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Action of a Frobenius-like group with fixed-point free kernel

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Abstract. We call a finite group Frobenius-like if it has a nontrivial nilpotent normal subgroup F possessing a nontrivial complement H such that $[F, h] = F$ for all nonidentity elements $h \in H$. We prove that any irreducible nontrivial FH-module for a Frobenius-like group FH of odd order over an algebraically-closed field has an H-regular direct summand if either F is fixed-point free on V or F acts nontrivially on V and the characteristic of the field is coprime to the order of F. Some consequences of this result are also derived.

1 Introduction

Let F be a finite group acted on by a finite group H via automorphisms. This action is said to be Frobenius if $C_F(h) = 1$ for all nonidentity elements $h \in H$. Accordingly the semidirect product FH is called a Frobenius group with kernel F and complement H whenever F and H are nontrivial. It is well known that Frobenius actions are coprime actions and the kernel F is nilpotent. In [3] a semidirect product with normal nilpotent subgroup F and complement H was called a Frobenius-like group with kernel F and complement H if the condition $C_F(h) = 1$ is replaced by $[F, h] = F$ for all nonidentity elements $h \in H$. Observe that H has the structure of a Frobenius complement and the orders of F and H are coprime since $(FH)/F'$ is a Frobenius group and $\pi(F) = \pi(F/F')$.

There has been a lot of research on the structure of solvable groups admitting a Frobenius group FH of automorphisms (see [2, 5–7]). In particular under some additional hypothesis it was shown that various properties of G, for instance the nilpotent length of G, are close to those of $C_G(H)$. We see that these results are essentially due to the fact that any FH-module V on which F acts fixed-point freely is a free H-module. In [3] we have been able to observe that not exactly the freeness, but a result close to this, is also true for Frobenius-like groups of odd order. For the sake of easy reference we restate below the main result in [3].

Fact. *Let V be a nonzero vector space over an algebraically-closed field k and let FH be a Frobenius-like group of odd order acting on V as a group of linear* *transformations such that* char(k) *does not divide the order of* H. *Then* V_H *has a proper* H*-regular direct summand if one of the following holds:*

(i) $C_V(F) = 0$,

(ii) $[V, F] \neq 0$ *and* char(k) *does not divide the order of* F.

Actually, regular modules force the existence of large dimension and of fixed points and hence there is a wide range of possible applications. In the present paper we study the case where FH is Frobenius-like of odd order with an almost abelian kernel F, in the sense that F' is of prime order and is contained in $Z(FH)$. Note that this includes the important case where F is an extraspecial group of odd order and H acts on F via automorphisms such that $[Z(F), H] = 1$ and $[F, h] = F$ for all nonidentity elements $h \in H$. We observe that these strong conditions together with the oddness of the group FH lead to strong results. Namely, we prove the following theorem using the above fact as a key ingredient:

Theorem A. *Let* G *be a finite group admitting a Frobenius-like group of automorphisms* FH *of odd order such that* F' *is of prime order and* $[F', H] = 1$. *Assume further that* $(|G|, |H|) = 1$ *and* $C_G(F) = 1$. *Then:*

(i) *the Fitting series of* $C_G(H)$ *coincides with the intersections of* $C_G(H)$ *with the Fitting series of* G,

(ii) *the nilpotent length of* G *is equal to the nilpotent length of* $C_G(H)$.

The following example shows that Theorem A is not true if we drop the condition that FH is of odd order:

Let G be the elementary abelian group of order 5^2 and F a Sylow 2-subgroup of SL$(2, 5)$. If H denotes the Sylow 3-subgroup of the normalizer of F in SL$(2, 5)$, then the group FH is isomorphic to SL$(2, 3)$ and is a Frobenius-like group of automorphisms of G where F' is of prime order and $[F', H] = 1$. Furthermore the orders of G and H are relatively prime and $C_G(F) \leq C_G(F') = 1$. As H acts irreducibly on G, we have $C_G(H) = 1$. So the nilpotent length of $C_G(H)$ is equal to 0 and is not equal to the nilpotent length of G.

As a consequence of Theorem A, we obtain the following:

Corollary B. *Let* G *be a finite group admitting a Frobenius-like group of automorphisms* FH *of odd order such that* F' *is of prime order and* $[F', H] = 1$. *Assume further that* $(|G|, |H|) = 1$ *and* $C_G(F) = 1$. *Then we have:*

(i) $O_\pi(C_G(H)) = O_\pi(G) \cap C_G(H)$ *for any set of primes* π,

(ii) *the* π*-length of* G *is equal to the* π*-length of* $C_G(H)$,

(iii) $O_{\pi_1,\pi_2,\ldots,\pi_k}(C_G(H)) = O_{\pi_1,\pi_2,\ldots,\pi_k}(G) \cap C_G(H)$ *where* π_i *denotes a set of primes for each* $i = 1, \ldots, k$.

If a group A acts on a group B via automorphisms, then the subgroup consisting of elements acting trivially on B is denoted by $C_A(B)$ as well as by $\mathrm{Ker}(A \text{ on } B)$. Throughout the paper we prefer the notation $\mathrm{Ker}(A \text{ on } B)$ in order to avoid cumbersome expressions. The following proposition is established as a key result in proving Theorem A. We consider it to be of independent interest and would like to emphasize that it does not assume FH to be of odd order.

Proposition C. *Let* FH *be a Frobenius-like group such that* F' *is of prime order and* $[F', H] = 1$. *Suppose that* FH *acts on a* q*-group* Q *for some prime* q *coprime to the order of* H. *Let* V *be a* $kQFH$*-module where* k *is a field with characteristic not dividing* $|QH|$. *Suppose further that* F *acts fixed-point freely on the semidirect product* VQ. *Then we have*

$$\mathrm{Ker}(C_Q(H) \textit{ on } C_V(H)) = \mathrm{Ker}(C_Q(H) \textit{ on } V).$$

In this paper all groups are assumed to be finite. We denote the nilpotent length of a solvable group G by $f(G)$. It is the smallest nonnegative integer k such that the k-th Fitting subgroup $F_k(G)$ is equal to G.

2 Proof of Proposition C

It is well known that if a finite nilpotent group F acts fixed-point freely on a finite group G, then G is solvable by [1] and F is a Carter subgroup of the semidirect product GF. Let N be a normal subgroup of G which is F-invariant and set $\overline{G} = G/N$. Then F is a Carter subgroup of the semidirect product $\overline{G}F$ and hence F acts fixed-point freely on $\overline{G}$ also. We shall use this observation in the following parts of the paper without any reference.

Proof of Proposition C. Suppose the proposition is false and choose a counterexample with minimum $\dim_k V + |QFH|$. We split the proof into a sequence of steps. To simplify the notation we set $K = \mathrm{Ker}(C_Q(H) \text{ on } C_V(H))$.

(1) *We may assume that* k *is a splitting field for all subgroups of* QFH.

We consider the QFH-module $\bar{V} = V \otimes_k \bar{k}$ where $\bar{k}$ is the algebraic closure of k. Notice that $C_{\bar{V}}(H) = C_V(H) \otimes_k \bar{k}$ and also $C_{\bar{V}}(F) = C_V(F) \otimes_k \bar{k} = 0$ as $C_V(F) = 0$. Therefore, once the proposition has been proven for the group QFH on $\bar{V}$, it becomes true for QFH on V also.

(2) *Q acts faithfully on V.*

We set $\overline{Q} = Q/\mathrm{Ker}(Q \text{ on } V)$ and consider the action of the group $\overline{Q}FH$ on V assuming $\mathrm{Ker}(Q \text{ on } V) \neq 1$. Notice that $C_{\overline{Q}}(F)$ is trivial by the above remark. An induction argument gives

$$\mathrm{Ker}(C_{\overline{Q}}(H) \text{ on } C_V(H)) = \mathrm{Ker}(C_{\overline{Q}}(H) \text{ on } V) = 1.$$

This leads to a contradiction as $\overline{C_Q(H)} = C_{\overline{Q}}(H)$ due to the coprime action of H on Q. Thus we may assume that Q acts faithfully on V.

It should be noted that we need only to prove $K = 1$ due to the faithful action of Q on V. So we assume this to be false.

(3) *Set $L = K \cap Z(C_Q(H))$. Then $Q = \langle z^F \rangle$ for any nonidentity element z in L.*

Since $1 \neq K \trianglelefteq C_Q(H)$, the group $L = K \cap Z(C_Q(H))$ is nontrivial. Pick $1 \neq z \in L$ and consider the group $Q_0 = \langle z^F \rangle$. As $C_{Q_0}(F) = 1$, if $Q_0 \neq Q$, the proposition holds by induction for the group Q_0FH on V, that is,

$$\mathrm{Ker}(C_{Q_0}(H) \text{ on } C_V(H)) = \mathrm{Ker}(C_{Q_0}(H) \text{ on } V) = 1.$$

This leads to a contradiction since $z \in \mathrm{Ker}(C_{Q_0}(H) \text{ on } C_V(H))$. Therefore we have $Q = Q_0$ as desired.

(4) *V is an irreducible QFH-module.*

As $\mathrm{char}(k)$ is coprime to the order of Q and $K \neq 1$, there is a QFH-composition factor W of V on which K acts nontrivially. If $W \neq V$, then the proposition is true for the group QFH on W by induction. That is

$$\mathrm{Ker}(C_Q(H) \text{ on } C_W(H)) = \mathrm{Ker}(C_Q(H) \text{ on } W)$$

and, hence,

$$K = \mathrm{Ker}(K \text{ on } C_W(H)) = \mathrm{Ker}(K \text{ on } W),$$

which contradicts the assumption that K acts nontrivially on W. Hence $V = W$, establishing the claim.

By Clifford's theorem, the restriction of the QFH-module V to the normal subgroup Q is a direct sum of Q-homogeneous components.

(5) *Let Ω denote the set of Q-homogeneous components of V. Then F acts transitively on Ω and H fixes an element of Ω.*

Let Ω_1 be an F-orbit on Ω and set $H_1 = \mathrm{Stab}_H(\Omega_1)$. Suppose first that $H_1 = 1$. Pick an element W from Ω_1. Clearly, we have $\mathrm{Stab}_H(W) \leq H_1 = 1$ and hence the sum $X = \sum_{h\in H} W^h$ is direct. It is straightforward to verify that

$$C_X(H) = \left\{ \sum_{h\in H} v^h : v \in W \right\}.$$

By definition, K acts trivially on $C_X(H)$. Note also that K normalizes each W^h as $K \leq Q$. It follows now that K is trivial on X. Notice that the action of H on the set of F-orbits on Ω is transitive, and $K \leq C_Q(H)$. Hence K is trivial on the whole of V contrary to (2). Thus $H_1 \neq 1$.

The group H acts transitively on the set $\{\Omega_i : i = 1, 2, \ldots, s\}$, the collection of F-orbits on Ω. Let now $V_i = \bigoplus_{W\in\Omega_i} W$ for $i = 1, 2, \ldots, s$. Suppose that H_1 is a proper subgroup of H, equivalently, $s > 1$. By induction, the proposition holds for the group QFH_1 on V_1, that is,

$$\mathrm{Ker}(C_Q(H_1) \text{ on } C_{V_1}(H_1)) = \mathrm{Ker}(C_Q(H_1) \text{ on } V_1).$$

In particular, we have $\mathrm{Ker}(C_Q(H) \text{ on } C_{V_1}(H_1)) = \mathrm{Ker}(C_Q(H) \text{ on } V_1)$. On the other hand we observe that

$$C_V(H) = \{u^{x_1} + u^{x_2} + \cdots + u^{x_s} : u \in C_{V_1}(H_1)\}$$

where $x_1, \ldots, x_s$ is a complete set of right coset representatives of H_1 in H. By definition, K acts trivially on $C_V(H)$ and normalizes each V_i. Then K is trivial on $C_{V_1}(H_1)$ and hence on V_1. As K is normalized by H, we see that K is trivial on each V_i and hence on V contrary to (2). Therefore $H_1 = H$ and F acts transitively on Ω so that $\Omega = \Omega_1$ as desired.

Let now $S = \mathrm{Stab}_{FH}(W)$ and $F_1 = F \cap S$. Then $|F : F_1| = |\Omega| = |FH : S|$ and so $|S : F_1| = |H|$. Notice next that as $(|F_1|, |H|) = 1$, there exists a complement, say S_1, of F_1 in S with $|H| = |S_1|$ by the Schur–Zassenhaus Theorem. Therefore by passing, if necessary, to a conjugate of W in Ω, we may assume that $S = F_1 H$, that is, W is H-invariant. This establishes the claim.

From now on, W will denote an H-invariant element in Ω the existence of which is established by (5). It should be noted that the group $Z(Q/C_Q(W))$ acts by scalars on the homogeneous Q-module W, and so $[Z(Q), H] \leq C_Q(W)$ as W is stabilized by H. Recall that $L = K \cap Z(C_Q(H))$.

(6) *Set* $U = \sum_{x\in F'} W^x$. *Then* $[L, Z_2(Q)] \leq C_Q(U)$.

Note that

$$Z_2(Q) = [Z_2(Q), H] C_{Z_2(Q)}(H)$$

as $(|Q|, |H|) = 1$. We have

$$[Z_2(Q), L, H] \leq [Z(Q), H] \leq C_Q(W).$$

We also have $[L, H, Z_2(Q)] = 1$ as $[L, H] = 1$. It follows now by the three subgroup lemma that

$$[H, Z_2(Q), L] \le C_Q(W).$$

On the other hand $[C_{Z_2(Q)}(H), L] = 1$ by the definition of L. Thus

$$[L, Z_2(Q)] \le C_Q(W).$$

Since the group $[L, Z_2(Q)]$ is F'-invariant as $[F', H] = 1$, we conclude that $[L, Z_2(Q)] \le C_Q(U)$ as desired.

(7) *The subgroup* $F_2 = \mathrm{Stab}_F(U)$ *is a proper subgroup of* F *and* K^x *acts trivially on* U *for every* $x \in F - F_2$.

For $F_2 = \mathrm{Stab}_F(U)$, clearly we have $F' \le F_2$ and $F_1 = \mathrm{Stab}_F(W) \le F_2$. Assume that $F = F_2$. This forces the equality $V = U$ as F is transitive on Ω by part (5). Let F' be generated by a of order p. Then we have either $V = W$ or $V = \bigoplus_{i=0}^{p-1} W^{a^i}$. If the former holds, then $Z(Q)$ acts by scalars on the whole of V whence $[Z(Q), F] \le C_Q(V) = 1$, contrary to $C_Q(F) = 1$. Therefore we have $V = \bigoplus_{i=0}^{p-1} W^{a^i}$. Then $F_1 \cap F' = 1$. On the other hand the index of F_1 in F is p implying that $F_1 \trianglelefteq F$ and $F' \le F_1$ which is a contradiction. So $F \ne F_2$.

Pick an element $x \in F - F_2$ and suppose that there exists $1 \ne h \in H$ such that $(U^x)^h = U^x$ holds. Then we get $[h, x^{-1}] \in F_2$ and so $F_2x = F_2x^h = (F_2x)^h$ implying the existence of an element $g \in F_2x \cap C_F(h)$ by [4, Kapitel I, 18.6] by coprimeness. The Frobenius action of H on F/F_2 gives that $x \in F_2$, a contradiction. That is, for each $x \in F - F_2$, $\mathrm{Stab}_H(U^x) = 1$.

Set now $U_1 = U^x$ for some $x \in F - F_2$. The sum $Y = \sum_{h \in H} {U_1}^h$ is direct by the preceding paragraph. It is straightforward to verify that

$$C_Y(H) = \left\{ \sum_{h \in H} v^h : v \in U_1 \right\}.$$

By definition, K acts trivially on $C_Y(H)$. Note also that K normalizes each ${U_1}^h$ for every $h \in H$ as $K \le Q$. It follows now that K is trivial on Y and hence trivial on U^x for every $x \in F - F_2$ which is equivalent to that K^x acts trivially on U for all $x \in F - F_2$ as desired.

(8) *The subgroup* Q *is abelian.*

By (3), we have $Q = L^F$. It follows by (7) that $Q = L^{F_2} C_Q(U)$. If $F_1 \ne F_2$, then $U = \bigoplus_{i=0}^{p-1} W^{a^i}$ where a generates F' and $F' \cap F_1 = 1$. We also have $p = |F_2 : F_1|$ and hence $F_2 = F'F_1$. In particular, we have either $F_1 = F_2$ or $F_2 = F'F_1$. Thus in any case $Q = L^{F_1} C_Q(U)$ since $L^{F'} = L$. Notice next that $[L, Z_2(Q)] \le C_Q(U)$ by (6). Then we have

$$[L^{F_1}, Z_2(Q)] \le C_Q(U)$$

as U is F_1-invariant, which yields that $[Q, Z_2(Q)] \le C_Q(U)$. Thus

$$[Q, Z_2(Q)] \le \bigcap_{f \in F} C_Q(U)^f = C_Q(V) = 1$$

and hence Q is abelian.

(9) *Final Contradiction.*

By (8), the subgroup Q is abelian and hence acts by scalars on the homogeneous Q-module W. Recall that $F_1 = \mathrm{Stab}_F(W)$. Assume first that $F_1 = F_2$ so that $U = W$. As Q is abelian, it follows that $\prod_{f \in F} z^f$ is a well-defined element of Q which lies in $C_Q(F) = 1$. Thus, by (7), we have

$$1 = \prod_{f \in F} z^f = \Big(\prod_{f \in F_1} z^f\Big)\Big(\prod_{f \in F - F_1} z^f\Big) \in \Big(\prod_{f \in F_1} z^f\Big) C_Q(W).$$

On the other hand the scalar action of Q on W gives that $[Q, F_1] \le C_Q(W)$. That is $(\prod_{f \in F_1} z^f) C_Q(W) = z^{|F_1|} C_Q(W)$ and so $z \in C_Q(W)$ as $|F_1|$ is coprime to $|z|$. This contradiction shows that $F_1 \ne F_2$.

We observe now that $C_W(F_1) = 1$ as $C_V(F) = 1$. On the other hand observe that the group F_1 is H-invariant and the group $F_1 H$ is Frobenius with kernel F_1 since we have $C_{F_1}(h) \le F' \cap F_1 = 1$ for all nonidentity elements $h \in H$. Applying [6, Lemma 1.3] to the action of $F_1 H$ on W, we see that $W|_H$ is free. Since Q, and hence K, acts by scalars on W, and K acts trivially on the nontrivial subspace $C_W(H)$, we see that K acts trivially on W. On the other hand, $K = L$ as Q is abelian, and so $Q = K^{F_1} C_Q(U)$. It follows now that Q itself is also trivial on W. Then Q acts trivially on W^{a^i} for each $i = 0, \dots, p-1$ and hence so does on U. We already know from (7) that the group K acts trivially on U^x for every element $x \in F - F_2$. Consequently we obtain $K \le C_Q(V) = 1$, and the proposition follows. □

We give an example which shows that the above proposition is not true if one drops the condition that the group F act fixed-point freely on VQ or replaces it by the condition $C_Q(F) = 1$. In contrast to [2] one cannot even weaken the condition $C_{VQ}(F) = 1$ to the condition $C_{C_{VQ}(F)}(h) = 1$ for all nonidentity elements $h \in H$.

Example. Let V_1 be the cyclic group of order 43 and let Q_1 and H be the subgroups of the full automorphism group of V_1, which is cyclic of order 42, of orders 7 and 3 respectively. We consider the semidirect product $V_1(Q_1 \times H)$ which is a Frobenius group of order $43 \cdot 21$ with kernel V_1 and complement $Q_1 H$. Let F be the extraspecial group of order 5^3 and exponent 5. The group F admits an automorphism of order 3 which centralizes $Z(F)$ and which acts regularly and irreducibly on $F/Z(F)$. We may assume that this automorphism generates H. The semidirect product FH is then a Frobenius-like group which is not Frobenius.

Let us consider the wreath product $(V_1 Q_1) \wr F$ which is the semidirect product BF of the base group $B = \{(a_f)_{f\in F} : a_f \in V_1 Q_1,\ f \in F\}$ with the group F. We have $B \cong (V_1 Q_1) \times (V_1 Q_1) \times \cdots \times (V_1 Q_1)$ ($|F|$-copies) and F acts on B by conjugation as follows: for any $x \in F$ and any $(a_f)_{f\in F} \in B$ we have

$$(a_f)^x = (b_f)_{f\in F} \quad \text{with} \quad b_f = a_{f\cdot x^{-1}}$$

for any $f \in F$. We next define an action of H on BF, using the action of H on $V_1 Q_1$ and on F, as follows: for $x \in F$, $(a_f)_{f\in F} \in B$ and $h \in H$ we have

$$[(a_f)_{f\in F}\, x]^h = (b_f)_{f\in F}\, y \quad \text{with} \quad b_f = (a_{hfh^{-1}})^h \in V_1 Q_1$$

(according to the action of H on $V_1 Q_1$) for any $f \in F$ and $y = x^h$ (according to the action of H on F).

We now set $V = \{(a_f)_{f\in F} : a_f \in V_1\}$ and $Q_0 = \{(a_f)_{f\in F} : a_f \in Q_1\}$ and consider the semidirect product of BF with H. Then V is an FH-invariant normal subgroup of $B = VQ_0$. Furthermore Q_0 is normalized by FH. Notice that $C_V(F) = \{(a)_{f\in F} : a \in V_1\}$. Let

$$K_0 = \{(a_f)_{f\in F} : a_f = 1 \text{ if } f \in F - F' \text{ and } a_f \in Q_1 \text{ if } f \in F'\}.$$

Then $K_0 \leq C_{Q_0}(H)$. On the other hand $(a_f)_{f\in F} \in V$ is centralized by every nonidentity element $h \in H$. That is

$$((a_f)_{f\in F})^h = (a_f)_{f\in F} \iff (a_{hfh^{-1}})^h = a_f \quad \text{for any } f \in F.$$

If $f \in F'$, then we get $a_f^h = a_f$, and hence $a_f = 1$, as H acts fixed-point freely on V_1. Thus

$$C_V(h) \leq \{(a_f)_{f\in F} \in V : a_f = 1 \text{ for } f \in F'\} \quad \text{for any nonidentity } h \in H.$$

In particular $[C_V(H), K_0] = 1$ and $C_{C_V(F)}(h) = 1$ for any nonidentity element $h \in H$. We also have

$$C_{Q_0}(F) = \{(a)_{f\in F} : a \in Q_1\}$$

and

$$C_{Q_0}(H) = K_0 \oplus \bigoplus_{i=1}^{40} \{(a_f)_{f\in F} \in Q_0 : a_f = 1 \text{ if } f \notin T_i \text{ and } a_f = a \in Q_1 \text{ if } f \in T_i\}$$

where T_i, $i = 1, 2, \ldots, 40$, are the orbits of H on $F - F'$.

Now let $Q = [Q_0, F]$ and consider the group $VQFH$. We have

$$[Q, F] = Q, \quad C_{VQ}(F) = C_V(F)$$

and

$$C_{C_{VQ}(F)}(h) = 1 \quad \text{for any nonidentity } h \in H.$$

Furthermore $C_V(h) \leq \{(a_f)_{f \in F} \in V_0 : a_f = 1 \text{ for } f \in F'\}$ for any $1 \neq h \in H$.

Now pick $1 \neq x \in F'$, $y \in Q_1$, and $u = (a_f)_{f \in F} \in Q_0$ defined as $a_1 = y$ and $a_f = 1$ for every nonidentity element $f \in F$. Then $[u, x] \in [Q_0, F] = Q$. Here $[u, x] = (b_f)_{f \in F}$ with $b_1 = y^{-1}$, $b_x = y$ and $b_f = 1$ for all $f \in F - \{1, x\}$. So we see that $K_0 \cap Q \neq 1$. It is also easy to check that $K_0 \cap Q$ is not contained in the Fitting subgroup of VQ but is contained in the Fitting subgroup of $C_{VQ}(H)$.

3 Main result

In this section we present our main result.

Lemma 3.1. *Suppose that a Frobenius-like group FH acts on the finite group G by automorphisms so that $C_G(F) = 1$. Then there is a unique FH-invariant Sylow p-subgroup of G for each prime p dividing the order of G.*

Proof. The proof of [7, Lemma 2.6] applies also to this statement. □

Proof of Theorem A. We already know that G is solvable due to the nilpotency of F and the assumption $C_G(F) = 1$ by [1].

First we will prove that the equality $F(C_G(H)) = F(G) \cap C_G(H)$ is true under the hypothesis of the theorem. It is straightforward to verify that

$$F(G) \cap C_G(H) \leq F(C_G(H)).$$

To prove the reversed inclusion $F(C_G(H)) \leq F(G)$ we proceed by induction on the order of G. Now consider the nontrivial group $\overline{G} = G/F(G)$. By the remark above $C_{\overline{G}}(F)$ is trivial. Then an induction argument yields that

$$F(C_{\overline{G}}(H)) \leq F(\overline{G}) \leq \overline{F_2(G)}.$$

Notice that $C_{\overline{G}}(H) = \overline{C_G(H)}$. If $F_2(G) \neq G$, another induction argument, applied to the action of FH on $F_2(G)$, implies that the desired inclusion is true. Thus we may assume that $F_2(G) = G$. It is clear that there exist distinct primes p and q such that $[O_q(C_G(H)), O_p(G)]$ is nontrivial. The group $G/O_{p'}(G)$ is a counterexample, whence $F(G) = O_p(G)$ and $\overline{G}$ is a q-group. By Lemma 3.1 there is a unique FH-invariant Sylow q-subgroup Q of G. Then we have $\overline{G} = \overline{Q}$, that is, $G = F(G)Q$. As $C_{Q/\Phi(Q)}(F) = C_Q(F)\Phi(Q)/\Phi(Q)$ is trivial, by [3, Theorem A] applied to the action of FH on $Q/\Phi(Q)$, we get $C_{Q/\Phi(Q)}(H)$ is nontrivial. Notice that $C_{Q/\Phi(Q)}(H) = C_Q(H)\Phi(Q)/\Phi(Q)$ as H acts coprimely on G whence $C_Q(H)$ is also nontrivial. Since $C_G(H) = C_{F(G)}(H)C_Q(H)$, we see that

$$F(C_G(H)) = C_{F(G)}(H)\mathrm{Ker}(C_Q(H) \text{ on } C_{F(G)}(H)).$$

On the other hand, applying Proposition C to the action of the group QFH on $V = F(G)/\Phi(G)$ we get

$$\mathrm{Ker}(C_Q(H) \text{ on } C_V(H)) = \mathrm{Ker}(C_Q(H) \text{ on } V) = 1$$

This establishes the desired equality.

To prove (i) is equivalent to showing that $F_k(C_G(H)) = F_k(G) \cap C_G(H)$ for each natural number k. This is true for $k = 1$ by the preceding paragraph. Assume that $F_k(C_G(H)) = F_k(G) \cap C_G(H)$ holds for a fixed but arbitrary $k > 1$. Due to the coprime action of H on G we have $C_{G/F_k(G)}(H) = C_G(H)F_k(G)/F_k(G)$ and hence

$$F_{k+1}(C_G(H))F_k(G)/F_k(G) \leq F(C_{G/F_k(G)}(H)) \leq F(G/F_k(G)).$$

This forces $F_{k+1}(C_G(H)) \leq F_{k+1}(G) \cap C_G(H)$, as desired.

Let now $f(C_G(H)) = n$. Then

$$C_G(H) = F_n(C_G(H)) \leq F_n(G).$$

Suppose that $F_n(G) \neq G$. Then the group $C_{Z(F_{n+1}(G)/F_n(G))}(H)$ is nontrivial by [3, Theorem A]. It follows now by the coprime action of H on G that $C_G(H)$ is not contained in $F_n(G)$. This contradiction completes the proof. □

Proof of Corollary B. The proof of [6, Corollary 1.4] applies also to this statement if one replaces [6, Theorem 2.1] by Theorem A. □

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