A GENERALIZED FIXED POINT FREE AUTOMORPHISM OF PRIME POWER ORDER

GÜLİN ERCAN AND İSMAİL Ş. GÜLOĞLU

(Day Month Year) (Day Month Year) A. Olshanskii

Abstract. Let G be a finite group and α be an automorphism of G of order p^n for an odd prime p. Suppose that α acts fixed point freely on every α-invariant p'-section of G, and acts trivially or exceptionally on every elementary abelian α-invariant p-section of G. It is proved that G is a solvable p-nilpotent group of nilpotent length at most $n+1$, and this bound is best possible.

noncoprime automorphism; nilpotent length; p-length; exceptional action.

Mathematics Subject Classification 2010: 20D10, 20D15, 20D45

1. Introduction

Let G be a finite group and $\alpha \in Aut\ G$. When G and α have coprime orders there exist certain very useful relations between G and α making some inductive arguments easy to apply. If the coprimeness condition is missing it becomes rather difficult, sometimes impossible to use this kind of arguments and the situation changes dramatically.

The major purpose of this paper is to investigate the structure of finite groups admitting a noncoprime automorphism α of prime power order, the action of which can be considered as a generalized fixed point free action. It is well known that if α is a fixed point free automorphism of G of order p^n for some prime p, then $|G|$ is not divisible by p and by [5] and [2], G is a solvable group of nilpotent length bounded by n. In [1] Espuelas gives an example of a noncoprime, generalized fixed point free action. More precisely, he studies the influence of an automorphism α of order p^n for an odd prime p on the structure of a p-solvable group G if it acts fixed point freely on every α-invariant p'-section of G. He proves under these conditions that the p-length of G is bounded by $n+1$ and the nilpotent length of G is bounded by $2n+1$, these bounds being best possible. In the present paper we refine this result by imposing an additional assumption on the action of α. Namely we study the structure of groups G satisfying the following

Hypothesis. *Let G be a finite group and α an automorphism of G of order p^n for a prime p and a positive integer n, where:*

(i) α acts fixed point freely on every α-invariant p'-section of G;

(ii) α acts either trivially; or exceptionally on every elementary abelian α-invariant p-section V of G, that is, the degree of the minimum polynomial of the operator α_V induced by α on V is less than the order of α_V.

Remark 1.1. The second condition of the Hypothesis is actually equivalent to the condition that the degree of the minimal polynomial of the operator α_V induced by α on V is at most $p-1$: If (ii) is satisfied then α acts on every elementary abelian α-invariant p-section V of G either trivially or as an automorphism of order p. Because if the order of α_V is p^k with $k>1$ we can find an α-invariant p-section U of V such that α_U is of order p and U is a free $\langle\alpha_U\rangle$-module contradicting (ii). The converse is trivial to see.

Our main result can be stated as follows:

For an odd prime p, let G be a finite group admitting an automorphism α of order p^n satisfying the above Hypothesis. Then G is a solvable p-nilpotent group of nilpotent length at most $n+1$. This bound is best possible.

The following [4, Lemma 2.3] is essentially due to Gross and serves as the key lemma in proving the results of this paper.

Lemma 1.2. *Let $Q\langle\alpha\rangle$ be a group where α is a p-element of order p^n and Q is a nontrivial normal q-subgroup of a group for distinct primes p and q where p is odd or $p^n=4$. Assume that $\alpha^{p^{n-1}}$ does not centralize Q. Let V be a faithful and irreducible $kQ\langle\alpha\rangle$-module where k is a field of characteristic r. Then $C_Q(\alpha)\neq 1$ provided one of the following conditions hold:*
(i) $r\neq p$, and α acts fixed point freely on V.
(ii) $r=p$, and the minimum polynomial of α on V is not $(X-1)^{p^n}$.

The notation used is that of Isaacs' book [3], together with the convention that all groups considered are finite.

2. Main Results

In this section we present the proofs of our results. It should be noted that if G is a group satisfying the Hypothesis then every α-invariant subgroup of G and every factor group of G by an α-invariant normal subgroup satisfy also the Hypothesis. This makes inductive reasoning possible for groups satisfying the Hypothesis.

Theorem 2.1. *Let G be a p-solvable group admitting an automorphism α of order p^n where p is an odd prime or $p^n=4$. Assume that α acts on G in such a way that the Hypothesis holds. Then G is p-nilpotent.*

Proof. To simplify the notation we set $H=G\langle\alpha\rangle$, the semidirect product of G by $\langle\alpha\rangle$. We assume that the theorem is false and choose a counterexample with $|H|$ minimum. Let K be a proper α-invariant subgroup of G. If α is trivial on K then K is a p-group and the conclusion of the theorem holds for K. If α is not trivial on K, and p is odd or the order of the automorphism induced by α on K is 4 then induction applies to the action of α on K . Suppose now that $p^n=4$, α is not trivial on K and α^2 acts trivially on K. As $K/O_{2',2}(K)$ is faithfully represented on the Frattini factor group V of $O_{2',2}(K)/O_{2'}(K)$ and the minimal polynomial of α_V has degree less than 2 we see that α acts trivially on V. This implies by the three-subgroup lemma that $K/O_{2',2}(K)$ is centralized by α and hence its order cannot be divisible by an odd prime. Thus we have $K=O_{2',2}(K)$. Note that a similar argument applies to every factor group of G by an α-invariant normal subgroup.

So it follows by induction that $O_{p'}(G) = 1$ and $G = O_{p,p'}(G)$. Furthermore we may assume that $\Phi(O_p(G)) = 1$. Since G is p-solvable and $O_{p'}(G)) = 1$, the p'-group $\overline{G} = G/O_p(G)$ acts faithfully on $M = O_p(G)$ and hence on $[M, G]$. Let now N be a minimal normal subgroup of H contained in $[M, G]$. As G/N is p-nilpotent by induction we get $[M, G] = N$, that is, $[M, G]$ is a minimal normal subgroup of G. If $H/O_p(G)$ acts faithfully on $[M, G]$ it follows by Lemma 2 that $C_{\overline{G}}(\alpha) \neq 1$ which is impossible. Thus $\langle\beta\rangle = C_{\langle\alpha\rangle}(\overline{G}) \neq 1$. Set $Y = C_M(\beta)\overline{G}$, the semidirect product of $C_M(\beta)$ by $\overline{G}$. Notice that α induces an automorphism α_Y on Y of order $|\langle\alpha\rangle/\langle\beta\rangle|$, the action of which satisfies the Hypothesis. Then Y is p-nilpotent by induction. This implies that $\overline{G}$, and hence $\overline{G}\langle\beta\rangle$ centralizes $C_M(\beta)$. As a consequence, $\overline{G}$ centralizes the whole of the group M by Thompson's $P \times Q$-theorem [3, Theorem 4.31] applied to the action of $\overline{G}\langle\beta\rangle$ on M. This final contradiction completes the proof. □

Theorem 2.2. *Let G be a group admitting an automorphism α of order p^n for an odd prime p. Assume that α acts on G in such a way that the Hypothesis holds. Then G is solvable.*

Proof. We argue by induction on $|G|$. Then every nontrivial α-invariant proper normal subgroup N and the factor group G/N satisfy the theorem and hence both N and G/N are solvable. This yields that G is characteristically simple with the property that every α-invariant proper subgroup of G is solvable. Pick a Sylow p-subgroup P of $G\langle\alpha\rangle$ containing α. Now $P_0 = P \cap G$ is an α-invariant Sylow p-subgroup of G. If $P_0 \neq 1$ then the subgroups $N_G(J(P_0))$ and $C_G(Z(P_0))$ are both solvable. By Theorem 3 we see that they are both p-nilpotent. As a consequence, G is p-nilpotent by [3, Theorem 7.1]. This forces that G is a p'-group on which α acts fixed point freely. Then the solvability of G is known by [5]. □

We are now ready to prove the main results of this paper.

Theorem 2.3. *Let G be a group admitting an automorphism α of order p^n where p is an odd prime. Assume that α acts on G in such a way that the Hypothesis holds. Then G is a solvable p-nilpotent group of nilpotent length at most $n+1$. This bound is best possible.*

Proof. G is p-nilpotent by Theorem 3 and Theorem 4. Then $G = O_{p',p}(G)$. More precisely, $G = O_{p'}(G)P$ where P is an α-invariant Sylow p-subgroup of G. Owing to the fixed point free action of α on $O_{p'}(G)$, the nilpotent length of $O_{p'}(G)$ is at most n by [2]. It is straightforward now to verify that G has nilpotent length at most $n+1$.

It is well known that there exists a solvable group H of nilpotent length n admitting a fixed point free automorphism β of order p^n, where p is a prime. Let $G = H\langle\beta\rangle$ be the semidirect product of H by $\langle\beta\rangle$ and α be the inner automorphism of G induced by β. It is straightforward to verify that G is a group of nilpotent length $n+1$ and the action of α on G satisfies the Hypothesis. This example demonstrates that the bound of the theorem is best possible. □

The proof of the above result is also valid for the next result which deals with the case $p^n = 4$.

Theorem 2.4. *Let G be a solvable group admitting an automorphism α of order* 4. *Assume that α acts on G in such a way that the Hypothesis holds. Then G is a* 2*-nilpotent group of nilpotent length at most* 3.

Acknowledgments

We thank the anonymous referee for his/her valuable suggestions drawing our attention to the fact expressed in the above remark. This work has been supported by the research project TÜBİTAK 109T664.

References

[1] A. Espuelas, A Noncoprime Shult Type Theorem, *Math. Z.* **196** (1987) 323-329.

[2] F. Gross, Solvable groups admitting a fixed point free automorphism of prime power order, *Proc. Amer. Math. Soc.* **17** (1966) 1440-1446.

[3] I. M. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics, 92.

[4] T. Meixner, The Fitting length of solvable H_{p^n}-groups, *Israel J. Math.* **51** (1985) 68-78.

[5] P. Rowley, Finite groups admitting a fixed-point-free automorphism group, *J.Algebra* **174** (1995) 724-727.

Department of Mathematics, Middle East Technical University, Ankara, 06800 Turkey,

Department of Mathematics, Doğuş University, Istanbul, Turkey,