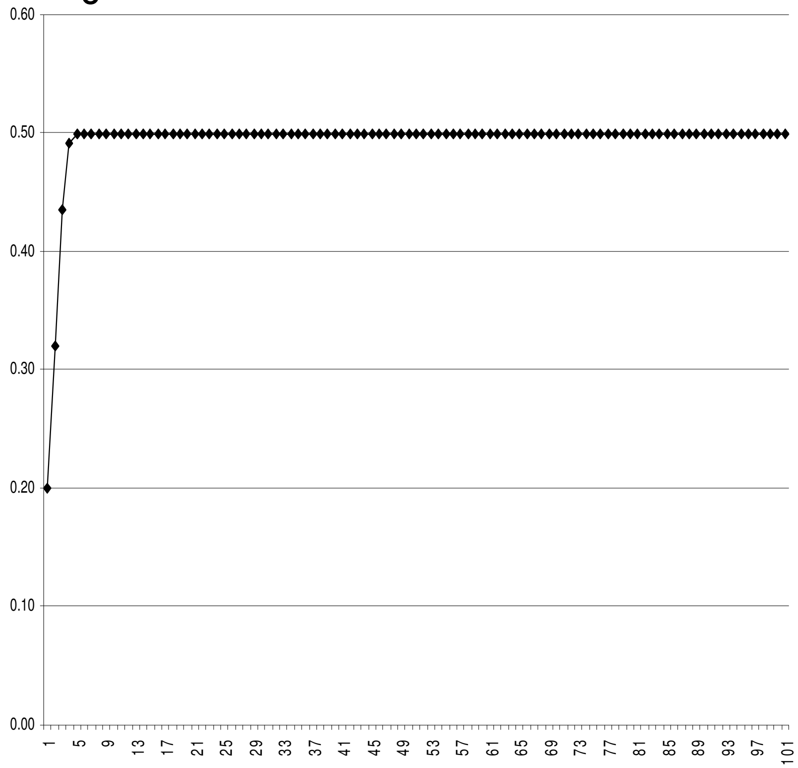




Nonlinear dynamical systems

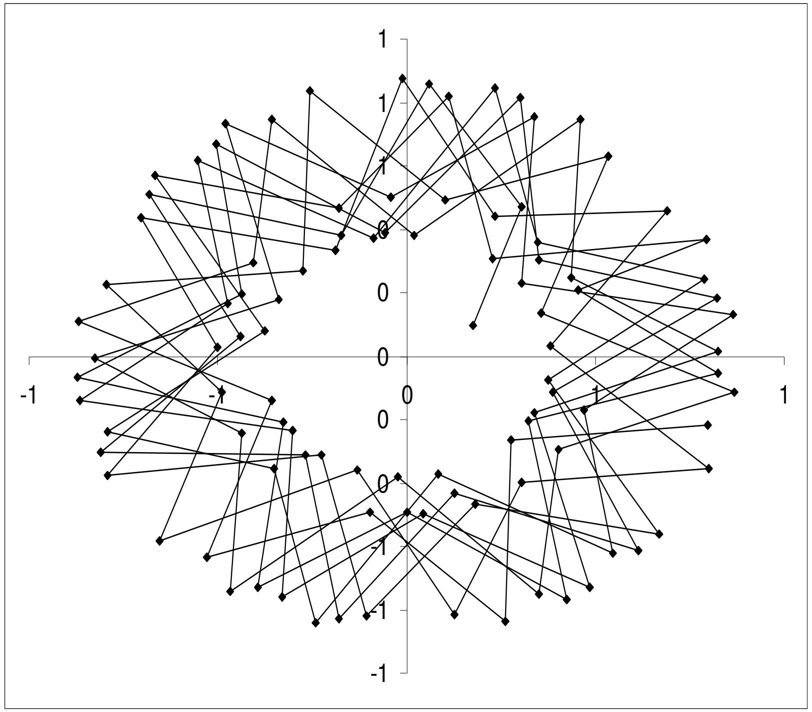
- $\lambda = 2$, $\omega = 0.5$, $x_0 = 0.2$, $y_0 = 0$

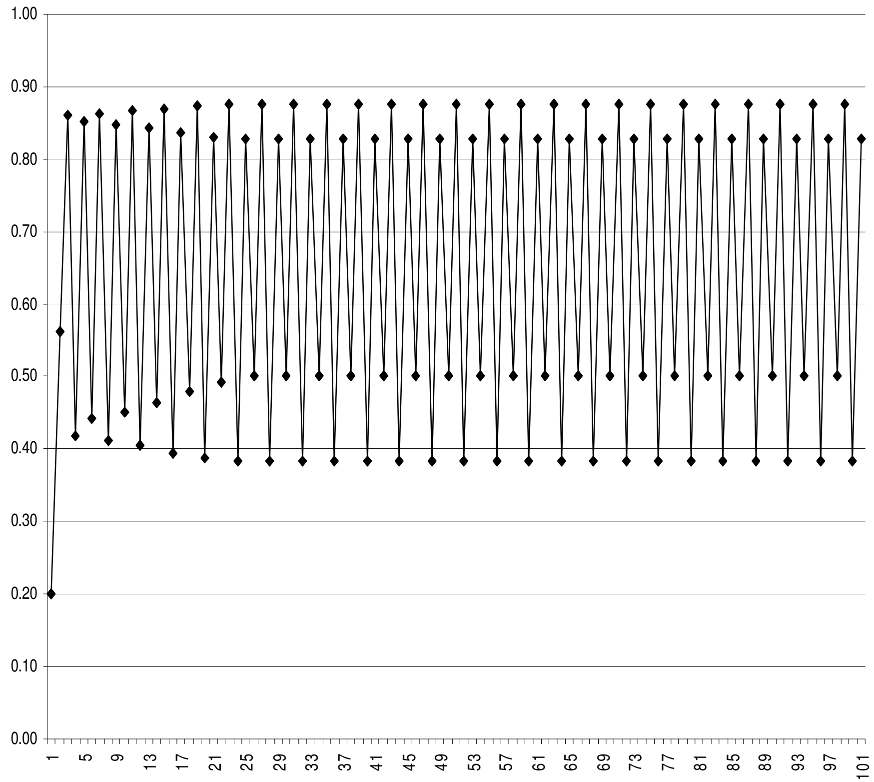




Nonlinear dynamical systems

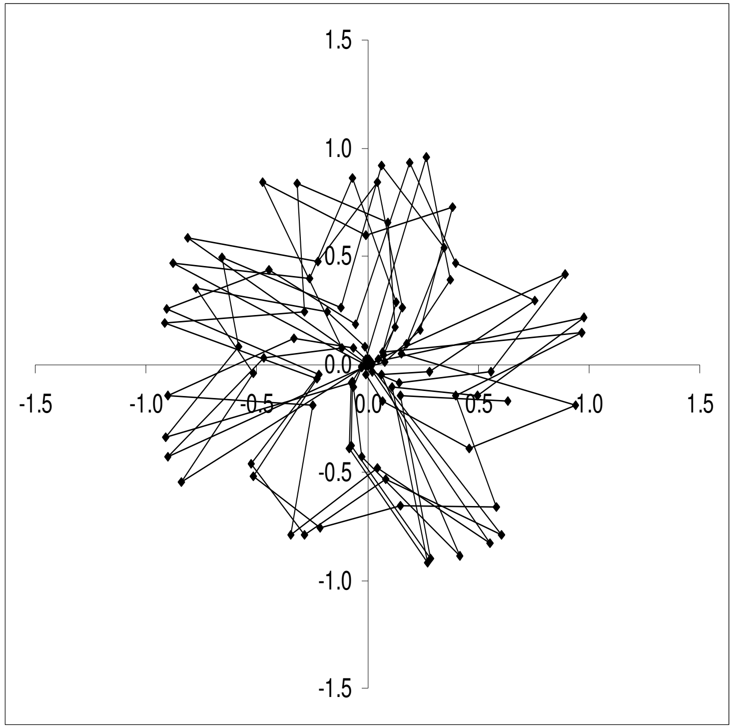
- $\lambda = 3.5$, $\omega = 0.5$, $x_0 = 0.2$, $y_0 = 0$

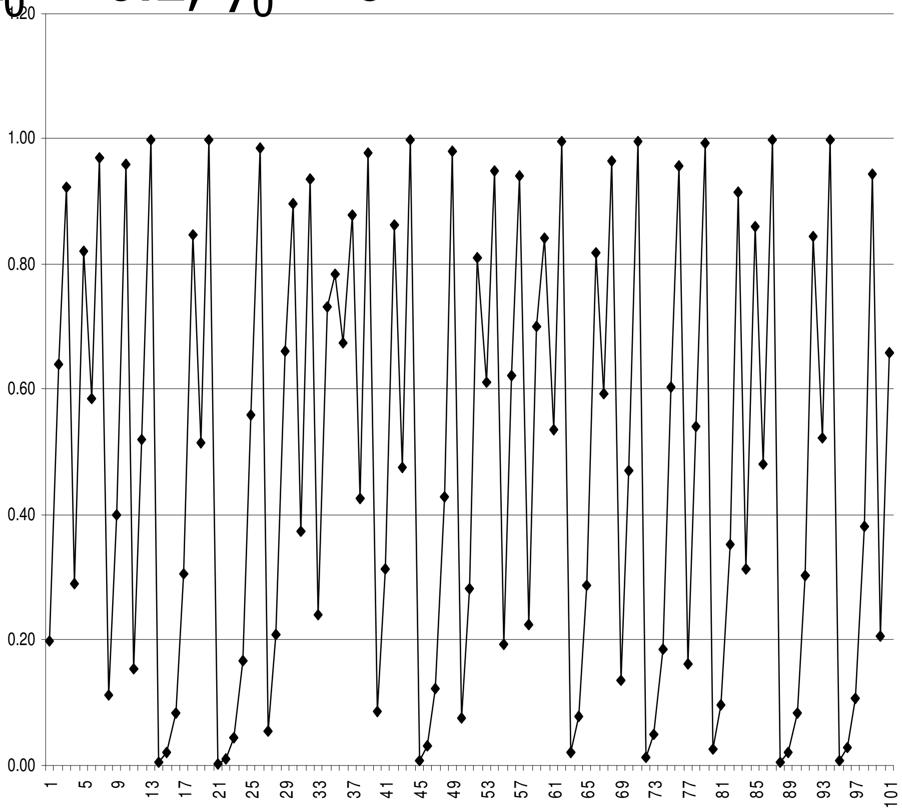




Nonlinear dynamical systems

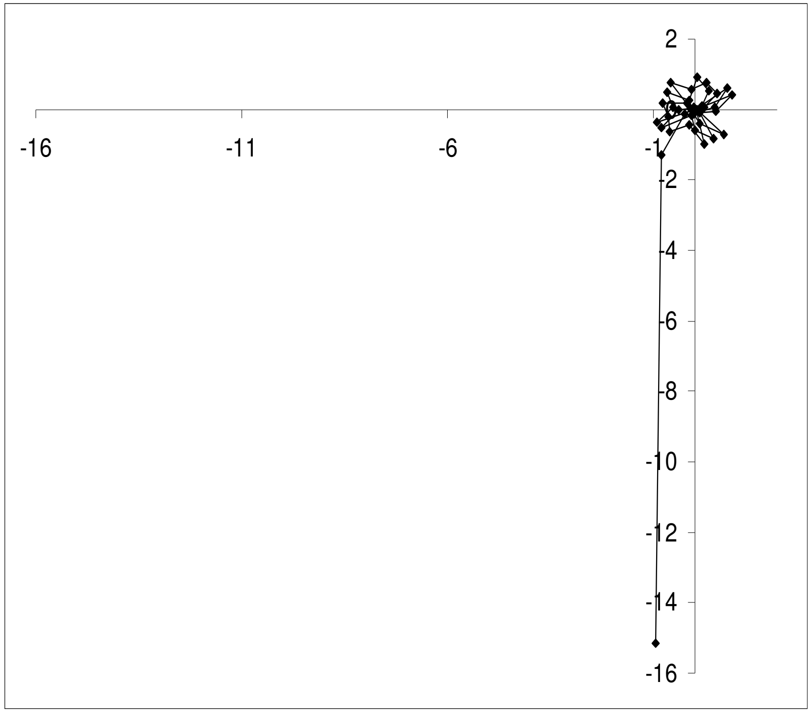
- $\lambda = 4.001, \omega = 0.5, x_0 = 0.2, y_0 = 0$

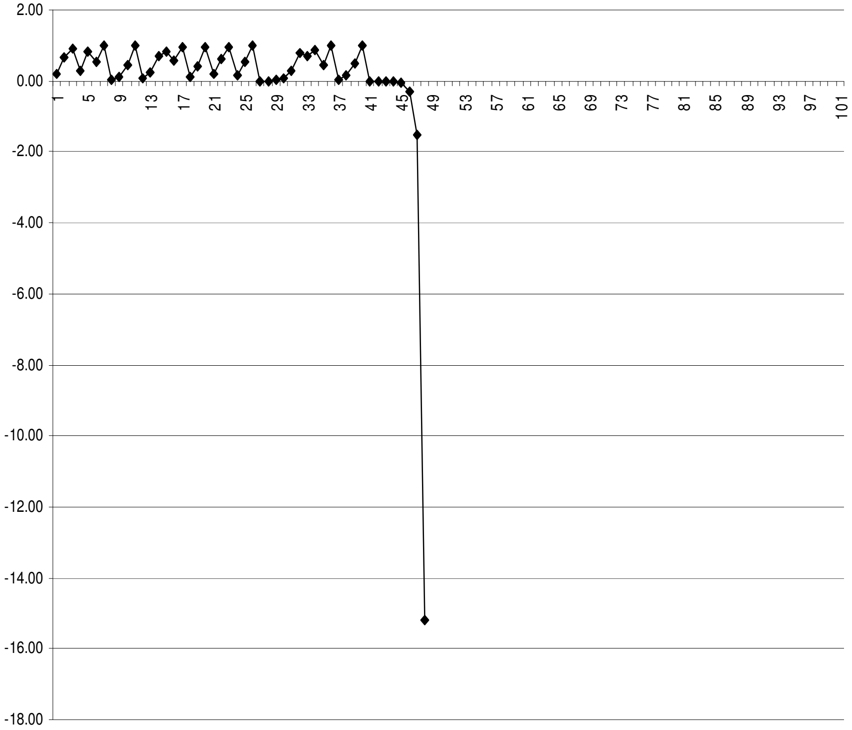




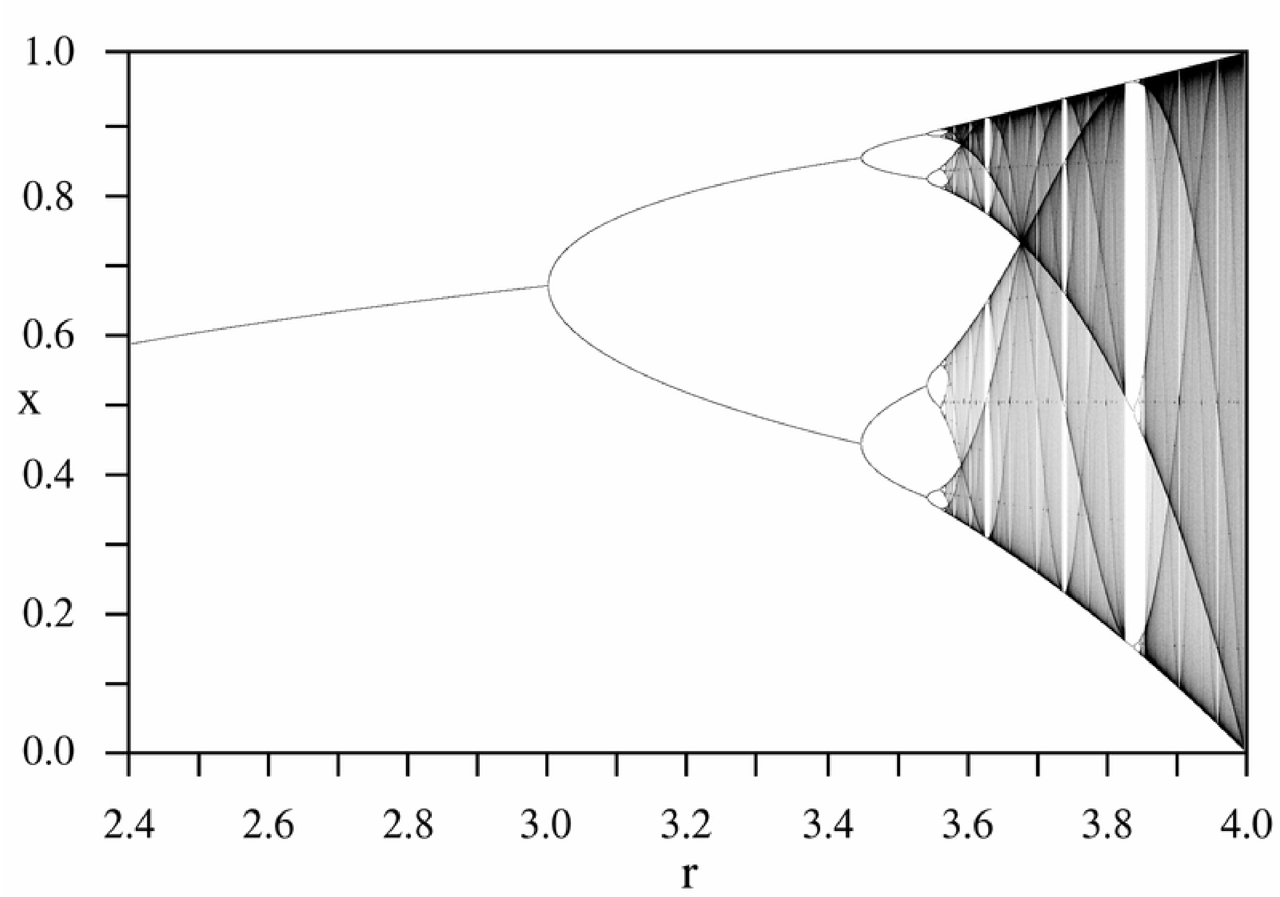
Nonlinear dynamical systems

- $\lambda = 4.001$, $\omega = 0.5$, $x_0 = 0.201$, $y_0 = 0$





Nonlinear dynamical systems



Nonlinear dynamical systems

By varying the parameter λ, the following behaviour is observed:

- With λ between 0 and 1, the population will eventually die, independent of the initial population.
- With λ between 1 and 2, the population will quickly stabilize on the value $(\lambda-1)/\lambda$, independent of the initial population.
- With λ between 2 and 3, the population will also eventually stabilize on the same value, but first oscillates around that value for some time.
- With λ between 3 and $1+\sqrt{6}$ (approximately 3.45), the population will oscillate between two values forever. These two values are dependent on λ but independent of the initial population.
- With λ between 3.45 and 3.54 (approximately), the population will oscillate between four values forever; again, this behavior does not depend on the initial population.

Nonlinear dynamical systems

- With λ slightly bigger than 3.54, the population will oscillate between 8 values, then 16, 32, etc. The lengths of the parameter intervals which yield the same number of oscillations decrease rapidly; the ratio between the lengths of two successive such bifurcation intervals approaches the Feigenbaum constant δ = 4.669. None of these behaviors depend on the initial population.
- At λ = 3.57 (approximately) is the onset of chaos, at the end of the period-doubling cascade. We can no longer see any oscillations. Slight variations in the initial population yield dramatically different results over time, a prime characteristic of chaos.
- Most values beyond 3.57 exhibit chaotic behaviour, but there are still certain isolated values of λ that appear to show non-chaotic behavior; these are sometimes called islands of stability. These behaviours are again independent of the initial value.
- Beyond λ = 4, the values eventually leave the interval [0,1] and diverge for almost all initial values.

Nonlinear dynamical systems

A chaotic system

- "Globally stable"
- Never exactly repeats its previous trajectory
- Locally predictable (if the system returns to a position in state space close to an earlier trajectory, its trajectory will follow for a while the previous trajectory closely)
- Exhibits regime shifts (what happens if r=1?)
- Trajectories depend on initial conditions/ parameter values (one trajectory, starting from one set of initial conditions, may be very different in detail from the trajectory starting from a very slightly different set of initial conditions)

Complexity

Types of complexity

- Type 1. Convergence
- Type 2. Repetitive patterns
- Type 3. Chaotic
- Type 4. Chaotic with regime shifts

Nonlinear dynamical systems

Examples of chaotic systems?

Nonlinear dynamical systems

Solovian neo-classical growth model

Assumptions

- Competitive markets
- Constant returns to scale
- Constant growth rate of labor, n
- Constant savings rate, S = sQ

Nonlinear dynamical systems

Equilibrium

$k^* = k$ ($k = K/L$)

Growth rate of K = Growth rate of labor (n)

$sQ/K = n$

$sq = nk$

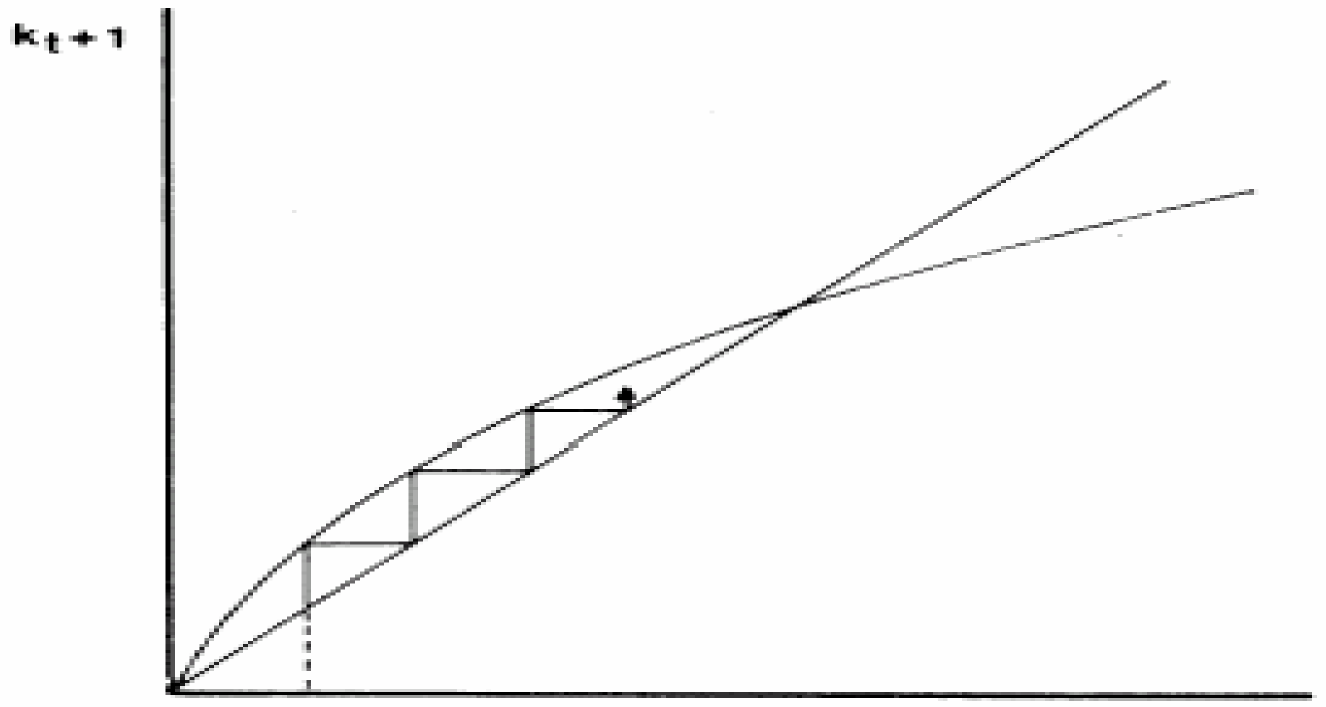
Dynamics

$k_{t+1} = s\, f(k_t)/(1+n)$

Assume a Cobb Douglas PF

$k_{t+1} = s\, Bk_t^{\beta}/(1+n)$

Nonlinear dynamical systems



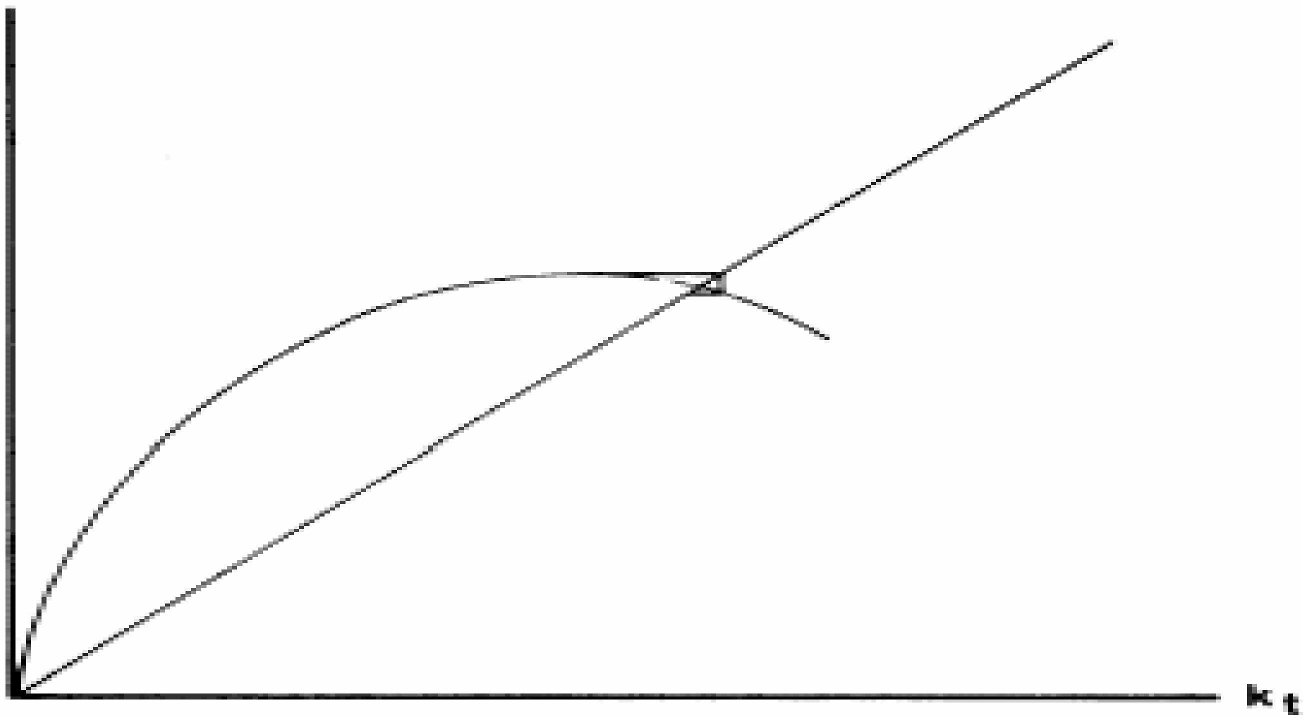
(a) Monotonic Convergence to a Steady State

Nonlinear dynamical systems

Assume the "pollution effect"
$f^P(k) = f(k)(m - k)^\gamma$

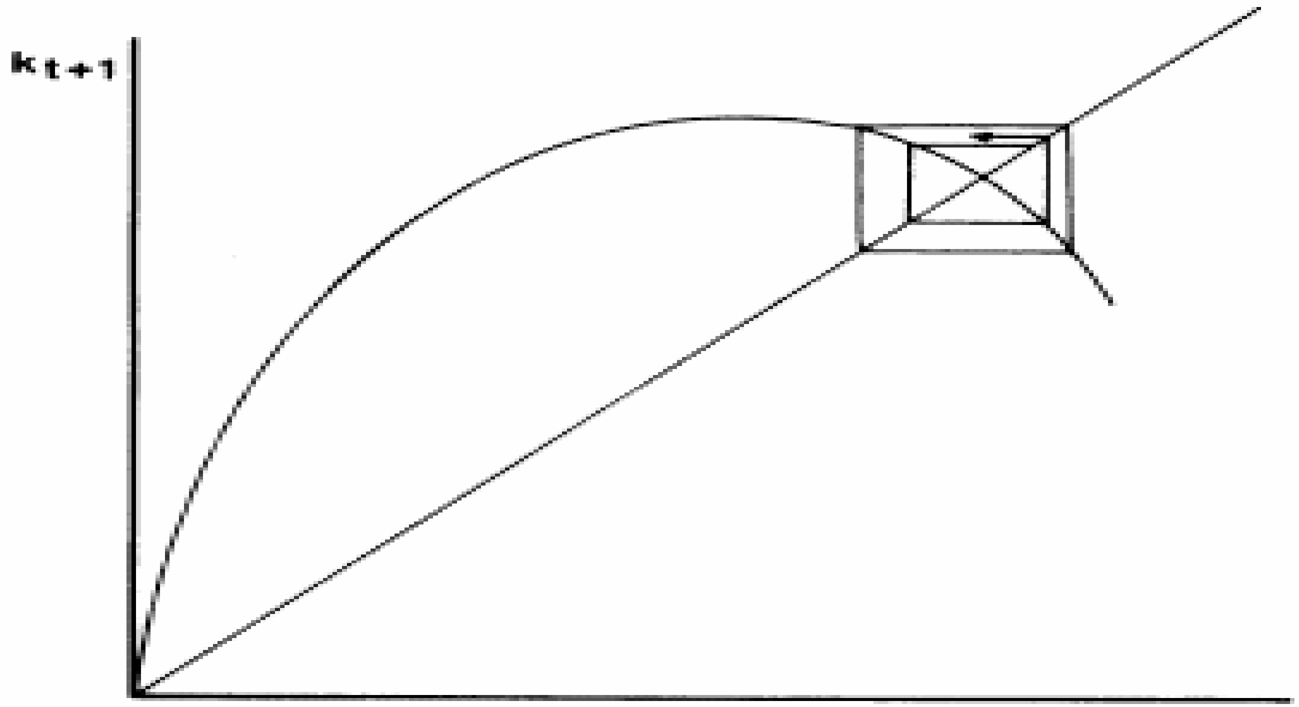
Then,
$k_{t+1} = s\ Bk_t^\beta(m - k_t)^\gamma/(1+n)$

Nonlinear dynamical systems



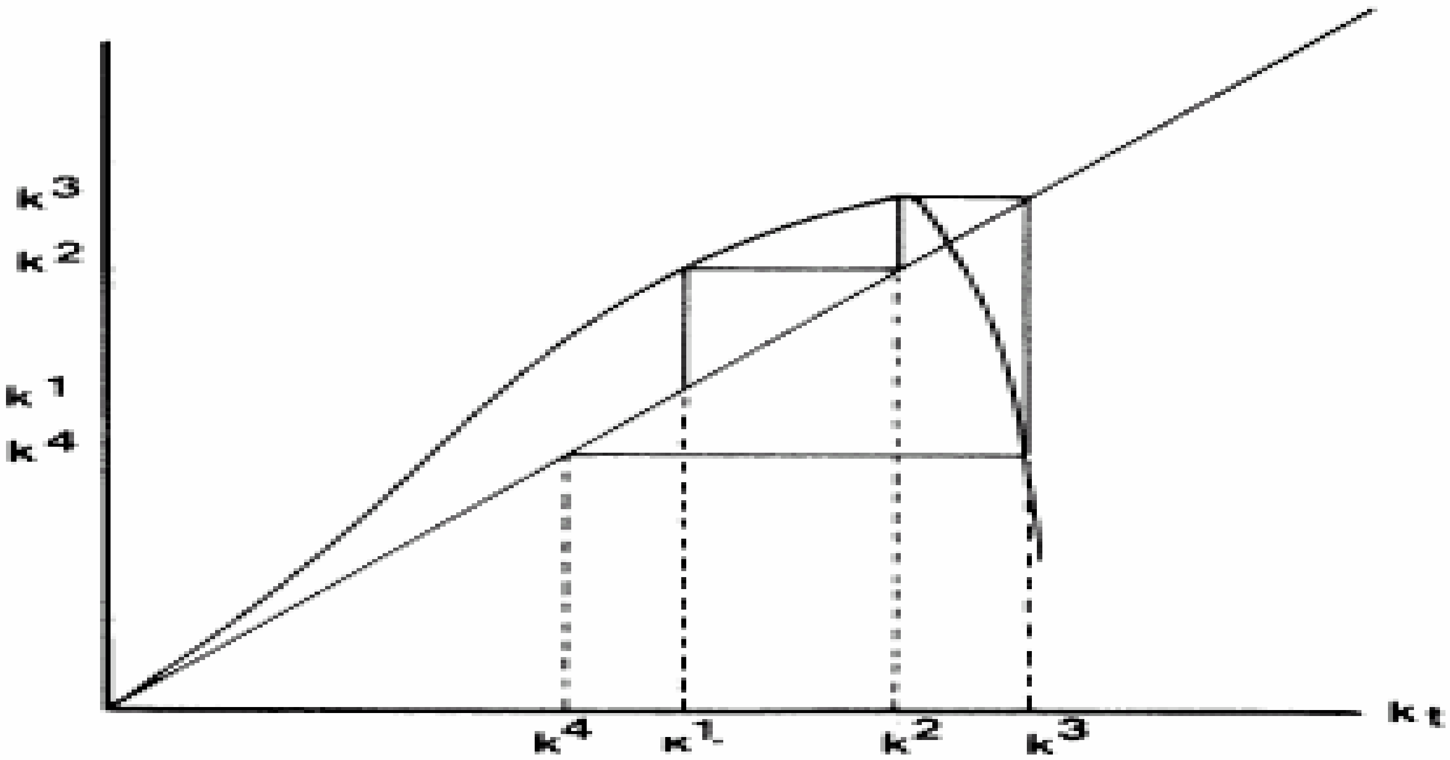
(b) Damped Cycles

Nonlinear dynamical systems



(c) Stable Two-Period Cycles

Nonlinear dynamical systems



(d) A More Complex Trajectory

Cellular automata

- The Cellular Automaton is an array of cells with a particular geometry (arranged on a line segment, a circle, a segment of the 2-dimensional plane, or a torus).
- In each period of simulation each cell is characterized by a certain state, chosen from a finite set of possibilities.
- The state of each cell in the next period depends on the current state of cells in the system as a whole.
- In most of the CA, the evolution of a single cells depends only on the past states of the cells in a neighborhood containing it.

Cellular automata

A simple case

- The CA might consist of n cells arranged on a line segment.
- Each cell can be in one of two states (on or off).
- The neighborhood of each cell contains one cell on either side of it.
- The current configuration of neighboring cells is described by each of the eight possibilities (000, 001, 010, ..., 111)

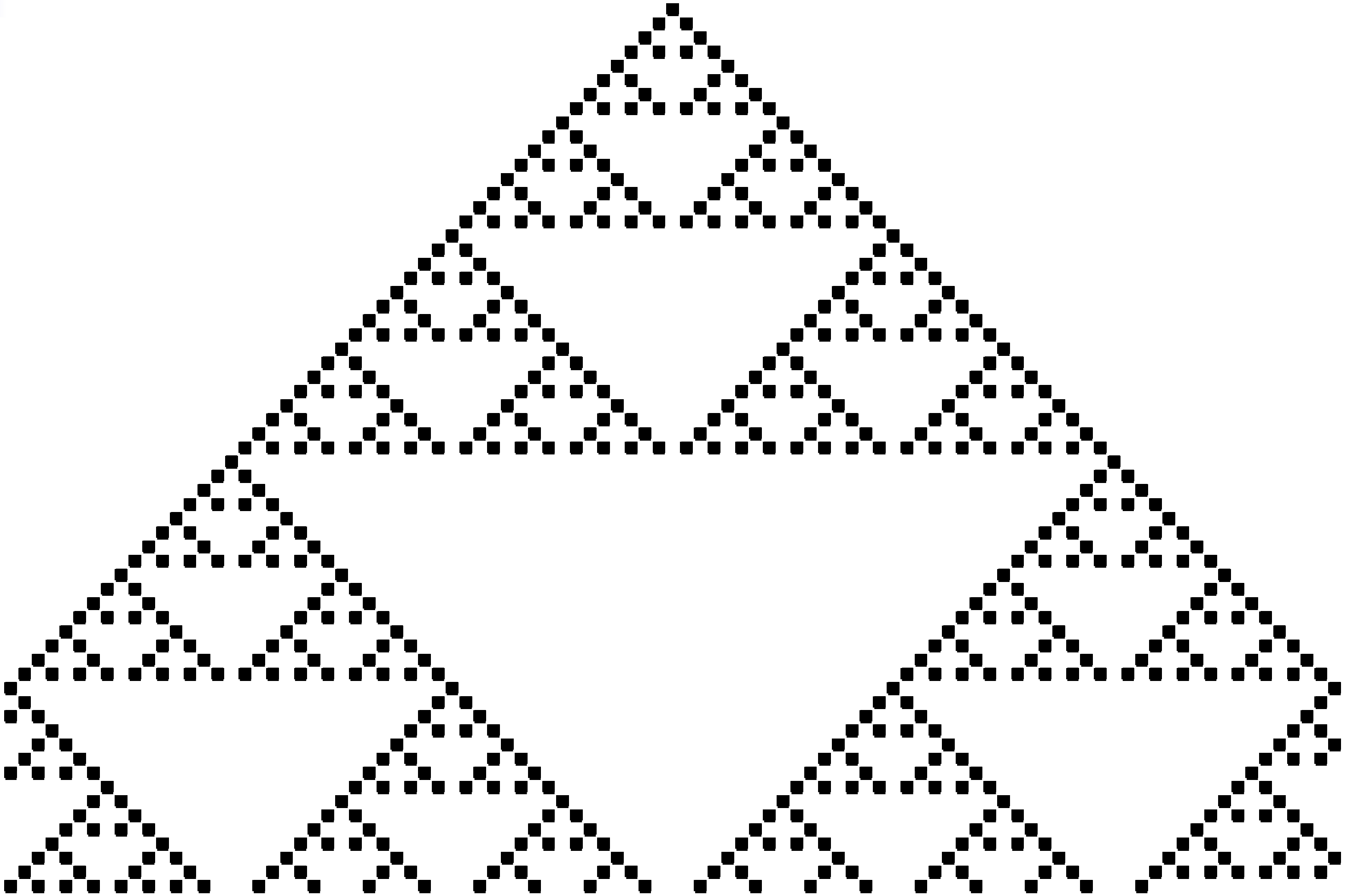
Cellular automata

A simple case

- $x_{it+1} = 1$ if $(x_{i-1t} = 1$ and $x_{it} = 1$ and $x_{i+1t} = 0)$
  or $(x_{i-1t} = 0$ and $x_{it} = 1$ and $x_{i+1t} = 1)$
  or $(x_{i-1t} = 0$ and $x_{it} = 0$ and $x_{i+1t} = 1)$
  or $(x_{i-1t} = 1$ and $x_{it} = 0$ and $x_{i+1t} = 0)$

  $= 0$ otherwise

Cellular automata



Cellular automata

A simple case

- $x_{it+1} = 1$ if $(x_{i-1t} = 0$ and $x_{it} = 1$ and $x_{i+1t} = 0)$
  or $(x_{i-1t} = 0$ and $x_{it} = 1$ and $x_{i+1t} = 1)$
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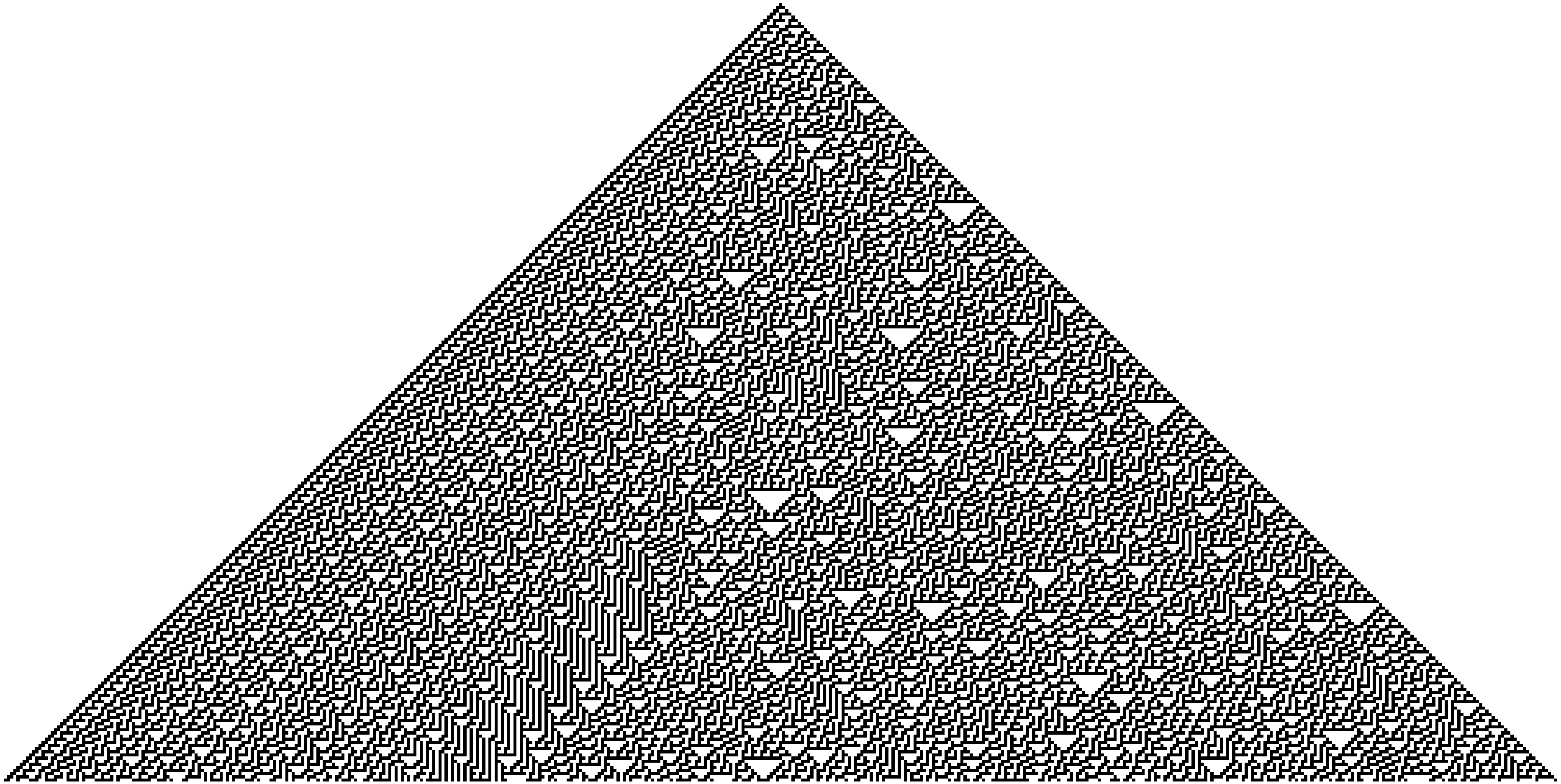
  or $(x_{i-1t} = 1$ and $x_{it} = 0$ and $x_{i+1t} = 0)$

  $= 0$ otherwise

Cellular automata



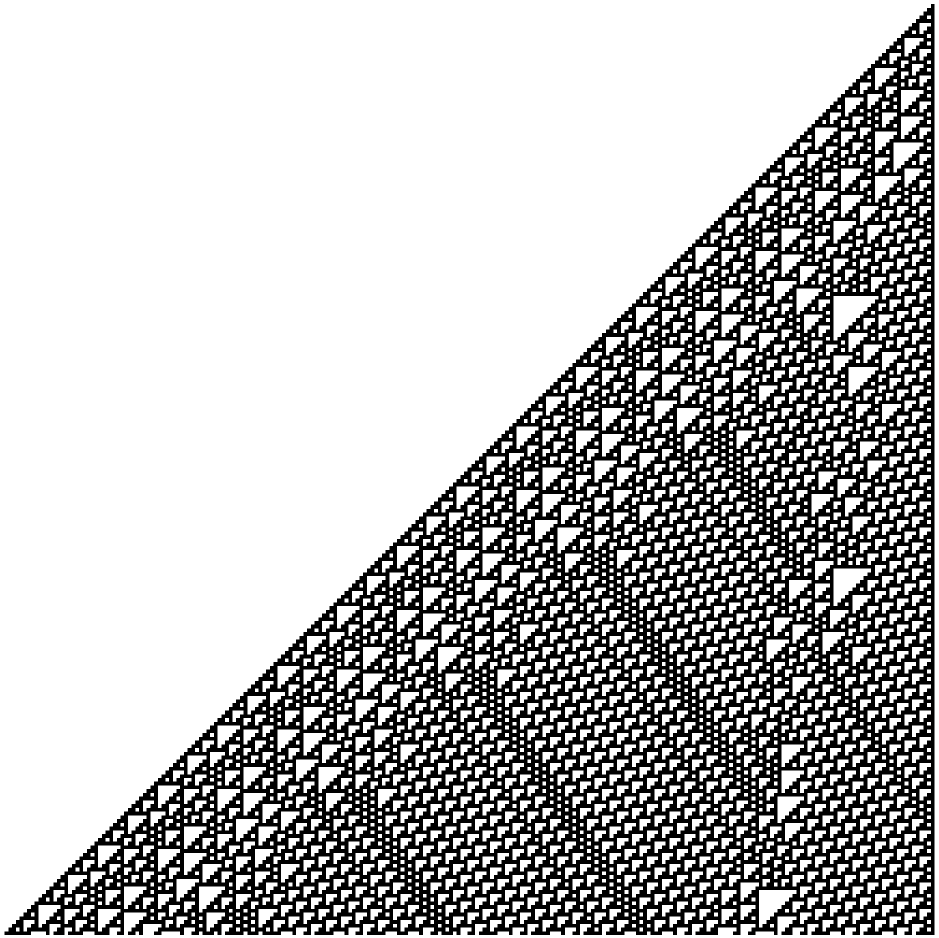
Cellular automata

A simple case

- $x_{it+1} = 1$ if ($x_{i-1t} = 1$ and $x_{it} = 1$ and $x_{i+1t} = 0$)
  or ($x_{i-1t} = 0$ and $x_{it} = 1$ and $x_{i+1t} = 1$)
  or ($x_{i-1t} = 0$ and $x_{it} = 1$ and $x_{i+1t} = 0$)
  or ($x_{i-1t} = 1$ and $x_{it} = 0$ and $x_{i+1t} = 1$)
  or ($x_{i-1t} = 0$ and $x_{it} = 0$ and $x_{i+1t} = 1$)

  $= 0$ otherwise

Cellular automata

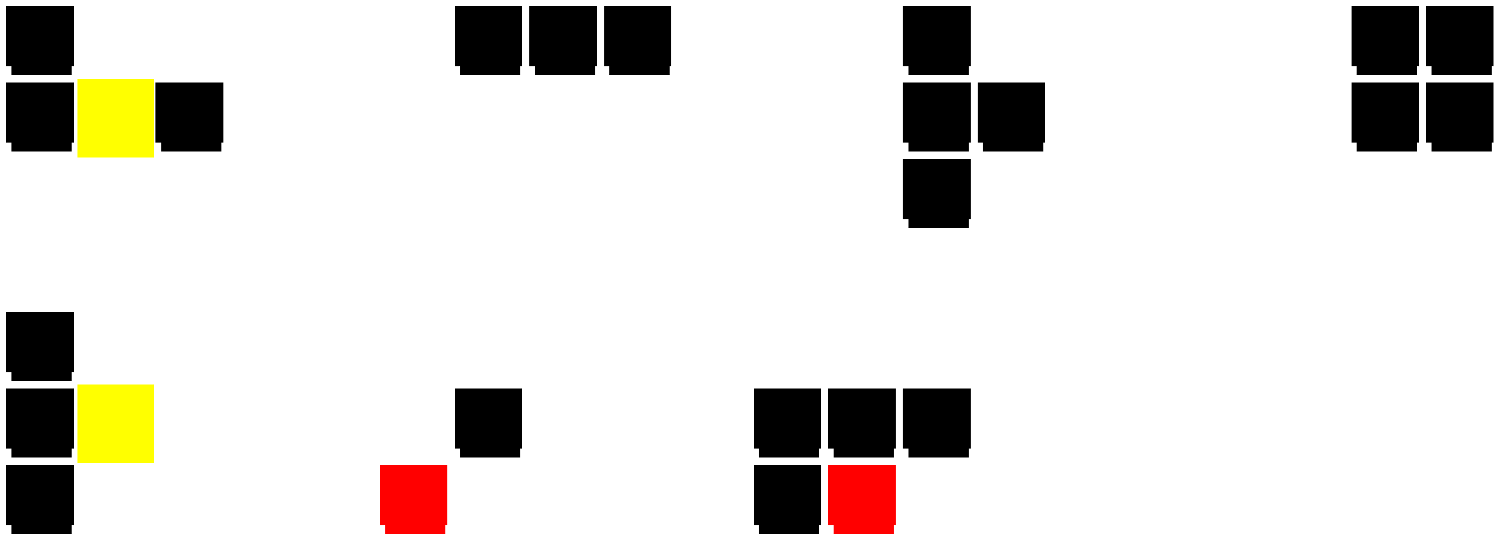


Cellular automata

2-dimensional CA

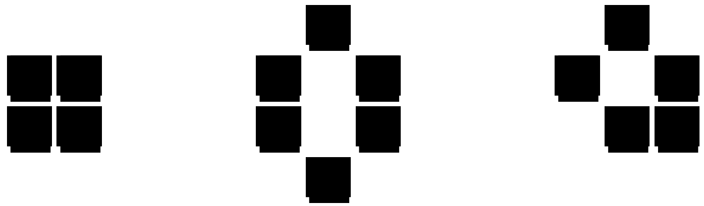
- Game of Life
- Rules
  - A dead cell with exactly three live neighbors becomes a live cell (birth).
  - A live cell with two or three live neighbors stays alive (survival).
  - In all other cases, a cell dies or remains dead (overcrowding or loneliness).

Cellular automata

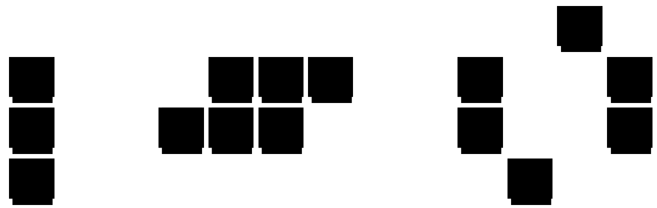


Cellular automata

- Still life objects

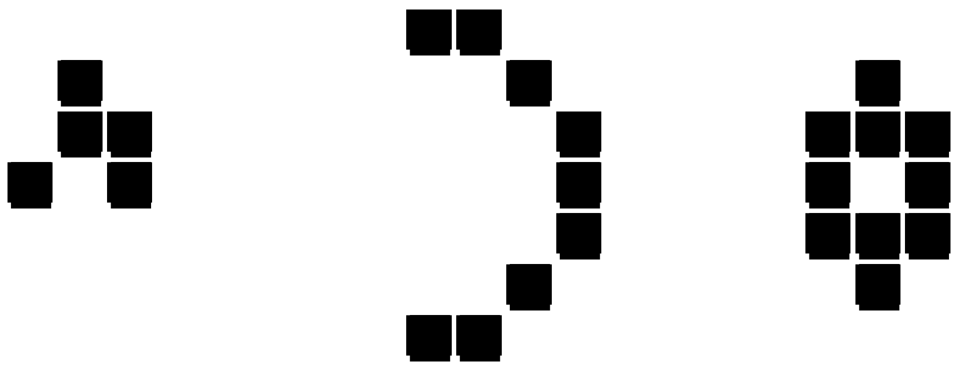


- Oscillators

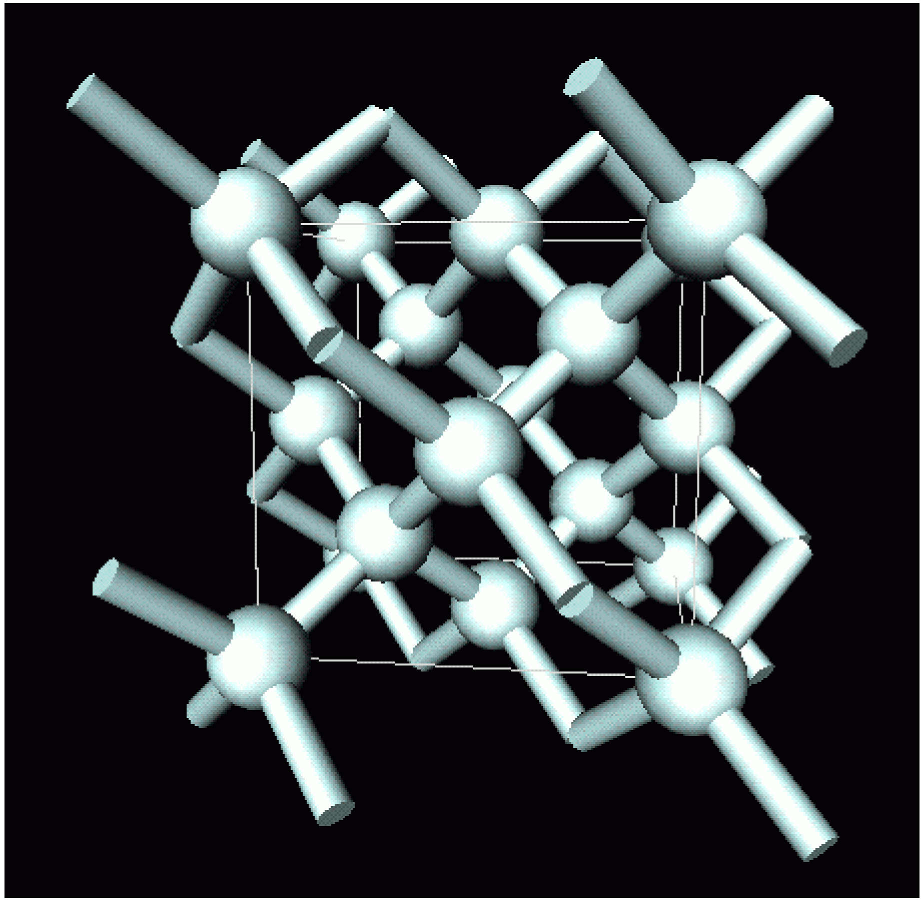


Cellular Automata

- Gliders/Shuttles/Pulsars

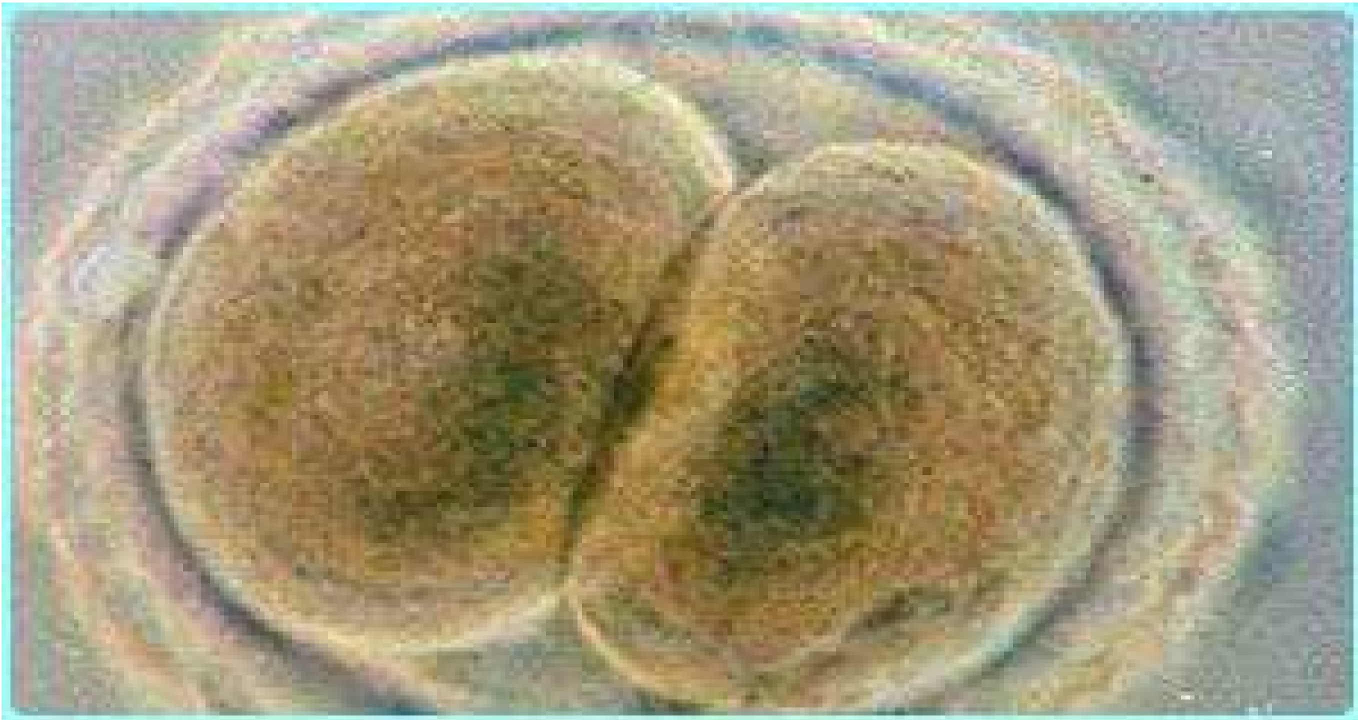


Cellular Automata



Cellular Automata

Cellular Automata



Cellular Automata