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Finite groups admitting fixed-point free automorphisms of order *pqr*

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1 Introduction

Let G be a finite group and A be a group of operators of G with $C_G(A) = 1$. In [14], Turull proved that if $(|G|, |A|) = 1$ then (with certain exceptions for A), the Fitting height of G is bounded by the length of the longest chain of subgroups of A. We expect a similar bound for the Fitting height of G when the assumption $(|G|, |A|) = 1$ is replaced by the assumption that A is nilpotent (see [1]). In [9], Cheng Kei-Nah showed that G is metanilpotent if A is a cyclic group whose order is a product of two distinct primes. Here we obtain a result that takes Kei-Nah's work one step further:

Theorem. *Let G be a finite group admitting a fixed-point free automorphism α of order pqr for pairwise distinct primes p, q and r. Then G has Fitting height at most* 3.

2 Preliminary results

First we state a well-known fact which is frequently used in this paper.

Lemma 1 (see [5]). *Let p, q, r be distinct primes and $G = QA$ where A is cyclic of order p and Q is a q-group with $[Q, A] = Q$. Assume further that G acts on a vector space V over a field k of characteristic r in such a way that $[V, A] = V$. If $[V, Q] \neq 0$, then $q = 2$.*

Lemma 2. *Let $H = ST$, where $S \lhd H$, S is a p-group and T is a t-group for distinct primes p and t, and let α be an automorphism of H of order p^n which leaves T invariant. Assume that $C_{T/T_0}(z) = 1$ where $T_0 = C_T(S)$ and $z = \alpha^{p^{n-1}}$. Let V be a $kH\langle\alpha\rangle$-module on which S acts faithfully, and k a field of characteristic different from p. If $[C_V(z), C_S(z)] = 1$, then $[S, T] = 1$.*

Proof. We set $G = H\langle\alpha\rangle$ and argue by induction on $|G| + \dim_k V$. We may assume that $n = 1$.

$[T/T_0, z] = T/T_0$

(1) *$S/\Phi(S)$ is an irreducible $T\langle\alpha\rangle$-module with $[S,T]=S$, $[\Phi(S),T]=1$ and S is special. Moreover $\Phi(T/T_0)=1$ and $\langle\alpha\rangle$ acts irreducibly on T/T_0.*

Let S_1 be a minimal element of

$$\{A \mid A \leqslant S, A \text{ is } T\langle\alpha\rangle\text{-invariant and } [A,T] \neq 1\}.$$

Then $S_1/\Phi(S_1)$ is an irreducible $T\langle\alpha\rangle$-module with $[S_1,T]=S_1$, $[\Phi(S_1),T]=1$ and S_1 is special. By induction we see that $S_1=S$.

Next let T_1/T_0 be a minimal proper $\langle\alpha\rangle$-invariant subgroup of T/T_0 on which $\langle\alpha\rangle$ acts non-trivially. Then $\Phi(T_1/T_0)=1$. Induction applied to $S[T_1,\alpha]$ on V gives that $[T_1,\alpha] \leqslant T_0$, which is impossible. Hence $[T_1,\alpha]=T=T_1$.

(2) *S is abelian.*

Assume the contrary. Then $\Phi(S)\neq 1$ and so $C_{\Phi(S)}(\alpha)\neq 1$. Now

$$U=[V, C_{\Phi(S)}(\alpha)] \neq 1 \quad \text{and} \quad C_U(C_{\Phi(S)}(\alpha))=1.$$

This shows that $C_U(\alpha)=1$, as $[C_U(\alpha), C_{\Phi(S)}(\alpha)]=1$.

If char $k=t$ then for any irreducible G-submodule W of U, $[W,T_0]$ is G-invariant and properly contained in W. Hence $[W,T_0]=1$. Set $K=\mathrm{Ker}(G \text{ on } W)$ and $\bar{G}=G/K$. Now $\bar{T}$ is abelian and $[\bar{T},\alpha]=\bar{T}\leqslant[\bar{H},\alpha]$. In this case, we may consider the action of $\bar{G}$ on W and apply [4, Lemma 1.1]. We conclude that $[[\bar{H},\alpha],\bar{S}/\Phi(\bar{S})]=1$, which is impossible as $[\bar{S},\bar{T}]=\bar{S}$. Thus char $k \nmid |G|$.

Consider $T\langle\alpha\rangle$ on U. Now $[U,T]\neq 1$, because otherwise $[U,S]=1$ and so $[V,S,C_{\Phi(S)}(\alpha)]=1$ which is not the case. Using Lemma 1 we get $t=2$ as $C_U(\alpha)=1$. On the other hand, let R be a maximal abelian normal subgroup of the p-group $S\langle\alpha\rangle$. Now $R=C_{S\langle\alpha\rangle}(R)$. If $\alpha\notin R$, then $[R,\alpha]\neq 1$. If $Y=[U,[R,\alpha]]=1$, then $[R,\alpha]\leqslant C_S(U)\leqslant\Phi(S)$ and so $1\neq C_{[R,\alpha]}(\alpha)\leqslant C_{\Phi(S)}(\alpha)$. It follows that

$$C_V(C_{\phi(S)}(\alpha)) \leqslant C_V(C_{[R,\alpha]}(\alpha)).$$

As $V=U\oplus C_V(C_{\Phi(S)}(\alpha))$, we get $[V,C_{[R,\alpha]}(\alpha)]=1$ which is a contradiction. Thus $Y\neq 1$. As $[Y,[R,\alpha]]=Y$, by [6, Lemma 4.5] we have $|Y|=|C_Y(\alpha)|^p$, which is impossible since $C_Y(\alpha)=1$. Therefore $[R,\alpha]=1$, that is, $\alpha\in R$. It follows that

$$[S\langle\alpha\rangle,\alpha,\alpha]\leqslant[S\langle\alpha\rangle,R,\alpha]\leqslant[R,\alpha]=1$$

and so the minimal polynomial of α on $S/\Phi(S)$ has degree at most 2. On the other hand, [7, Theorem IX. 1.10] gives that the minimal polynomial of α on $S/\Phi(S)$ is x^p-1 as T/T_0 is abelian. Consequently $p=2=t$, a contradiction. Therefore S is abelian.

As V is a completely reducible S-module we have $V|_S=V_1\oplus\cdots\oplus V_l$, where $V_1,\ldots,V_l$ are the homogeneous S-components of V and $T\langle\alpha\rangle$ permutes $V_1,\ldots,V_l$.

(3) *There exists an $\langle\alpha\rangle$-invariant homogeneous S-component.*

If no V_i is α-invariant, put

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for $i=1,\ldots,l$, the subgroup
not the case.

(4) *Each $T\langle\alpha\rangle$-orbit of $\{V_1,$*

Let U and W belong
fixed by α. Since $U^y=W$
$yN_T(U)\in C_{T/N_T(U)}(\alpha)$. As

(5) *S acts trivially on each T*

Let V_i be a $T\langle\alpha\rangle$-invaria
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(6) *$C_S(\alpha)$ centralizes every*
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An orbit having no $\langle\alpha\rangle$-in
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(7) *Let $\{y_0=1,y_1,\ldots,y_m\}$*
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Consider the $T\langle\alpha\rangle$-orbit
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$C_S(\alpha)\leqslant\mathrm{Ker}(S$ on $U^{y_i})$ for

(8) Finally, let L be the int
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we have $C_S(\alpha)\cap L=1$ by (

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This gives that $C_S(\alpha)^{y_i}\cap\langle C$

a $T\langle\alpha\rangle$-submodule of S. W

If no V_i is α-invariant, put $X_i = V_i \oplus V_i^{\alpha} \oplus \cdots \oplus V_i^{\alpha^{p-1}}$ for $i = 1, \ldots, l$. Since

$$\{v + v^{\alpha} + \cdots + v^{\alpha^{p-1}} \mid v \in V_i\} \leqslant C_{X_i}(\alpha)$$

for $i = 1, \ldots, l$, the subgroup $C_S(\alpha)$ acts trivially on each V_i and hence on V, which is not the case.

(4) *Each $T\langle\alpha\rangle$-orbit of $\{V_1, \ldots, V_l\}$ contains at most one $\langle\alpha\rangle$-invariant element.*

Let U and W belong to the same $T\langle\alpha\rangle$-orbit, and suppose that both are fixed by α. Since $U^y = W$ for some $y \in T\langle\alpha\rangle$, we have $[y, \alpha] \in N_T(U)$ and so $yN_T(U) \in C_{T/N_T(U)}(\alpha)$. As $C_T(\alpha) \leqslant T_0 \leqslant N_T(U)$, we see that $y \in N_T(U)$.

(5) *S acts trivially on each $T\langle\alpha\rangle$-invariant homogeneous S-component.*

Let V_i be a $T\langle\alpha\rangle$-invariant subspace on which S acts non-trivially. Choose an irreducible $ST\langle\alpha\rangle$-submodule M of V_i on which S acts non-trivially. Now $M|_S$ is homogeneous and so $[S/\mathrm{Ker}(S \text{ on } M), T] = 1$, which is not the case.

(6) *$C_S(\alpha)$ centralizes every element of a $T\langle\alpha\rangle$-orbit containing no $\langle\alpha\rangle$-invariant element.*

An orbit having no $\langle\alpha\rangle$-invariant element can be written as a union of $\langle\alpha\rangle$-orbits, and the same argument as in (3) gives the result.

(7) *Let $\{y_0 = 1, y_1, \ldots, y_m\}$ be a transversal to T_0 in T and let U be a $T_0\langle\alpha\rangle$-invariant homogeneous S-component. Then $C_S(\alpha)^{y_j^{-1}} \leqslant \mathrm{Ker}(S \text{ on } U)$ for $j = 1, \ldots, m$.*

Consider the $T\langle\alpha\rangle$-orbit $\{U, U^{y_1}, \ldots, U^{y_m}\}$ containing U. As $\{U^{y_1}, \ldots, U^{y_m}\}$ is $\langle\alpha\rangle$-invariant, it is a union of $\langle\alpha\rangle$-orbits and the same argument as in (3) gives that $C_S(\alpha) \leqslant \mathrm{Ker}(S \text{ on } U^{y_j})$ for $j = 1, \ldots, m$, that is, $C_S(\alpha)^{y_j^{-1}} \leqslant \mathrm{Ker}(S \text{ on } U)$.

(8) Finally, let L be the intersection of the kernels of the actions of S on all $\langle\alpha\rangle$-invariant homogeneous S-components which are not $T\langle\alpha\rangle$-invariant. As

$$[V, C_S(\alpha)] \neq 1,$$

we have $C_S(\alpha) \cap L = 1$ by (5) and (6). Also (7) shows that

$$\langle C_S(\alpha)^t \mid t \in T - T_0\rangle \leqslant L.$$

Hence

$$C_S(\alpha) \cap \langle C_S(\alpha)^t \mid t \in T - T_0\rangle \leqslant C_S(\alpha) \cap L = 1.$$

This gives that $C_S(\alpha)^{y_i} \cap \langle C_S(\alpha)^{y_j} \mid y_i \neq y_j\rangle = 1$ and so

$$\sum_{\bar{t} \in T/T_0} C_S(\alpha)^{\bar{t}} = \bigoplus_{\bar{t} \in T/T_0} C_S(\alpha)^{\bar{t}},$$

a $T\langle\alpha\rangle$-submodule of S. We conclude that

$$S = \bigoplus_{\bar{t} \in T/T_0} C_S(\alpha)^{\bar{t}}$$

as S is irreducible, and so $|S| = |C_S(\alpha)|^{|T/T_0|}$. On the other hand, $[S, [T/T_0, \alpha]] = S$ and so $|S| = |C_S(\alpha)|^p$ by [6, Lemma 4.5]. As $t \neq p$ we get a contradiction and this completes the proof.

We write $f(G)$ for the Fitting length of a group G.

Corollary 3 (Cheng Kei-Nah [9]). *Let G be a finite group admitting a fixed-point free automorphism α of order pq, where p and q are distinct primes. Then $f(G) \leqslant 2$.*

Proof. Set $\langle\alpha\rangle = \langle\alpha_p\rangle \times \langle\alpha_q\rangle$ where $|\alpha_p| = p$ and $|\alpha_q| = q$. We argue by induction on $|G|$. As $C_G(\alpha) = 1$, for any prime dividing $|G|$ we have a unique $\langle\alpha\rangle$-invariant Sylow subgroup of G and so we obtain an $\langle\alpha\rangle$-tower (C_i) $(i = 1, 2, 3)$ in the sense of [13] except that we have reversed the order of the indices, that is, we let C_j normalize C_i for $i < j$. By induction we have $G = C_1 C_2 C_3$ where $C_1 = F(G)$ is the unique minimal normal subgroup of G. By the Fong–Swan Theorem we may assume that $(|C_1|, |C_2 C_3\langle\alpha\rangle|) = 1$. If $pq \nmid |C_2 C_3|$, then we know the result. Hence assume that p divides $|C_2 C_3|$. If G is a q'-group, [13, Theorem 3.1] implies that

$$[C_{C_1}(\alpha_q), C_{C_3}(\alpha_q)] \neq 1.$$

As $C_G(\alpha_q)$ is nilpotent, we get that $\pi(C_1) = \pi(C_3)$, which is not the case. Hence p and q both divide $|C_2 C_3|$. Let $\pi(C_2) = \{p\}$ and $\pi(C_3) = \{q\}$. Now $C_{C_3}(\alpha_p) = 1$. As $[C_{C_1}(\alpha_p), C_{C_2}(\alpha_p)] = 1$, we may apply Lemma 2 to $C_2 C_3 \langle\alpha_p\rangle$ on C_1 and obtain that $[C_2, C_3] = 1$, a contradiction.

Lemma 4. *Let VST be an $\langle\alpha\rangle$-tower in the sense of* [13] *where*

(i) $\langle\alpha\rangle = \langle\alpha_p\rangle \times \langle\alpha_q\rangle$ *with* $|\alpha_p| = p^n$ *and* $|\alpha_q| = q$ *for distinct primes p and q,*

(ii) $\pi(S) = \{p\}$, $\pi(T) = \{t\}$ *with* $t \notin \{p, q\}$, $\pi(V) \notin \{p, q\}$,

(iii) $\Phi(\Phi(S)) = 1$, $[S, T] = S$, $[\Phi(S), T] = 1$,

(iv) $C_V(\alpha) = 1$, $[S, \alpha_q] = S$ *and* $[T, z] = T$ *where* $z = \alpha_p^{p^{n-1}}$.

Then $[C_V(\alpha_q), C_T(\alpha_q)] \neq 1$.

Proof. Let $T_0 = C_T(S)$. If there exists $\bar{t} \in C_{T/T_0}(\alpha_q)$ with $|\bar{t}| > 2$, then we get $[C_V(\alpha_q), C_T(\alpha_q)] \neq 1$ as desired, by [13, Theorem 2.1.B]. So either $C_T(\alpha_q) \leqslant T_0$ or $C_{T/T_0}(\alpha_q)$ is an elementary abelian 2-group. The first is impossible since

$$C_{S/\Phi(S)}(\alpha_q) = 1 \quad \text{and} \quad [S, T] = S.$$

Hence $t = 2$, that is, $p \neq 2$. As $t \neq q$ and $C_T(\alpha_q) \nleqslant T_0$, we may assume that $[T, \alpha_q] = 1$.

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T_0, we may assume that

Now we consider $VST\langle z\rangle$ as an $\langle\alpha_q\rangle$-tower. Since $p \neq 2$, by [13, Theorem 3.1], the only possible towers for centralizers inside this tower are the following:

(a) $C_S(\alpha_q)C_T(\alpha_q)\langle z\rangle$; (b) $C_V(\alpha_q)C_S(\alpha_q)\langle z\rangle$; (c) $C_V(\alpha_q)C_T(\alpha_q)\langle z\rangle$.

Now (a) is impossible since $C_S(\alpha_q) \leqslant \Phi(S)$ and $[\Phi(S), T] = 1$. Assume that (b) holds. Then $[[C_S(\alpha_q), z], C_V(\alpha_q)] \neq 1$. As α_p acts fixed-point freely on $C_V(\alpha_q)$, an application of [3, Lemma 1.1] to $C_S(\alpha_q)\langle\alpha_p\rangle$ on $C_V(\alpha_q)$ leads to a contradiction. Thus (c) holds and so $[C_V(\alpha_q), C_T(\alpha_q)] \neq 1$.

3 Proof of the Theorem

Set $\langle\alpha\rangle = \langle\alpha_p\rangle \times \langle\alpha_q\rangle \times \langle\alpha_r\rangle$ where $|\alpha_p| = p$, $|\alpha_q| = q$ and $|\alpha_r| = r$. Let G be a minimal counter-example to the theorem. As $C_G(\alpha) = 1$, for any prime dividing $|G|$ we have a unique $\langle\alpha\rangle$-invariant Sylow subgroup of G, and so we obtain an irreducible $\langle\alpha\rangle$-tower (C_i) $(i = 1, 2, 3, 4)$ in the sense of [13] satisfying the following:

(i) $\pi(C_i) = \{p_i\}$ consists of a single prime for $i = 1, 2, 3, 4$ and $p_i \neq p_{i+1}$ for $i = 1, 2, 3$;

(ii) C_i is $\langle\alpha\rangle$-invariant for $i = 1, 2, 3, 4$ and C_i is normalized by C_j for $j > i$ and $i = 1, 2, 3$;

(iii) $\bar{C}_i = C_i/D_i$ is a special group on the Frattini factor group of which $(\prod_{j>i} C_j)\langle\alpha\rangle$ acts irreducibly for $i = 1, 2, 3$ where $D_i = C_{C_i}(C_{i-1}/D_{i-1})$ for $i > 1$ and $D_1 = 1$;

(iv) $[C_i, C_{i+1}] = C_i$ for $i = 1, 2, 3$.

Then by induction we see that $G = C_1C_2C_3C_4$ where $C_1 = F(G)$ is the unique minimal normal subgroup of G. Put $H = C_2C_3C_4$. Now $C_H(C_1) = 1$. As C_1 is an irreducible $H\langle\alpha\rangle$-module, by the Fong–Swan Theorem, we may assume that $(|C_1|, |H\langle\alpha\rangle|) = 1$.

Let W be a homogeneous H-component of C_1 on which C_2 acts non-trivially. Put $B = N_{\langle\alpha\rangle}(W)$ and $\bar{H} = H/\mathrm{Ker}(H \text{ on } W)$. Then W is a homogeneous and faithful $\bar{H}$-module and $C_W(B) = 0$ as $C_{C_1}(\alpha) = 1$. Therefore $B \neq 1$.

If $(|H|, |B|) = 1$, we see that $C_W(C_B(\mathrm{supp}_B(\bar{H})) = 0$ by [12, Proposition 4.5]. Then $C_B(\mathrm{supp}_B(\bar{H})) \neq 1$ and $1 \neq \overline{C_2} \leqslant \mathrm{supp}_B(\bar{H})$. It follows that $C_B(\mathrm{supp}_B(\bar{H})) \leqslant C_B(\overline{C_2})$. But $C_B(\overline{C_2}) = 1$, because otherwise $[\bar{H}, C_B(\overline{C_2})] = 1$, which is not the case as the centralizer of a Sylow subgroup of $\langle\alpha\rangle$ has Fitting height at most 2 by the Corollary and since $f(\bar{H}) = 3$. This contradiction shows that $(|H|, |B|) \neq 1$.

On the other hand if $B = \langle\alpha\rangle$, then C_1 is a homogeneous H-module and so it is irreducible by [10, Theorem B.7.11]. Then we apply [8, Theorem] and get $C_H(\alpha) \neq 1$, a contradiction. Thus $1 \neq B < \langle\alpha\rangle$. Set $\langle\alpha\rangle = B \oplus B'$ and consider $Y = C_1C_H(B')$. Now $C_Y(B) = 1$. If $|B|$ is a prime, then Y is nilpotent. It follows that $C_H(B') = 1$ as $C_H(C_1) = 1$, and now the Corollary gives that $f(H) \leqslant 2$ which is not the case. Thus $|\pi(B)| = 2$. Let $|B| = pq$ and let $\overline{C_i/D_i}$ denote the Frattini factor group of C_i/D_i. Observe that $D_2 = 1$ and that C_4/D_4 is elementary abelian. Also note that $C_H(\alpha_r)$ is nilpotent because $C_1 C_H(\alpha_r)$, as a group on which $\alpha_p\alpha_q$ acts fixed-point freely, is of Fitting height at most 2.

We shall frequently use [13, Theorem 3.1], which implies the following:

Let A be a group of prime order acting on a group G with $(|G|, |A|) = 1$. Let (C_i) $(i = 1, \ldots, h)$ be an A-tower and assume that A centralizes C_k (possibly with $k = h + 1$ and $C_{h+1} = 1$). Then for some $j \leqslant k$ the tower

$$(C_{C_i}(A)) \quad (i = 1, \ldots, j-1, j+1, \ldots, h)$$

satisfies $[C_{C_s}(A), C_{C_t}(A)] \neq 1$ for $s \neq t$. If $|C_k|$ is odd, we may take $j < k$.

We call $(C_{C_i}(A))$ $(i = 1, \ldots, j-1, j+1, \ldots, h)$ a possible A-tower inside $C_G(A)$.

(1) *p and q divide $|H|$.*

Suppose that p divides $|H|$ but q does not. Then [13, Theorem 3.1] shows that the only possible tower inside $C_G(\alpha_q)$ is $C_{C_1}(\alpha_q)C_{C_2}(\alpha_q)C_{C_4}(\alpha_q)$ and $\pi(C_2) = \pi(C_4)$ as $f(C_G(\alpha_q)) \leqslant 2$, by the Corollary.

First assume that $\pi(C_2) = \pi(C_4) = \{p\}$. Here we have $D_4 = C_{C_4}(C_2)$ and $[C_4, \alpha_p] \leqslant D_4$. If $C_{\overline{C_3/D_3}}(\alpha_p) \neq 1$, then α_p acts trivially on $C_2(C_3/D_3)(C_4/D_4)$ and so $\alpha_q\alpha_r$ acts fixed-point freely on it, and this is impossible by the Corollary. Hence $C_{\overline{C_3/D_3}}(\alpha_p) = 1$. Also observe that $[C_4, \alpha_q] \leqslant D_4$: otherwise $C_{C_4}(\alpha_q) \leqslant D_4 = C_{C_4}(C_2)$ and this is not the case as $[C_{C_2}(\alpha_q), C_{C_4}(\alpha_q)] \neq 1$. We may assume that $[C_4, \alpha_q] = 1$ as $\pi(C_4) \neq \{q\}$ which implies that $C_{C_4}(\alpha_r) = 1$. If $\pi(C_3) \neq \{r\}$, then G is an r'-group and applying [13, Theorem 3.1] we see that the only possible tower inside $C_G(\alpha_r)$ is $C_{C_1}(\alpha_r)C_{C_2}(\alpha_r)C_{C_4}(\alpha_r)$. Consequently $C_{C_4}(\alpha_r) \neq 1$ which is not the case. Thus $\pi(C_3) = \{r\}$. If $r = 2$ then, as α_r acts fixed-point freely on $C_{C_2}(\alpha_q)C_{C_4}(\alpha_q)$, we must have $[C_{C_2}(\alpha_q), C_{C_4}(\alpha_q)] = 1$ and this is not the case. Hence $r \neq 2$. Now an application of Lemma 2 to $\overline{C_2}(C_3/D_3)C_4\langle\alpha_r\rangle$ gives that

$$[C_{\overline{C_2}}(\alpha_r), C_{C_3/D_3}(\alpha_r)] \neq 1.$$

If $[C_{\overline{C_2}}(\alpha_r), [C_{C_3/D_3}(\alpha_r), \alpha_q]] \neq 1$, then $r = 2$ by Lemma 1, which is not the case. Hence

$$[C_{\overline{C_2}}(\alpha_r), C_{C_3/D_3}(\alpha_r\alpha_q)] \neq 1 \quad \text{as} \quad C_{C_3/D_3}(\alpha_r) = [C_{C_3/D_3}(\alpha_r), \alpha_q]C_{C_3/D_3}(\alpha_r\alpha_q).$$

Now $C_{C_3/D_3}(\alpha_r\alpha_q) = C_{X/D_3}(\alpha_q) = C_X(\alpha_q)D_3/D_3$ where $C_{C_3/D_3}(\alpha_r) = X/D_3$. So

$$[C_{\overline{C_2}}(\alpha_r), C_X(\alpha_q)] \neq 1.$$

Note that X normalizes $C_{C_2}(\alpha_r)$ by the three subgroup lemma as $[\alpha_r, C_{C_2}(\alpha_r), X] = 1$ and $[X, \alpha_r, C_{C_2}(\alpha_r)] \leqslant [D_3, C_{C_2}(\alpha_r)] = 1$. But then $C_1 C_{C_2}(\alpha_r)C_X(\alpha_q)$ is a group of Fitting height 3 on which $\alpha_p\alpha_q$ acts fixed-point freely, a contradiction by the Corollary. Thus $\{s\} = \pi(C_2) = \pi(C_4) \neq \{p\}$ and $\pi(C_3) = \{p\}$. Now $C_{C_4}(\alpha_p\alpha_q) \neq 1$, because otherwise α_p acts fixed-point freely on $C_{C_1}(\alpha_q)C_{C_4}(\alpha_q)$ which is impossible as $[C_{C_1}(\alpha_q), C_{C_4}(\alpha_q)] \neq 1$. Then $s \neq r$ and $[C_4, \alpha_r] \neq 1$.

It follows that G is an r'-grou
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(2) *r divides $|H|$.*

Assume the contrary. Then
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and $\pi(C_3) = \{q\}$.

If $[C_4, \alpha_r] \nleqslant D_4$, we get C_C
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On the other hand Lemma 2

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As $C_{\overline{C_2}}(\alpha_r) \neq 1 \neq C_{\overline{C_2}}(\alpha_q)$,
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It follows that $[C_{\overline{C_2}}(\alpha_r), \Phi(C$

$1 \neq$

a contradiction.

(3) $\pi(C_2) \subset \{p, q\}$.

Now H is a $\{p, q, r\}$-gr
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(4) $\pi(C_3) \subset \{p, q\}$.

Assume that $\pi(C_2) = \{$
$[C_4, \alpha_r] = 1$, then $C_{C_4}(\alpha_p) =$
that $C_{C_1}(\alpha_p)C_{C_3}(\alpha_p)$ is the c

ı $\alpha_p\alpha_q$ acts fixed-point freely, is of

h implies the following:
group G with $(|G|,|A|)=1$. *Let*
t A centralizes C_k *(possibly with*
tower

$i+1,\dots,h)$

ld, we may take $j<k$.

ι possible A-tower inside $C_G(A)$.

en [13, Theorem 3.1] shows that
$\alpha_q)C_{C_4}(\alpha_q)$ and $\pi(C_2)=\pi(C_4)$ as

e we have $D_4=C_{C_4}(C_2)$ and
ly on $C_2(C_3/D_3)(C_4/D_4)$ and so
ossible by the Corollary. Hence
herwise $C_{C_4}(\alpha_q)\leqslant D_4=C_{C_4}(C_2)$
Ve may assume that $[C_4,\alpha_q]=1$
$\pi(C_3)\neq\{r\}$, then G is an r'-
the only possible tower inside
$(\alpha_r)\neq 1$ which is not the case.
nt freely on $C_{C_2}(\alpha_q)C_{C_4}(\alpha_q)$, we
he case. Hence $r\neq 2$. Now an
hat

1.

ı 1, which is not the case. Hence

$C_{C_3/D_3}(\alpha_r),\alpha_q]C_{C_3/D_3}(\alpha_r\alpha_q)$.

e $C_{C_3/D_3}(\alpha_r)=X/D_3$. So

lemma as $[\alpha_r,C_{C_2}(\alpha_r),X]=1$
$C_{C_2}(\alpha_r)C_X(\alpha_q)$ is a group of
', a contradiction by the Cor-
$\{p\}$. Now $C_{C_4}(\alpha_p\alpha_q)\neq 1$, be-
$C_{C_4}(\alpha_q)$ which is impossible as

It follows that G is an r'-group and $C_{C_4}(\alpha_r)\leqslant D_4=C_{C_4}(C_2)$. But an application of [13, Theorem 3.1] shows that $C_{C_1}(\alpha_r)C_{C_2}(\alpha_r)C_{C_4}(\alpha_r)$ is the only possible tower inside $C_G(\alpha_r)$, that is, $[C_{C_2}(\alpha_r),C_{C_4}(\alpha_r)]\neq 1$, a contradiction.

(2) *r divides* $|H|$.
Assume the contrary. Then as in the above argument, $C_{C_1}(\alpha_r)C_{C_2}(\alpha_r)C_{C_4}(\alpha_r)$ is a tower inside $C_G(\alpha_r)$ implying that $[C_{C_2}(\alpha_r),C_{C_4}(\alpha_r)]\neq 1$ and $\pi(C_2)=\pi(C_4)$ as $f(C_G(\alpha_r))\leqslant 2$ by the Corollary. Now H is a $\{p,q\}$-group; let $\pi(C_2)=\pi(C_4)=\{p\}$ and $\pi(C_3)=\{q\}$.

If $[C_4,\alpha_r]\nleqslant D_4$, we get $C_{C_4}(\alpha_r)\leqslant C_{C_4}(C_2)=D_4$ which is not the case. Thus we may assume that $[C_4,\alpha_r]=1$ as $\pi(C_4)\neq\{r\}$ and so $C_{C_4}(\alpha_q)=1$. We have $q\neq 2$, since $C_{C_2C_4}(\alpha_q\alpha_r)=1$ and $C_{C_2C_4}(\alpha_r)$ is non-abelian. Moreover $C_{\overline{C_3/D_3}}(\alpha_r)=1$, since otherwise $[C_3,\alpha_r]=1$ as $\pi(C_3)\neq\{r\}$ and so $\alpha_p\alpha_q$ acts fixed-point freely on $C_1C_3C_4$, which is impossible by the Corollary. Then by Lemma 1 we have $C_{\overline{C_2}}(\alpha_r)\neq 1$ as $q\neq 2$. On the other hand Lemma 2 applied to $\overline{C_2}(C_3/D_3)C_4\langle\alpha_q\rangle$ gives that $C_{\overline{C_2}}(\alpha_q)\neq 1$.

If C_3/D_3 is abelian, then $C_{C_3/D_3}(\alpha_r)=1$ and so $\alpha_q\alpha_r$ acts fixed-point freely on $\overline{C_2}(C_3/D_3)(C_4/D_4)$ which is not the case by the Corollary. Therefore C_3/D_3 is non-abelian. Let W be a homogeneous $\Phi(C_3/D_3)$-component of the irreducible $(C_3/D_3)(C_4/D_4)\langle\alpha\rangle$-module $\overline{C_2}$. Set $N=N_{\langle\alpha\rangle}(W)$. Note that N coincides with the stabilizer of any other homogeneous component and $C_{\overline{C_2}}(N)=1$ and $N\neq 1$. Also $N\neq\langle\alpha\rangle$, because otherwise $1\neq\Phi(C_3/D_3)\leqslant C_{C_3/D_3}(\alpha)$, a contradiction.

As $C_{\overline{C_2}}(\alpha_r)\neq 1\neq C_{\overline{C_2}}(\alpha_q)$, either $N=\langle\alpha_q\alpha_r\rangle$ or $\alpha_p\in N$. If $\alpha_p\in N$, then $\langle\alpha_p\rangle$ is properly contained in N, as $\pi(C_2)=\{p\}$, i.e. either $N=\langle\alpha_p\alpha_q\rangle$ or $N=\langle\alpha_p\alpha_r\rangle$. In the first case $C_{\overline{C_2}}(\alpha_q)=1$ and in the latter case $C_{\overline{C_2}}(\alpha_r)=1$; and both are impossible by our observations above. Therefore $N=\langle\alpha_q\alpha_r\rangle$ and so $[\Phi(C_3/D_3),\alpha_q\alpha_r]=1$, implying that

$$C_{C_3/D_3}(\alpha_r)=\Phi(C_3/D_3)=C_{C_3/D_3}(\alpha_q\alpha_r).$$

It follows that $[C_{\overline{C_2}}(\alpha_r),\Phi(C_3/D_3)]=1$ as $C_H(\alpha_r)$ is nilpotent. Thus

$$1\neq C_{\overline{C_2}}(\alpha_r)\leqslant C_{\overline{C_2}}(\Phi(C_3/D_3))=1,$$

a contradiction.

(3) $\pi(C_2)\subset\{p,q\}$.
Now H is a $\{p,q,r\}$-group. If $\pi(C_2)=\{r\}$, then $C_1C_2C_{C_3}(\alpha_r)$ is a group on which $\alpha_p\alpha_q$ acts fixed-point freely. It follows that $[C_2,C_{C_3}(\alpha_r),C_1]=1$ and so $[C_2,C_{C_3}(\alpha_r)]=1$ by the Corollary, that is, $C_{C_3}(\alpha_r)\leqslant D_3$. Thus $[C_4,\alpha_r]\leqslant D_4$. As $\pi(C_4)\neq\{r\}$, we may assume that $[C_4,\alpha_r]=1$. Then $C_{C_4}(\alpha_p\alpha_q)=1$ and so $\alpha_p\alpha_q$ acts fixed-point freely on $C_1C_2C_4$ which is impossible by the Corollary.

(4) $\pi(C_3)\subset\{p,q\}$.
Assume that $\pi(C_2)=\{p\}$, $\pi(C_3)=\{r\}$ and $\pi(C_4)=\{q\}$. Then $[C_4,\alpha_q]\leqslant D_4$. If $[C_4,\alpha_r]=1$, then $C_{C_4}(\alpha_p)=1$ and [12, Theorem 3.1] applied to $C_3C_4\langle\alpha_p\rangle$ on C_1 gives that $C_{C_1}(\alpha_p)C_{C_3}(\alpha_p)$ is the only tower inside $C_{C_1C_3C_4}(\alpha_p)$, that is, $[C_{C_1}(\alpha_p),C_{C_3}(\alpha_p)]\neq 1$.

This contradicts the fact that $C_{C_1C_3}(\alpha_p\alpha_q) = 1$ as $\pi(C_3) = \{r\}$. Therefore $[C_4, \alpha_r] \neq 1$ which implies that $[C_4, \alpha_r] = C_4$, i.e. $C_{C_4/D_4}(\alpha_r) = 1$.

Now Lemma 2 applied to $\overline{C_2}(C_3/D_3)C_4\langle\alpha_r\rangle$ shows that $[C_{\overline{C_2}}(\alpha_r), C_{C_3/D_3}(\alpha_r)] \neq 1$. It follows that $[C_{C_2}(\alpha_r), X] \neq 1$ where $X/D_3 = C_{C_3/D_3}(\alpha_r)$. We observe that $C_{C_2}(\alpha_r)$ is normalized by X and $C_1C_2(\alpha_r)X$ is a group of Fitting height 3. But this is not the case by the Corollary since $\alpha_p\alpha_q$ acts fixed-point freely on it.

(5) The final contradiction.

By (3) and (4), we see that $\pi(C_4) = \{r\}$. Let $\pi(C_2) = \{p\}$ and $\pi(C_3) = \{q\}$. If $[C_4, \alpha_q] \leqslant D_4$, we may assume that $[C_4, \alpha_q] = 1$ as $\pi(C_4) \neq \{q\}$. It follows that $C_{C_4}(\alpha_p) = 1$ and $[\overline{C_3/D_3}, \alpha_q] = 1$ and so $[C_3/D_3, \alpha_q] = 1$ by the three subgroup lemma as $\pi(C_3) = \{q\}$. Then $C_{\overline{C_2}}(\alpha_q) = 1$ because otherwise $[\overline{C_2}(C_3/D_3)C_4, \alpha_q] = 1$ and so $\langle\alpha_p\rangle \times \langle\alpha_r\rangle$ acts fixed-point freely on $\overline{C_2}(C_3/D_3)C_4$ which is impossible by the Corollary.

Now $C_1C_2C_4$ is an $(\langle\alpha_p\rangle \times \langle\alpha_q\rangle)$-tower, where $[C_2, \alpha_q] = C_2$ and $[C_4, \alpha_p] = C_4$. We choose an irreducible $(\langle\alpha_p\rangle \times \langle\alpha_q\rangle)$-tower $E_1E_2E_4$ where $[E_2, \alpha_q] = E_2$ and $[E_4, \alpha_p] = E_4$ inside this tower and apply Lemma 3. If follows that

$$[C_{C_1}(\alpha_q), C_{C_4}(\alpha_q)] \neq 1,$$

which is impossible as α_p acts fixed-point freely on $C_{C_1C_4}(\alpha_q)$. Thus $[C_4, \alpha_q] = C_4$.

Also observe that $[C_3/D_3, \alpha_p] \neq 1$, because otherwise $\overline{C_2}(C_3/D_3)C_4$ is centralized by α_p, which is impossible by the Corollary.

If $[C_{C_4}(\alpha_p), [C_3, \alpha_p]] \neq 1$, we consider $C_1[C_3, \alpha_p]C_{C_4}(\alpha_p)$ as an $(\langle\alpha_p\rangle \times \langle\alpha_q\rangle)$-tower and pass to an irreducible tower $E_1E_3E_4$ where $[E_4, \alpha_q] = E_4$, $[E_3, \alpha_p] = E_3$. Now $[E_4, \alpha_p] = 1$ and $C_{E_1E_4}(\alpha_p\alpha_q) = 1$. But an application of Lemma 3 gives that $[C_{E_1}(\alpha_p), C_{E_4}(\alpha_p)] \neq 1$, a contradiction. Hence $[C_{C_4}(\alpha_p), [C_3, \alpha_p]] = 1$.

On the other hand, if $[C_{C_4}(\alpha_p), C_{C_3}(\alpha_p)] \nleqslant D_3$, then $\overline{C_2}C_{C_3}(\alpha_p)C_{C_4}(\alpha_p)$ is a group of Fitting height 3 on which $\alpha_q\alpha_r$ acts fixed-point freely. This contradicts the Corollary. Therefore $[C_{C_4}(\alpha_p), C_3/D_3] = 1$, that is, $C_{C_4}(\alpha_p) \leqslant D_4$.

We also observe that $[C_3/D_3, \alpha_r] = C_3/D_3$, because otherwise α_r centralizes $\overline{C_3/D_3}$ as it centralizes C_4/D_4, and so α_p acts fixed-point freely on $(\overline{C_3/D_3})(C_4/D_4)$, which is impossible.

Next assume that C_3/D_3 is non-abelian and consider the Wedderburn decomposition of $\overline{C_2}$ with respect to $(C_3/D_3)C_4$. Now $\alpha_r \notin N = N_{\langle\alpha\rangle}(W)$ for any homogeneous component W, because otherwise $[\Phi(C_3/D_3), \alpha_r] = 1$ as C_3/D_3 acts faithfully on $\overline{C_2}$. Then

$$C_{\overline{C_2}}(\alpha_r) \leqslant C_{\overline{C_2}}(\Phi(C_3/D_3)) = 1,$$

because $C_H(\alpha_r)$ is nilpotent as $f(C_1C_H(\alpha_r)) \leqslant 2$ by the Corollary and $C_H(C_1) = 1$. It follows that α_r acts fixed-point freely on $\overline{C_2}C_{C_3}(\alpha_p)$ and so $C_{C_3}(\alpha_p) \leqslant D_3$. But then α_p acts fixed-point freely on $(C_3/D_3)(C_4/D_4)$, which is impossible.

This shows that $\alpha_r \notin N$. Also $N \neq \langle\alpha_p\rangle$ since $C_{\overline{C_2}}(\alpha_p) \neq 1$. Therefore either $N = \langle\alpha_q\rangle$ or $N = \langle\alpha_p\rangle \times \langle\alpha_q\rangle$, each of which implies that $C_{\overline{C_2}}(\alpha_q) = 1$. But

$[C_{\overline{C_2}}(c$

and hence $C_{\overline{C_2}}(\alpha_q) \neq 1$ by Lemm
as $C_{C_4/D_4}(\alpha_q) = 1$. This contradic
$C_{C_3/D_3}(\alpha_r) = 1$. Set $X/C_{C_4}(\overline{C_2}) =$
subgroup of C_4 such that

$C_{C_4}(\alpha_r) \leqslant X \nleqslant C$

We also observe that $C_{C_2}(\alpha_r)$ is
on $C_1C_{C_2}(\alpha_r)X$, its Fitting height
izes $C_{C_2}(\alpha_r)$.
If $C_1C_2C_3X < G$, then $f(C_1C$
is, $X \leqslant D_4$. Then X stabilizes eve
observe that $[W, C_3/D_3] = W$ fo
follows that there is no α_r-invari
$[C_3/D_3, \alpha_r] = C_3/D_3$ must act tri

$Y =$

where W is a homogeneous C
group X acts trivially on

$C_Y(\alpha_r) =$

and hence on W, as $[X, \alpha_r] \leqslant C$
case. It follows that $C_1C_2C_3X =$
Next observe that $[C_3/D_3, \alpha_q$
$(C_3/D_3)C_4(\langle\alpha_q\rangle \times \langle\alpha_r\rangle)$-subm
Now $[V, C_3/D_3] = V$ and we h
are homogeneous C_3/D_3-comp
$[W_i, C_3/D_3] = W_i$ for each i an

Y_i

for each i. Thus $V = Y_1 \oplus \cdots$

$C_{Y_i}(\alpha_r) =$

is centralized by C_4, we ha
$Y_i = Y_i^x$ for all $x \in C_4$. Let
element of Ω. Hence

and so t is either 1 or q.

$C_3) = \{r\}$. Therefore $[C_4, \alpha_r] \neq 1$

ws that $[C_{\overline{C_2}}(\alpha_r), C_{C_3/D_3}(\alpha_r)] \neq 1$.
$_3(\alpha_r)$. We observe that $C_{C_2}(\alpha_r)$ is
ing height 3. But this is not the
ly on it.

$C_2) = \{p\}$ and $\pi(C_3) = \{q\}$. If
s $\pi(C_4) \neq \{q\}$. It follows that
1 by the three subgroup lemma
e $[\overline{C_2}(C_3/D_3)C_4, \alpha_q] = 1$ and so
which is impossible by the Cor-

$C_2, \alpha_q] = C_2$ and $[C_4, \alpha_p] = C_4$.
E_2E_4 where $[E_2, \alpha_q] = E_2$ and
follows that

$_{C_4}(\alpha_q)$. Thus $[C_4, \alpha_q] = C_4$.
se $\overline{C_2}(C_3/D_3)C_4$ is centralized

$_{C_4}(\alpha_p)$ as an $(\langle\alpha_p\rangle \times \langle\alpha_q\rangle)$-
re $[E_4, \alpha_q] = E_4$, $[E_3, \alpha_p] = E_3$.
ation of Lemma 3 gives that
$[C_3, \alpha_p]] = 1$.
$_2C_{C_3}(\alpha_p)C_{C_4}(\alpha_p)$ is a group of
his contradicts the Corollary.

therwise α_r centralizes $\overline{C_3/D_3}$
on $(\overline{C_3/D_3})(C_4/D_4)$, which is

the Wedderburn decomposi-
$_{\langle\alpha\rangle}(W)$ for any homogeneous
C_3/D_3 acts faithfully on $\overline{C_2}$.

1,

orollary and $C_H(C_1) = 1$. It
o $C_{C_3}(\alpha_p) \leqslant D_3$. But then α_p
ossible.
$(\alpha_p) \neq 1$. Therefore either
at $C_{\overline{C_2}}(\alpha_q) = 1$. But

$$[C_{\overline{C_2}}(\alpha_q), C_{C_3/D_3}(\alpha_q)] \neq 1$$

and hence $C_{\overline{C_2}}(\alpha_q) \neq 1$ by Lemma 2 applied to $(C_3/D_3)C_4(\langle\alpha_p\rangle \times \langle\alpha_q\rangle)$ on $\overline{C_2}$ as $C_{C_4/D_4}(\alpha_q) = 1$. This contradiction shows that C_3/D_3 is abelian. In this case $C_{C_3/D_3}(\alpha_r) = 1$. Set $X/C_{C_4}(\overline{C_2}) = C_{C_4/C_{C_4}(\overline{C_2})}(\alpha_r)$ Obviously X is an $\langle\alpha\rangle$-invariant subgroup of C_4 such that

$$C_{C_4}(\alpha_r) \leqslant X \nleqslant C_{C_4}(\overline{C_2}) \quad \text{and} \quad [X, \alpha_r] \leqslant C_{C_4}(\overline{C_2}).$$

We also observe that $C_{C_2}(\alpha_r)$ is normalized by X. Since $\alpha_p\alpha_q$ acts fixed-point freely on $C_1C_{C_2}(\alpha_r)X$, its Fitting height is at most 2 by the Corollary. Therefore X centralizes $C_{C_2}(\alpha_r)$.

If $C_1C_2C_3X < G$, then $f(C_1C_2C_3X) \leqslant 3$ by induction and so $[C_3, X] \leqslant D_3$, that is, $X \leqslant D_4$. Then X stabilizes every homogeneous C_3/D_3 component of $\overline{C_2}$. We also observe that $[W, C_3/D_3] = W$ for any such component W as $[\overline{C_2}, C_3/D_3] = \overline{C_2}$. It follows that there is no α_r-invariant homogeneous C_3/D_3-component of $\overline{C_2}$ because $[C_3/D_3, \alpha_r] = C_3/D_3$ must act trivially on any such component. Now put

$$Y = W \oplus W^{\alpha_r} \oplus \cdots \oplus W^{\alpha_r^{r-1}}$$

where W is a homogeneous C_3/D_3-component of $\overline{C_2}$. Since $[C_{C_2}(\alpha_r), X] = 1$, the group X acts trivially on

$$C_Y(\alpha_r) = \{w + w^{\alpha_r} + \cdots + w^{\alpha_r^{r-1}} \mid w \in W\}$$

and hence on W, as $[X, \alpha_r] \leqslant C_{C_4}(\overline{C_2})$. Thus X acts trivially on $\overline{C_2}$, which is not the case. It follows that $C_1C_2C_3X = G$, that is, $X = C_4$. Now C_4 centralizes $C_{C_2}(\alpha_r)$.

Next observe that $[C_3/D_3, \alpha_q] \neq 1$ since $[C_4, \alpha_q] = C_4$ and let V be an irreducible $(C_3/D_3)C_4(\langle\alpha_q\rangle \times \langle\alpha_r\rangle)$-submodule of $\overline{C_2}$ on which $[C_3/D_3, \alpha_q]$ acts non-trivially. Now $[V, C_3/D_3] = V$ and we have $V|_{C_3/D_3} = W_1 \oplus \cdots \oplus W_s$ where the modules W_i are homogeneous C_3/D_3-components of V. We see that no W_i is α_r-invariant, since $[W_i, C_3/D_3] = W_i$ for each i and $[C_3/D_3, \alpha_r] = C_3/D_3$. Put

$$Y_i = W_i \oplus W_i^{\alpha_r} \oplus \cdots \oplus W_i^{\alpha_r^{r-1}}$$

for each i. Thus $V = Y_1 \oplus \cdots \oplus Y_t$. Since

$$C_{Y_i}(\alpha_r) = \{w + w^{\alpha_r} + \cdots + w^{\alpha_r^{r-1}} \mid w \in W_i\}$$

is centralized by C_4, we have $C_{Y_i}(\alpha_r) \leqslant Y_i \cap Y_i^x$ for all $x \in C_4$. This gives that $Y_i = Y_i^x$ for all $x \in C_4$. Let $\Omega = \{Y_1, \ldots, Y_t\}$. We observed that $C_4\langle\alpha_r\rangle$ fixes every element of Ω. Hence

$$t = |\Omega| = |\langle\alpha_q\rangle : N_{\langle\alpha_q\rangle}(Y_1)|$$

and so t is either 1 or q.

If $t = 1$, then $V = Y_1$ and so $s = r$; thus α_q stabilizes each W_i. It follows that $[C_3/D_3, \alpha_q]$ acts trivially on each W_i and hence on V, which is not the case. Thus $t = q$, so that no W_i is α_q-invariant. Then

$$C_V(\alpha_q\alpha_r) = \{u + u^{\alpha_q} + \cdots + u^{\alpha_q^{q-1}} \mid u \in C_{Y_1}(\alpha_r)\} \neq 1,$$

a contradiction which completes the proof.

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