THE INFLUENCE OF HUGHES TYPE ACTION

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1. Introduction

Let G be a finite group. For a prime number p, $H_p(G) = \langle x \in G | x^p \neq 1 \rangle$ is called the Hughes subgroup of G. It is well known that $[G : H_p(G)] = p$ and $H_p(G)$ is nilpotent if G is not a p-group and $H_p(G)$ is properly contained in G. In this case $G = H_p(G) \langle \alpha \rangle$ for some element α of order p and α induces on $H_p(G)$ a so called splitting automorphism of order p, that is, α acts on $H_p(G)$ so that

$$xx^{\alpha}x^{\alpha^2} \cdots x^{\alpha^{p-1}} = 1 \text{ for any } x \in H_p(G).$$

The information about the structure of $H_p(G)$ is then a consequence of

Thompson-Hughes-Kegel Theorem. [7,9] *If G is a finite group admitting a splitting automorphism of prime order then G is nilpotent.*

This is clearly a generalization of a well known result due to Thompson [11] about finite groups admitting a fixed point free automorphism of prime order. In this sense the concept of a splitting automorphism can be considered as a generalization of fixed point free action. If α is an automorphism of prime order p of the finite group G then

$$(xx^{\alpha}x^{\alpha^2} \cdots x^{\alpha^{p-1}} = 1 \text{ for any } x \in G\ ) \ \Leftrightarrow\ (\ |y| = p \text{ for any } y \in G\langle \alpha \rangle - G).$$

Therefore one can study extensions of the fixed point free action by putting conditions on the orders of elements outside a proper subgroup. The papers [1,2,3] of the authors are such examples. We call the action of an automorphism α of G a **Hughes type action** if it is described by conditions on the orders of elements of $G\langle \alpha \rangle - G$.

In the present paper we study the structure of finite groups G admitting an automorphism α of prime order p so that the orders of elements in $G\langle \alpha \rangle - G$ are not divisible by p^2. This is of course a very weak condition as it is trivially satisfied in the case that G is a p'-group independent of the action of α on G. So one cannot hope to get much information under such a condition. We have mainly proven the following:

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Let G be a finite group admitting an automorphism α of prime order p so that $x^N = 1$ for every element x in $G\langle\alpha\rangle - G$ where N is a positive integer not divisible by p^2. Suppose that G is π-separable where $\pi = \pi(N)$. Then G is of π'-length 1 and has a nilpotent Hall π'-subgroup. Furthermore p does not divide the index of $O_{\pi,\pi'}(G)$.

We would like to thank Jürgen Müller for pointing out the following example which shows that π-separability of G in the hypothesis is indespensable:

Example 1.1. Let $G = Sz(8)$ be the smallest Suzuki group. Then $|G| = 2^6 \cdot 5 \cdot 7 \cdot 13$, and G has an automorphism α of order 3. Set $H = G\langle\alpha\rangle$. Since $|G|$ is not divisible by 3, there are no elements of order 3^2 in H. Moreover, the elements of order 7 and 13 are self-centralizing in G, and each of them are distributed into 3 conjugacy classes which are fused under α. Hence there cannot possibly be elements of order 21 or 39 in H. As there exist elements of order 15 in $H - G$ we can take $N = 2^6 \cdot 3 \cdot 5$ and hence $\pi = \{2, 3, 5\}$. It follows that G is not π-separable.

2. Main Result

In this section we shall present a proof of the main result of this paper.

Theorem 2.1. *Let p be a prime number and N be a positive integer not divisible by p^2. Let G be a finite group admitting an automorphism α of prime order p, and H denote the semidirect product of G by $\langle\alpha\rangle$. Assume that $x^N = 1$ for every element x in $H - G$. If G is π-separable, where $\pi = \pi(N)$, then the following hold:*

(a) G has π'-length 1,

(b) $O_{\pi,\pi'}(G)/O_\pi(G)$ is nilpotent and $G/O_{\pi,\pi'}(G)$ is a p'-group,

(c) If $2 \notin \pi$, then $[G, \alpha] = O_{\pi,\pi'}([G, \alpha])$,

(d) If G is a solvable group and 4 does not divide N, then G has p-length 1.

Proof. We should note first that the following three facts are immediate consequences of the Hughes type action of α on G (see [1, 2, 3])

(1) If $x \in H - G$, for any x-invariant section S of G, the exponent of $C_S(x)$ divides N.

(2) Every element x in $H - G$ of order p acts trivially or exceptionally on every elementary abelian x-invariant p-section S of G, that is, the degree of the minimum polynomial of x on S is less than the order of the linear operator induced by x on S.

(3) For any subgroup Q of G with $N_H(Q) \not\leq G$, there exists $\beta \in N_H(Q) - G$ of order p so that $N_H(Q) = N_G(Q)\langle\beta\rangle$.

Let now G be a minimal counterexample to the *Claim(a)* of the theorem. We shall proceed towards a contradiction in a series of steps. First three of them can be easily deduced from the minimality of G.

(4) Every α-invariant proper subgroup of G and every factor group of G by a nontrivial α-invariant normal subgroup satisfy Claim(a).

(5) $O_\pi(G) = 1$ and $O^\pi(G) = G$.

(6) G has a unique minimal normal α-invariant subgroup M which is contained in $O_{\pi'}(G)$.

(7) $G = O_{\pi',\pi,\pi'}(G)$ and $G/O_{\pi',\pi}(G)$ is an elementary abelian r-group for some prime r not in π, on which α acts fixed point freely and nontrivially.

By (4), we have the equality $G = O_{\pi',\pi,\pi'}(G)$. We may choose an α-invariant Sylow r-subgroup $\overline{R} = R/O_{\pi',\pi}(G)$ of $G/O_{\pi',\pi}(G)$ by [6, 6.2.2]. Set $\overline{X} = \Omega_1(Z(\overline{R}))$. It follows by (4) that $G = O_{\pi',\pi}(G)X$ where $C_{\overline{X}}(\alpha) = 1$.

(8) $M = O_{\pi'}(G)$ is a self-centralizing elementary abelian s-group for some prime s, on which α acts fixed point freely and nontrivially.

We observe that $O_{\pi'}(G)$ is nilpotent as α acts fixed point freely on $O_{\pi'}(G)$ by a well known result due to Thompson [11, Theorem 1]. In fact, $O_{\pi'}(G)$ is an s-group for some prime s by (6). We may also assume that it is elementary abelian. Set $\overline{G} = G/M$. By (4), *Claim(a)* of the theorem holds in the factor group $\overline{G}$ and hence we have the equality $\overline{G} = O_{\pi,\pi',\pi}(\overline{G})$. It follows by (5) that $\overline{G} = O_{\pi,\pi'}(\overline{G})$. Let now H be a Hall π-subgroup of G. Then $O_{\pi'}(G)H = O_{\pi',\pi}(G)$ and hence $C_{O_{\pi'}(G)}(H) = Z(O_{\pi',\pi}(G)) = Z$ is a normal subgroup of G. If $Z \neq 1$, then $M \leq Z$ by (6). This leads to the contradiction that $H \leq C_G(O_{\pi'}(G)) \leq O_{\pi'}(G)$ since G is π-separable. Therefore $Z = 1$, and hence $O_{\pi'}(G) = [O_{\pi'}(G), H]$ by [6, 5.3.5]. Set $K/M = O_\pi(\overline{G})$. As $H \leq O_{\pi',\pi}(G) \cap K$, we have $[H, O_{\pi'}(G)] \leq M$. Thus $M = O_{\pi'}(G)$ and $C_G(M) = M$, as desired.

(9) $G = MQR$ where Q and R are α-invariant Sylow q and r-subgroups of G, respectively, for a prime q in π.

Let Q be a Sylow q-subgroup of G for a prime q in π. Then $Q \leq O_{\pi',\pi}(G)$ and hence $H = O_{\pi',\pi}(G)N_H(Q)$ by the Frattini argument. It follows by (3) that $N_H(Q) = N_G(Q)\langle\beta\rangle$ for some $\beta \in H - G$ of order p. Set $L = MN_G(Q)$. It is straightforward to verify that the action of β on L satisfies the hypothesis and hence L is of π'-length 1 by induction. That is

$L = O_{\pi,\pi',\pi}(L)$. Notice that $O_\pi(L) \le C_L(M) = M$ and hence $O_\pi(L) = 1$. Then $L = O_{\pi',\pi}(L)$.

Let now S be a Sylow r-subgroup of $N_G(Q)$. It follows that $G = O_{\pi',\pi}(G)S$ and $S \le O_{\pi'}(L)$, whence $[S,Q] \le Q \cap O_{\pi'}(L) = 1$. As a consequence, we get $G = O_{\pi',\pi}(G)C_G(Q)$ for each q in π and for every Sylow q-subgroup Q of G.

Set $\overline{G} = G/M$ and pick a Sylow r-subgroup R of $C_G(Q)$. Then $G = O_{\pi',\pi}(G)R$ by (7) and hence $\overline{R}$ is a Sylow r-subgroup of $\overline{G}$. In fact, $\overline{G} = O_\pi(\overline{G})\overline{R} = O_\pi(\overline{G})C_{\overline{G}}(\overline{Q})$. Let $\overline{R}_Q$ be a Sylow r-subgroup of $C_{\overline{G}}(\overline{Q})$. Then $\overline{G} = O_\pi(\overline{G})\overline{R}_Q$, implying that $|\overline{R}_Q| = |\overline{R}|$. Hence $\overline{R}$ is a Sylow r-subgroup of $C_{\overline{G}}(\overline{Q})$, and so $\overline{Q} \le C_{\overline{G}}(\overline{R})$. As q and Q are arbitrary, we see that $|O_\pi(\overline{G})|$ divides $|C_{\overline{G}}(\overline{R})|$, leading to the contradiction that $\overline{G} = C_{\overline{G}}(\overline{R})$. Thus we have $L = G$; more precisely, $H = MN_H(Q)$. Recall that Q is β-invariant for some $\beta \in H - G$ of order p where $N_H(Q) = N_G(Q)\langle\beta\rangle$, and that R is a Sylow r-subgroup of $N_G(Q)$. Now $K = N_G(Q)\langle\beta\rangle = N_G(Q)N_K(R)$ by the Frattini argument. Then, by (3), there exists $\gamma \in N_G(Q)\langle\beta\rangle - N_G(Q)$ of order p so that R is γ-invariant and hence Q is also γ-invariant. Without loss of generality we may assume that Q and R are both α-invariant. By the minimality of G, it follows that $G = MQR$, as desired.

(10) Claim(a) follows.

We observe first that $s = r$, because otherwise $[M,R] = 1$ as $C_{MR}(\alpha) = 1$. In case where G is a p'-group we obtain $C_M(\alpha) \neq 1$ by [12, Theorem 2.1.A] applied to the action of $QR\langle\alpha\rangle$ on M, which is not the case. Thus $|G|$ is divisible by p, whence $q = p$. Applying [4, Lemma 1] to the action of $QR\langle\alpha\rangle$ on M we obtain again $C_M(\alpha) \neq 1$. This contradiction completes the proof of *Claim(a)*. As a consequence we have $G = O_{\pi,\pi',\pi}(G)$.

As α acts fixed point freely on $O_{\pi,\pi'}(G)/O_\pi(G)$, it is a nilpotent group, and hence the first part of *Claim (b)* follows. We prove next that $G/O_{\pi,\pi'}(G)$ is a p'-group: Set $\overline{G} = G/O_\pi(G)$. Let P be a Sylow p-subgroup of H containing α. Then $P \cap G$ is an α-invariant Sylow p-subgroup of G. Pick now a nontrivial element x from $C_{\overline{P\cap G}}(\alpha)$ of order p and form the elementary abelian p-group $X = \langle x, \alpha\rangle$. Due to the fact that $C_{\overline{G}}(O_{\pi'}(\overline{G})) \le O_{\pi'}(\overline{G})$ we can choose an α-invariant Sylow s-subgroup V of the Frattini factor group of $O_{\pi'}(\overline{G})$ for a prime s not in π so that $[V,x] \neq 1$. It follows by [6, 5.3.16] that $V = \langle C_V(t) \mid 1 \neq t \in X\rangle$. On the other hand we have $C_V(t) = 1$ for each $t \in X - \langle x\rangle$ by (1). This contradiction completes the proof of *Claim(b)*.

We are now ready to establish *Claim(c)*. Suppose that $2 \notin \pi$. The proof of the equality $[G,\alpha] = O_{\pi,\pi'}([G,\alpha])$ goes by induction on $|G|$. We may assume that $[G,\alpha] = O_{\pi',\pi}([G,\alpha])$. By similar reduction arguments as in the proof of *Claim (a)*, it can be shown that $[G,\alpha] = MQ$ where M is an elementary abelian α-invariant s-group for some prime s which is not in π, and Q is an α-invariant q-group for and odd prime q in π so that $[Q,\alpha]=Q$ and $[M,Q]=M$. As $C_M(\alpha)=1$, it follows by [5, Lemma 2.2] that Q is a 2-group, which is impossible. Hence we have $[G,\alpha]=O_{\pi,\pi'}([G,\alpha])$ as claimed.

Finally we observe that *Claim(d)* holds: Recall that $G/O_{\pi,\pi'}(G)$ is a p'-group by *Claim(b)*. Hence it suffices to show that $O_\pi(G)$ has p-length 1. Assume the contrary. As $O_\pi(G)$ is solvable in this case, there exist subgroups S,T and U of $O_\pi(G)$ and an element $\beta\in H-G$ of order p by (3) satisfying the following (see also [3, Lemma 1]):

(i) S and U are p-groups, T is a q-group for some prime q in π-$\{p\}$;

(ii) $\langle\beta\rangle$ normalizes S,T and U; U normalizes S and T and T normalizes S;

(iii) STU is a group of nilpotent length 3.

We may also assume that $\overline{T}=T/C_T(S)$ is a special q-group so that $U\langle\beta\rangle$ acts irreducibly on its Frattini factor group, $TU\langle\beta\rangle$ acts irreducibly on $S/\Phi(S)$ and $U/C_U(\overline{T})$ is an elementary abelian p-group on which $\langle\beta\rangle$ acts irreducibly. It is straightforward to verify that β is trivial on $U/C_U(\overline{T})$. It should also be noted that $[S/\Phi(S),C_U(\overline{T})]=1$ because $C_{S/\Phi(S)}(C_U(\overline{T}))\neq 1$ as S and U are both p-groups. Thus we have $C_U(\overline{T})= C_{C_U(\overline{T})}(S/\Phi(S))$. To simplify the notation we set $X=STU/\Phi(S)C_T(S)C_U(\overline{T})$. If $[\overline{T},\beta]=1$, then $[S/\phi(S),\beta]=1$ holds, and so the group X is centralized by β. It follows by (1) that a Sylow p-subgroup of X has exponent p and hence X has p-length 1 by [8, IX.4.3]. This contradiction shows that $[\overline{T},\beta]=\overline{T}$. Applying now [5, Lemma 2.2] to the action of $\overline{T}\langle\beta\rangle$ on $S/\Phi(S)$, we see that $\overline{T}$ is a nonabelian special group. Using [5] and [8, IX.3.2] and with some effort one gets $q=2$. Set $Y=U/C_U(\overline{T})\langle\beta\rangle$. Using [6, 5.3.16] we see that $\overline{T}/\phi(\overline{T})=\langle C_{\overline{T}/\phi(\overline{T})}(y)\,|\,1\neq y\in Y\rangle$. Then there exists $y\in Y\langle\beta\rangle-Y$ such that $[\overline{T}/\phi(\overline{T}),y]=1$. By the hypothesis of *Claim(d)*, we see that $\overline{T}/\phi(\overline{T})$ is of exponent 2 and hence abelian. This contradiction completes the proof.
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References

[1] Ercan, G., Güloğlu İ. Ş.: On the Fitting length of generalized Hughes subgroup, Arch. Math., **55** (1990), 5-9.

[2] Ercan, G., Güloğlu İ. Ş.: On the Fitting length of $H_{pq}(G)$, Arch. Math., **56** (1991) 214-217.

[3] Ercan, G., Güloğlu İ. Ş.: On the Fitting length of $H_n(G)$, Rend. Sem. Mat. Univ. Padova, **89** (1993) 171-175.

[4] Ercan, G., Güloğlu İ. Ş.: Finite groups admitting fixed point free automorphisms of order pqr, J. Group Theory **7** (2004) 437-446.

[5] Gagola S, Jr.: Solvable groups admitting an almost fixed point free automorphism of prime order, Illinois J. Math, **22** (1978) 191-207.

[6] Gorenstein, D.: Finite Groups, New York (1969)
[7] Hughes, D. R., Thompson, J.G.: The H_p-problem and the structure of H_p-groups, Pacific J. Math., **9** (1959) 1097-1101.
[8] Huppert, B., Blackburn, N.: Finite groups II, Berlin-New York, (1982).
[9] Kegel, O.H.: Die nilpotenz der H_p-gruppen, Math. Z. **75** (1960/61) 373-376.
[10] Meixner, T.: Israel J. Math., **51** (1985), no. 1-2, 68-78.
[11] Thompson, J.G.: Finite groups with fixed point free automorphisms of prime order, Proc. Nat. Acad. Sciences (USA) **45** (1959) 578-581.
[12] Turull, A.: Fitting height of groups and of fixed points, J. Algebra, **86** (1984) 555-566.

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