RANK AND ORDER OF A FINITE GROUP ADMITTING A FROBENIUS-LIKE GROUP OF AUTOMORPHISMS

G. ERCAN, İ. Ş. GÜLOĞLU, AND E. I. KHUKHRO

Abstract. A finite group FH is said to be Frobenius-like if it has a nontrivial nilpotent normal subgroup F with a nontrivial complement H such that $FH/[F,F]$ is a Frobenius group with Frobenius kernel $F/[F,F]$. Suppose that a finite group G admits a Frobenius-like group of automorphisms FH of coprime order with certain additional restrictions (which are satisfied, in particular, if either $|FH|$ is odd or $|H|=2$). In the case where G is a finite p-group such that $G=[G,F]$ it is proved that the rank of G is bounded above in terms of $|H|$ and the rank of the fixed-point subgroup $C_G(H)$, and that $|G|$ is bounded above in terms of $|H|$ and $|C_G(H)|$. As a corollary, in the case where G is an arbitrary finite group estimates are obtained of the form $|G|\leqslant |C_G(F)|\cdot f(|H|,|C_G(H)|)$ for the order, and $\mathbf{r}(G)\leqslant \mathbf{r}(C_G(F))+g(|H|,\mathbf{r}(C_G(H)))$ for the rank, where f and g are some functions of two variables.

1. Introduction

In several recent papers [1, 2, 3, 4, 5, 6, 7, 8], finite groups G admitting a Frobenius group of automorphisms FH with kernel F and complement H were considered in the case where the kernel F acts fixed-point-freely: $C_G(F)=1$. In these papers bounds were obtained for the order, rank, Fitting height, nilpotency class, and exponent of the group G in terms of the corresponding properties and parameters of $C_G(H)$ and $|H|$ (for nilpotency class and exponent under certain additional conditions). Similar restrictions for the order, rank, and Fitting height of G were later obtained in [9, 10] under weaker assumptions on the action of a Frobenius group of automorphisms FH of coprime order, without assuming that the action of F is fixed-point-free.

In the present paper we obtain estimates for the rank and order of a finite group G admitting a so-called Frobenius-like group of automorphisms FH of coprime order with certain additional restrictions (which are satisfied, in particular, if either $|FH|$ is odd or $|H|=2$) also without assuming that the action of F is fixed-point-free. A finite group FH is said to be *Frobenius-like* if it has a nontrivial nilpotent normal subgroup F called *kernel* which has a nontrivial complement H such that $FH/[F,F]$ is a Frobenius group with Frobenius kernel $F/[F,F]$. The results of the present paper use a theorem of the first two authors [11, Theorem A] on linear representations of Frobenius-like groups, which is in turn based on their generalization [11, Theorem B] of the well-known Hall–Higman type Satz 17.13 in [12] (attributed to Dade) about representations of a cyclic extension of an extraspecial group to a more general situation.

As in [9] the proofs are essentially reduced to studying Sylow p-subgroups of G, for various primes p. Since in the case where G is a p-group satisfying $G=[G,F]$ the results are most

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strong, it is convenient to state a separate theorem for p-groups. The *rank* of a finite group is the minimum number r such that every subgroup can be generated by r elements.

Theorem 1. *Let FH be a Frobenius-like group with kernel F and complement H such that a Sylow 2-subgroup of H is cyclic and normal and F has no extraspecial sections of order p^{2m+1}, where $p^m+1=|H_1|$ for some subgroup $H_1\leqslant H$. If a finite p-group P admits FH as a group of automorphisms of coprime order such that $P=[P,F]$, then*

(a) *the nilpotency class of P is at most $2\log_p|C_P(H)|$;*
(b) *the order of P is bounded above in terms of $|H|$ and $|C_P(H)|$;*
(c) *the rank of P is bounded above in terms of $|H|$ and the rank of $C_P(H)$.*

Note that the condition that F has no extraspecial sections of order p^{2m+1}, where $p^m+1=|H_1|$ for some subgroup $H_1\leqslant H$, is satisfied, for example, if $|FH|$ is odd, or if the orders of F and $|H|$ satisfy well-known restrictions related to powers of 2 and Fermat or/and Mersenne primes, or simply if $|H|=2$. As a corollary we obtain a result on the rank and order of an arbitrary finite group with a Frobenius-like group of automorphisms. Let $\mathbf{r}(K)$ denote the rank of a finite group K.

Theorem 2. *Let FH be a Frobenius-like group with kernel F and complement H such that a Sylow 2-subgroup of H is cyclic and normal, and F has no extraspecial sections of order p^{2m+1}, where $p^m+1=|H_1|$ for some subgroup $H_1\leqslant H$. If a finite group G admits FH as a group of automorphisms of coprime order, then*

(a) $|G|\leqslant|C_G(F)|\cdot f(|H|,|C_G(H)|)$ *for some function f of two variables;*
(b) $\mathbf{r}(G)\leqslant\mathbf{r}(C_G(F))+g(|H|,\mathbf{r}(C_G(H)))$ *for some function g of two variables.*

Compared to the results in [9], replacing the condition of FH being a Frobenius group by being a Frobenius-like group is a very significant relaxation of the hypotheses, while the additional conditions on the structure of FH are unavoidable in view of well-known examples related to so-called exceptional Hall–Higman–type situations.

All the functions mentioned in the theorems can be easily given explicit upper estimates.

2. Preliminaries

The induced group of automorphisms of an invariant section is often denoted by the same letter. We use the abbreviation, say, "(m,n)-bounded" for "bounded above by a function depending only on m and n".

Recall that if a group A is acting by automorphisms on a finite group G of coprime order, $(|A|,|G|)=1$, then the fixed points of the induced action of A on the quotient G/N by an A-invariant normal subgroup are covered by fixed points of A in G:

$$C_{G/N}(A)=C_G(A)N/N.$$

In particular, $[[G,A],A]=[G,A]$. For every prime p, the group G has an A-invariant Sylow p-subgroup. We shall use these well-known properties of coprime action without special references.

We now reproduce the statement of a theorem in [11].

Theorem 3 ([11, Theorem B])**.** *Let H be a finite group in which each Sylow subgroup is cyclic and $H/F(H)$ is not a nontrivial 2-group. Let P be an extraspecial group of order p^{2m+1} for some prime p not dividing $|H|$. Suppose that H acts on P in such a way that H centralizes $Z(P)$ and $[P,h]=P$ for any nonidentity element $h\in H$. Let k be an algebraically* *closed field of characteristic not dividing the order of $G = PH$ and let V be a kG-module on which $Z(P)$ acts nontrivially and P acts irreducibly. Let χ be the character of G afforded by V. Then $|H|$ divides $p^m - \delta$ and*

$$\chi_H = \frac{p^m - \delta}{|H|}\rho + \delta\mu,$$

where ρ is the regular character of H, μ is a linear character of H, and $\delta \in \{-1, 1\}$.

The following corollary was stated in [11] as Theorem A in the case of FH of odd order. Henceforth we use commutator notation $[U, G]$ for the submodule generated by all elements $-u + ug$, $u \in U$, $g \in G$, of a kG-module U. We also use the centralizer notation for the fixed-point subspace $C_V(A) = \{v \in V \mid va = v \text{ for all } a \in A\}$. It is convenient to introduce the following condition on a Frobenius-like group FH with kernel F and complement H in the hypotheses of Theorems 1 and 2:

$$(*) \qquad \begin{cases} \text{a Sylow 2-subgroup of } H \text{ is cyclic and normal,} \\ \text{and } F \text{ has no extraspecial sections of order } p^{2m+1} \\ \text{such that } p^m + 1 = |H_1| \text{ for some subgroup } H_1 \leqslant H. \end{cases}$$

Corollary 1. *Let FH be a Frobenius-like group with kernel F and complement H satisfying condition $(*)$. If FH acts by linear transformations on a vector space over an algebraically closed field k of characteristic coprime to $|FH|$ so that $[V, F] \neq 0$, then V_H has an H-regular direct summand; in particular, then $C_V(H) \neq 0$.*

Proof. Note that all Sylow p-subgroups of H are cyclic for $p \neq 2$ since H is Frobenius complement in $FH/[F, F]$, and for $p = 2$ by hypothesis. We can repeat word-for-word the proof of [11, Theorem A]; it is clear that any subgroups of H arising in all the inductive arguments will satisfy the hypotheses of Theorem 3. □

Note that a result like Corollary 1 cannot hold without additional conditions on FH: in the smallest example $F = Q_8$ is the quaternion group of order 8 and H is generated by its automorphism of order 3; then the Frobenius-like group FH has a faithful representation in coprime characteristic in which H acts without nontrivial fixed points.

We make use of the following theorem of Hartley and Isaacs [13].

Theorem 4 ([13, Theorem B])**.** *Let A be an arbitrary finite group. Then there exists a number $\delta = \delta(A)$ depending only on A with the following property. Let A act on G, where G is a finite soluble group such that $(|G|, |A|) = 1$, and let k be any field of characteristic not dividing $|A|$. Let V be any irreducible kAG-module and let S be any kA-module that appears as a component of the restriction V_A. Then $\dim_k V \leqslant \delta m_S$, where m_S is the multiplicity of S in V_A.*

Combining the Hartley–Isaacs Theorem 4 with Corollary 1 we obtain the following.

Corollary 2. *Let FH be a Frobenius-like group satisfying condition $(*)$. If FH acts by linear transformations on a vector space over a field k of characteristic coprime to $|FH|$ so that $V = [V, F]$, then $\dim V \leqslant \delta(H) \dim C_V(H)$, where $\delta(H)$ is a number depending only on H given by the Hartley–Isaacs Theorem 4.*

Proof. We may assume that the ground field is algebraically closed, since neither the hypothesis nor the conclusion is affected by field extensions. Let $V = \bigoplus V_i$, where the V_i are irreducible kFH-submodules. Clearly, $[V_i, F] = V_i$ for every i. By Corollary 1, the trivial kH-module appears as a component of V_{iH}; its multiplicity is exactly $\dim C_{V_i}(H)$. By Theorem 4, $\dim V_i \leqslant \delta(H) \dim C_{V_i}(H)$, whence $\dim V = \sum \dim V_i \leqslant \delta(H) \sum \dim C_{V_i}(H) = \delta(H) \dim C_V(H)$. □

A finite p-group P is said to be *powerful* if $[P, P] \leqslant P^p$ for $p \neq 2$, or $[P, P] \leqslant P^4$ for $p = 2$. (Here, $A^n = \langle a^n \mid a \in A \rangle$.) We recall the important connections of powerful p-groups with ranks of finite p-groups.

Lemma 2.1 ([14])**.** (a) *If a powerful p-group P is generated by d elements, then the rank of P is at most d and P is a product of d cyclic subgroups.*

(b) *If P is a finite p-group of rank r, then P contains a characteristic powerful subgroup of index at most $p^{r(\log_2 r+2)}$.*

Lemma 2.2. *If a finite p-group P has rank r and exponent p^n, then $|P| \leqslant p^{nf(r)}$ for some r-bounded number $f(r)$.*

Proof. The group P can be assumed to be powerful by Lemma 2.1(b); Lemma 2.1(a) completes the proof. □

The following result was obtained by Kovács [15] for soluble groups on the basis of Hall–Higman type theorems and extended, with the use of the classification, to arbitrary finite groups by Longobardi and Maj [16] (with the bound $2d$) and Guralnik [17].

Lemma 2.3. *If d is the maximum of the ranks of the Sylow p-subgroups of a finite group* (*over all primes p*), *then the rank of this group is at most $d + 1$.*

We shall also need the following well-known fact about nilpotent groups.

Lemma 2.4. *Let G be a nilpotent group of nilpotency class c.*

(a) *The order of G is bounded in terms of c and the order of $G/[G, G]$.*

(b) *The rank of G is bounded in terms of c and the rank of $G/[G, G]$.*

Proof. If $\gamma_i = \gamma_i(G)$ are terms of the lower central series of G, then there are homomorphisms

$$\underbrace{\gamma_1/\gamma_2 \otimes \cdots \otimes \gamma_1/\gamma_2}_{k} \;\to\; \gamma_k/\gamma_{k+1}$$

from the tensor power on the left onto γ_k/γ_{k+1}. Both parts of the lemma follow. □

3. Finite p-groups

Proof of Theorem 1. (a) Recall that P is a finite p-group admitting a Frobenius-like group FH of automorphisms of coprime order with kernel F and complement H satisfying condition $(*)$. Let $\gamma_i = \gamma_i(P)$ denote terms of the lower central series of P. Let $|C_P(H)| = p^n$. If V is an FH-invariant elementary abelian section such that $[V, F] \neq 1$, then $C_V(H) \neq 1$ by Corollary 1. Hence the group F can act nontrivially on at most n factors of the lower central series of P. Consequently, for some $i \leqslant 2n$ the group F acts trivially on the two consecutive factors γ_i/γ_{i+1} and $\gamma_{i+1}/\gamma_{i+2}$. Then $[F, \gamma_i, P] \leqslant [\gamma_{i+1}, P] = \gamma_{i+2}$ and $[\gamma_i, P, F] = [\gamma_{i+1}, F] \leqslant \gamma_{i+2}$. By the Three Subgroup Lemma we obtain $[[P, F], \gamma_i] = [P, \gamma_i] = \gamma_{i+1} \leqslant \gamma_{i+2}$. It follows that $\gamma_{i+1} = 1$, since P is a nilpotent group, so that P is nilpotent of class at most $2n$, as required.

(b) Given a bound for the nilpotency class obtained in (a), a bound for the order will follow by Lemma 2.4(a) if we obtain a bound for the order of P/γ_2. Since $P = [P, F]$, we also have $P/\gamma_2 = \big[P/\gamma_2,\, F\big]$, whence $C_{P/\gamma_2}(F) = 1$, so that also $U = [U, F]$ for every elementary abelian FH-invariant section of P/γ_2. Regarding U as an $\mathbb{F}_p FH$-module, we see that $\dim U \leqslant \delta(H) \dim C_U(H)$ by Corollary 2, whence $|U| \leqslant |C_U(H)|^{\delta(H)}$. Since the order of P/γ_2 is equal to the product of the orders of such suctions U, we obtain $|P/\gamma_2| \leqslant |C_{P/\gamma_2}(H)|^{\delta(H)} \leqslant |C_P(H)|^{\delta(H)}$, since the action is coprime. By part (a) the nilpotency class of P is at most $2 \log_p |C_P(H)| \leqslant 2 \log_2 |C_P(H)|$. Therefore the order $|P|$ is indeed bounded in terms of $|C_P(H)|$ and $|H|$ only.

(c) We now obtain a bound for the rank of P. The crucial step is to show that P has a powerful p-subgroup of bounded rank and 'co-rank'. The construction of a powerful subgroup is similar to how it was done in [9], [18], and [19]. First we need an estimate for the number of generators of P. Let $r = \mathbf{r}(C_P(H))$ denote the rank of $C_P(H)$ for brevity.

Lemma 3.1. *The group P is generated by at most $r\delta(H)$ elements.*

Proof. Consider the action of FH on the Frattini quotient $V = P/\Phi(P)$. We have already shown in part (b) that $\dim U \leqslant \delta(H) \dim C_U(H)$ by Corollary 2. The result follows, since $\dim C_U(H) \leqslant r$. □

Let M be a normal FH-invariant subgroup of P, which will be specified later. Consider the quotient $\overline{P} = P/M^p$ (or P/M^4 if $p = 2$); let the bar denote the images. Since $\overline{M} = M/M^p$ (or $\overline{M} = M/M^4$) has exponent p (or 4), the order of $C_{\overline{M}}(H)$ is at most p^f for some r-bounded number $f = f(r)$ by Lemma 2.2.

We denote terms of the upper central series by ζ_i, starting from the centre ζ_1.

Lemma 3.2. *We have $\overline{M} \leqslant \zeta_{2f+1}(\overline{P})$.*

Proof. Consider the following central series of $\overline{P}$:

$$M_1 = \overline{M} > M_2 > M_3 > \cdots > 1, \qquad \text{where } M_i = [...[\overline{M}, \underbrace{\overline{P}], \ldots, \overline{P}}_{i-1}].$$

All the M_i are normal FH-invariant subgroups of $\overline{P}$. Let $V_i = M_i/M_{i+1}$ and consider the action of FH on these sections.

Whenever $[V_i, F] \neq 1$ we have $C_{V_i}(H) \neq 1$ by Corollary 1. Since $|C_{\overline{M}}(H)| \leqslant p^f$, there can be at most f factors V_i with $[V_i, F] \neq 1$. Therefore for some $k \leqslant 2f + 1$ we must have both $[V_k, F] = 1$ and $[V_{k+1}, F] = 1$. In other words, we have $[[F, M_k], \overline{P}] \leqslant [M_{k+1}, \overline{P}] = M_{k+2}$ and $[[M_k, \overline{P}], F] = [M_{k+1}, F] \leqslant M_{k+2}$. Hence $[[\overline{P}, F], M_k] = [\overline{P}, M_k] = M_{k+1} \leqslant M_{k+2}$ by the Three Subgroup Lemma.

Then $M_{k+1} = 1$, since $\overline{P}$ is nilpotent: $M_{k+1} \leqslant M_{k+2}$ implies $[M_{k+1}, \overline{P}] \leqslant [M_{k+2}, \overline{P}]$, that is, $M_{k+2} \leqslant M_{k+3}$, and so on, which becomes eventually the trivial subgroup, since $\overline{P}$ is nilpotent. The equation $M_{k+1} = 1$ obtained above means precisely that $\overline{M} \leqslant \zeta_k(\overline{P}) \leqslant \zeta_{2f+1}(\overline{P})$. □

We continue proving that P has $(r, |H|)$-bounded rank. We put $M = \gamma_{2f+1} = \gamma_{2f+1}(P)$. It is convenient to introduce the unified notation $p_* = p$ if $p \neq 2$, and $p_* = 4$ if $p = 2$. Then by Lemma 3.2 we have $[\overline{M}, \overline{M}] \leqslant [\gamma_{2f+1}(\overline{P}),\, \zeta_{2f+1}(\overline{P})] = 1$, that is, $[M, M] \leqslant M^{p_*}$. This means precisely that $M = \gamma_{2f+1}(P)$ is a powerful p-subgroup of P.

The quotient $P/\gamma_{2f+1}^{p_*}$ is nilpotent of class $4f + 1$, since $\gamma_{2f+1}/\gamma_{2f+1}^{p_*} \leqslant \zeta_{2f+1}(P/\gamma_{2f+1}^{p_*})$ by Lemma 3.2 and by the choice of M. Since P is generated by at most $r\delta(H)$ elements by Lemma 3.1 and $P/\gamma_{2f+1}^{p_*}$ is nilpotent of class $4f+1$, the rank of $P/\gamma_{2f+1}^{p_*}$ is $(|H|, r)$-bounded by Lemma 2.4(b).

In particular, the rank of $\gamma_{2f+1}/\gamma_{2f+1}^{p_*}$ is $(|H|, r)$-bounded. Since $\gamma_{2f+1}^{p_*} \leqslant \Phi(\gamma_{2f+1})$, we obtain that the number of generators of γ_{2f+1} is $(|H|, r)$-bounded. But in a powerful p-group the number of generators is equal to its rank (Lemma 2.1(a)), so that the rank of γ_{2f+1} is $(|H|, r)$-bounded.

Thus, both the rank of P/γ_{2f+1} and the rank of γ_{2f+1} are $(|H|, r)$-bounded, whence the rank of P is $(|H|, r)$-bounded, as required. □

Remark. The functions in parts (b) and (c) of Theorem 1 can be assumed to be non-decreasing in each of their arguments. Actually, any function $f(x, y)$ of two positive integer variables can be replaced by the function $\bar{f}(x, y) = \sup\{f(u, v) \mid u \leqslant x,\ v \leqslant y\}$, which satisfies the required property.

4. General case

Proof of Theorem 2. Recall that G is a finite group admitting a Frobenius-like group of automorphisms FH of coprime order with kernel F and complement H satisfying condition $(*)$. We need to bound the order and rank of G.

For each prime p, let S_p be an FH-invariant Sylow p-subgroup of G (one for each p). We have $S_p = C_{S_p}(F)[S_p, F]$.

(a) By Theorem 1(b) we have $|[S_p, F]| \leqslant f_1(|H|, |C_{[S_p,F]}(H)|)$ for some function f_1 that is non-decreasing in each argument. Hence, $|S_p| \leqslant |C_{S_p}(F)| \cdot f_1(|H|, |C_{[S_p,F]}(H)|)$. Note also that $S_p = C_{S_p}(F)$ if $[S_p, F] = 1$. Since $|G| = \prod_p |S_p|$ and $|C_G(F)| = \prod_p |C_{S_p}(F)|$, we obtain

$$|G| \leqslant \prod_p |C_{S_p}(F)| \cdot \prod_{[S_p,F]\neq 1} f_1(|H|, |C_{[S_p,F]}(H)|) = |C_G(F)| \cdot \prod_{[S_p,F]\neq 1} f_1(|H|, |C_{[S_p,F]}(H)|).$$

But $C_{[S_p,F]}(H) \neq 1$ whenever $[S_p, F] \neq 1$ by Corollary 1. Hence in the product on the right-hand side the primes p divide $|C_G(H)|$. As a rough estimate, there are at most $\log_2 |C_G(H)|$ such primes. Therefore,

$$|G| \leqslant |C_G(F)| \cdot f_1(|H|, |C_G(H)|)^{\log_2 |C_G(H)|},$$

which is a required upper estimate for the order with the function $f(|H|, |C_G(H)|) = f_1(|H|, |C_G(H)|)^{\log_2 |C_G(H)|}$.

(b) For each prime p, by Theorem 1(c) we have $\mathbf{r}([S_p, F]) \leqslant f_2(|H|, \mathbf{r}(C_{[S_p,F]}(H)))$ for some function f_2 that is non-decreasing in each argument. Hence,

$$\mathbf{r}(S_p) \leqslant \mathbf{r}(C_{S_p}(F)) + f_2(|H|, \mathbf{r}(C_{[S_p,F]}(H))) \leqslant \mathbf{r}(C_G(F)) + f_2(|H|, \mathbf{r}(C_G(H))).$$

By Lemma 2.3, an upper estimate for the rank of G is obtained by adding 1 to the right-hand side. □

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Gülİn Ercan, Department of Mathematics, Middle East Technical University, Ankara, Turkey
E-mail address: ercan@metu.edu.tr

İsmaİl Ş. Güloğlu, Department of Mathematics, Doğuş University, Istanbul, Turkey
E-mail address: iguloglu@dogus.edu.tr

Evgeny Khukhro, Sobolev Inst. Math., Novosibirsk, 630090, Russia
E-mail address: khukhro@yahoo.co.uk