Asymptotic behavior of Lotz – Räbiger nets and of martingale nets

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Abstract

Based on the convergence theorem 1 obtained in [4] several results related to LR-nets and to M-nets are proved. Among them we mention theorems 2, 3, and 4. We also present a unified approach to both LR- and M-nets in the introduction. These nets appear as two extreme types of asymptotically abelian nets.

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1 Introduction

We begin with the following rather general discussion. Although we deal with LR- and M-nets in the core part of the paper, we prefer to introduce some important generalization of these nets. From our point of view this way of the presentation explains more transparently the nature of concepts of LR- and M-nets.

1.1 Nets of continuous functions

Let $\Lambda = (\Lambda, \succ)$ be a partially ordered directed set. This set assumed to be fixed through the paper. Let $X = (X, \tau)$ be a topological space. Denote by $C(X, X)$ the set of all continuous functions from X to X, and by $\mathcal{NC}(X) = \mathcal{NC}_\Lambda(X)$ the set of all nets in $C(X, X)$ indexed by Λ. The set $\mathcal{NC}(X)$ is equipped with the following natural operation

$$\mathcal{F} \bullet \mathcal{G} := (f_\lambda \circ g_\lambda)_{\lambda\in\Lambda} \qquad (\mathcal{F} = (f_\lambda)_{\lambda\in\Lambda}, \mathcal{G} = (g_\lambda)_{\lambda\in\Lambda} \in \mathcal{NC}(X)),$$

to which we may refer as to the *pointwise composition* of nets $\mathcal{F}$ and $\mathcal{G}$.

We identify a function $f \in C(X, X)$ with the net $\mathbf{f} := (f_\lambda)_{\lambda\in\Lambda}$ such that $f_\lambda = f$ for all $\lambda \in \Lambda$ if this cannot cause a confusion. The embedding $f \mapsto \mathbf{f}$

preserves the operation of composition in the sense that if $h = f \circ g$ in $C(X, X)$ then $\mathbf{h} = \mathbf{f} \bullet \mathbf{g}$ in $\mathcal{NC}(X)$.

Obviously, $\mathcal{NC}(X)$ is a semigroup with respect to the operation "$\bullet$". The net **id**, where $id(x) = x$ for all $x \in X$, is a unit of this semigroup. After the identification of functions with nets, $C(X, X)$ becomes a subsemigroup of $\mathcal{NC}(X)$.

Let x be an element of X then, for a net $\mathcal{F} = (f_\lambda)_{\lambda \in \Lambda} \in \mathcal{NC}(X)$, the net $(f_\lambda(x))_{\lambda \in \Lambda}$ of elements of X is called the *orbit of* $\mathcal{F}$ *starting at* x.

1.2 Asymptotic equivalence of nets

Given a net $(x_\lambda)_{\lambda \in \Lambda}$ in X and an element $x \in X$, we write $\tau\text{-}\lim\limits_{\lambda \to \infty} x_\lambda = y$ or $x_\lambda \xrightarrow{\tau} y$ if the net $(x_\lambda)_{\lambda \in \Lambda}$ converges to x topologically, that is, for every neighborhood U of x, there exists $\lambda_U \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \succ \lambda_U$.

How one can describe and investigate $\mathcal{NC}(X)$ asymptotically? For this purpose, a suitable concept of asymptotic equivalence between elements of $\mathcal{NC}(X)$ is needed. There are at least two approaches to this concept based on different possibilities of comparing of orbits of nets belonging to $\mathcal{NC}(X)$. Let $\mathcal{F} = (f_\lambda)_{\lambda \in \Lambda}$ and $\mathcal{G} = (g_\lambda)_{\lambda \in \Lambda}$ be such nets.

The first possibility of comparing of orbits of $\mathcal{F}$ and $\mathcal{G}$ arises if the difference of any two elements $a, b \in X$ exists as an element of X. This is the case, when (X, τ) is a topological group. Denote the group operation on X by $+$ and the unit element of X by $\mathbf{0}$. We say that $\mathcal{F}$ is *asymptotically equivalent* to $\mathcal{G}$ if

$$\tau - \lim_{\lambda \to \infty} (f_\lambda(x) - g_\lambda(x)) = \mathbf{0} \qquad (\forall x \in X)\,. \tag{1}$$

The second possibility takes place if we can compare the nearness between two points of X uniformly, for instance, if τ is a uniformizable topology on X (i.e. the topology of a uniform space $(X, \mathcal{U})$). Then we call the net $\mathcal{F}$ *asymptotically equivalent* to $\mathcal{G}$ if, for every $x \in X$, $V \in \mathcal{U}$, there exists $\lambda_V \in \Lambda$ satisfying

$$(f_\lambda(x), g_\lambda(x)) \in V \qquad (\forall \lambda \succ \lambda_V)\,. \tag{2}$$

In both cases, we write $\mathcal{F} \stackrel{a}{\approx} \mathcal{G}$ if $\mathcal{F}$ is asymptotically equivalent to $\mathcal{G}$. It is easy to see that $\stackrel{a}{\approx}$ is an equivalence relation on $\mathcal{NC}(X)$. We say that a net $\mathcal{F} = (f_\lambda)_{\lambda \in \Lambda}$ is asymptotically equivalent to a function $f \in C(X, X)$, and write $\mathcal{F} \stackrel{a}{\approx} f$ if $\mathcal{F} \stackrel{a}{\approx} \mathbf{f}$. Remark that $\mathcal{F} \stackrel{a}{\approx} f$ means exactly that the net $(f_\lambda)_{\lambda \in \Lambda}$ converges to the function f pointwise.

1.3 Asymptotically abelian nets

We say that a net $\mathcal{F}$ *commutes asymptotically* with a net $\mathcal{G}$ if

$$\mathcal{F} \bullet \mathcal{G} \stackrel{a}{\approx} \mathcal{G} \bullet \mathcal{F}$$

and say that a net $\mathcal{H} \in \mathcal{NC}(X)$ commutes asymptotically with a function $f \in C(X,X)$ if $\mathcal{H} \bullet \mathbf{f} \overset{a}{\approx} \mathbf{f} \bullet \mathcal{H}$. We call a net $\mathcal{F} = (f_\lambda)_{\lambda\in\Lambda}$ *asymptotically abelian* if

$$\mathcal{F} \bullet \mathbf{f}_\mu \overset{a}{\approx} \mathbf{f}_\mu \bullet \mathcal{F}$$

for every $\mu \in \Lambda$. Of course, any net consisting of pairwise commuting functions is asymptotically abelian. However, a net can be asymptotically abelian regardless of the pairwise commutativity of its elements. For example, any net $\mathcal{H}$ of continuous functions on a topological group (X,τ) such that $\mathcal{H} \overset{a}{\approx} \mathbf{0}$ is asymptotically abelian.

1.4 Two extreme types of asymptotically abelian nets

In the present paper, we study asymptotically abelian nets, say $\mathcal{F} = (f_\lambda)_{\lambda\in\Lambda}$, satisfying one of the following two extreme conditions

$$\mathbf{f}_\mu \bullet \mathcal{F} \overset{a}{\approx} \mathbf{f}_\mu \overset{a}{\approx} \mathcal{F} \bullet \mathbf{f}_\mu \qquad (\forall \mu \in \Lambda)\,, \tag{3}$$

$$\mathbf{f}_\mu \bullet \mathcal{F} \overset{a}{\approx} \mathcal{F} \overset{a}{\approx} \mathcal{F} \bullet \mathbf{f}_\mu \qquad (\forall \mu \in \Lambda)\,. \tag{4}$$

Roughly speaking, the idea behind (3) is that the net $\mathcal{F}$ is *absorbed asymptotically by each of its elements*, and the idea of the condition (4) is that the net $\mathcal{F}$ *absorbs asymptotically each of its elements*.

Remark that if the net $\mathcal{F}$ satisfies (3) then

$$\mathbf{h} \bullet \mathcal{F} \overset{a}{\approx} \mathbf{h} \overset{a}{\approx} \mathcal{F} \bullet \mathbf{h}$$

for any function h in the semigroup $SG(\mathcal{F})$ generated by elements of the net $\mathcal{F}$ in $C(X,X)$. If $\mathcal{F}$ satisfies (4) then

$$\mathbf{h} \bullet \mathcal{F} \overset{a}{\approx} \mathcal{F} \overset{a}{\approx} \mathcal{F} \bullet \mathbf{h}$$

for any $h \in SG(\mathcal{F})$.

1.5 Nets of continuous functions on a compact metric space

Now we illustrate the usefulness of conditions (3) and (4) by proving the following nice elementary fact.

Proposition 1. *Let $X = (X,\rho)$ be a compact metric space and $\mathcal{F} = (f_\lambda)_{\lambda\in\Lambda}$ be an equicontinuous net in $\mathcal{NC}(X)$ satisfying one of the conditions* (3) *and* (4) *above. Then $\mathcal{F}$ converges uniformly to some function $f \in C(X,X)$ satisfying $f \circ f = f$.*

Proof. Assume that $\mathcal{F}$ does not converge uniformly. Then, by the Arzéla – Ascoli theorem, there exist two subnets $\mathcal{F}_1 = (f_{\lambda_\alpha})_\alpha$ and $\mathcal{F}_2 = (f_{\lambda_\beta})_\beta$ of $\mathcal{F}$

which converge uniformly, as $\alpha \to \infty$ and $\beta \to \infty$, to functions $\phi \in C(X, X)$ and $\psi \in C(X, X)$ respectively, and $\phi \neq \psi$.

If the net $\mathcal{F}$ satisfies condition (3) then, for every $\mu \in \Lambda$, $x \in X$,

$$\rho(f_\mu \circ f_{\lambda_\alpha}(x), f_\mu(x)) \to 0 \quad \& \quad \rho(f_{\lambda_\alpha} \circ f_\mu(x), f_\mu(x)) \to 0\,,$$

as $\alpha \to \infty$. Passing to the limits gives us

$$f_\mu \circ \phi = \phi \circ f_\mu = f_\mu \qquad (\forall \mu \in \Lambda)\,. \tag{3a}$$

By passing to the limits in $(3a)$, as $\mu = \lambda_\beta \to \infty$, we obtain

$$\psi \circ \phi = \phi \circ \psi = \psi\,. \tag{3b}$$

By the switching of the roles of subnets $\mathcal{F}_1$ and $\mathcal{F}_2$ and their limits ϕ and ψ in (3a) and (3b), we arrive at

$$\phi \circ \psi = \psi \circ \phi = \phi\,,$$

which contradicts to (3b) since $\psi \neq \phi$. Therefore $\mathcal{F}$ converges uniformly to the function ϕ. Now, by passing to the limits in $(3a)$, as $\mu = \lambda_\alpha \to \infty$, we obtain $\phi \circ \phi = \phi$, which completes the proof if $\mathcal{F}$ satisfies (3).

If net $\mathcal{F}$ satisfies condition (4) then, for every $\mu \in \Lambda$, $x \in X$,

$$\rho(f_\mu \circ f_{\lambda_\alpha}(x), f_{\lambda_\alpha}(x)) \to 0 \quad \& \quad \rho(f_{\lambda_\alpha} \circ f_\mu(x), f_{\lambda_\alpha}(x)) \to 0\,,$$

as $\alpha \to \infty$ along Λ. By passing to the limits, as $\alpha \to \infty$, we have

$$f_\mu \circ \phi = \phi \circ f_\mu = \phi \qquad (\forall \mu \in \Lambda)\,. \tag{4a}$$

Passing to the limits in $(4a)$, as $\mu = \lambda_\beta \to \infty$, gives us

$$\psi \circ \phi = \phi \circ \psi = \phi\,. \tag{4b}$$

By the switching of the roles of subnets $\mathcal{F}_1$ and $\mathcal{F}_2$ and their limits ϕ and ψ in (4b), we arrive at

$$\phi \circ \psi = \psi \circ \phi = \psi\,.$$

Which contradicts to (4b) since $\psi \neq \phi$. Therefore $\mathcal{F}$ converges uniformly to the function ϕ. By passing to the limits in $(4a)$, as $\mu = \lambda_\alpha \to \infty$, we obtain $\phi \circ \phi = \phi$, which completes the proof if $\mathcal{F}$ satisfies (4).

Therefore $\mathcal{F}$ converges uniformly to an idempotent function in both cases. □

In the present paper, we illustrate a way, in which the asymptotic notions mentioned above may perform in the framework of linear bounded operators on a Banach space. This setting provides a plenty of interesting structures from the operator theory and supplies us with a lot of well studied examples of nets related to our topic.

2 Operator nets on a Banach space

2.1 Definitions and notations

Let X be a Banach space, then X becomes a topological group with respect to the norm-topology $\tau := \tau_{\|\cdot\|}$. The two definitions given in (1) and (2) agree for nets in $\mathcal{NC}(X)$. Denote by $L(X)$ the algebra of all bounded linear operators on X. In the following, we consider nets (indexed by Λ) consisting of operators from $L(X)$, which are continuous function from X to X, of course. Denote the set of all such nets by $\mathcal{NL}(X)$ and call elements of $\mathcal{NL}(X)$ *operator nets.* We will use the terminology and notations from the introduction, by replacing $C(X,X)$ by $L(X)$ and $\mathcal{NC}(X)$ by $\mathcal{NL}(X)$. Remark that $\mathcal{NL}(X)$ is an algebra with respect to the operation of pointwise addition

$$(T_\lambda)_{\lambda\in\Lambda} + (S_\lambda)_{\lambda\in\Lambda} = (T_\lambda + S_\lambda)_{\lambda\in\Lambda}$$

and the operation "$\bullet$" defined in the introduction. The Banach algebra $L(X)$ becomes a subalgebra of $\mathcal{NL}(X)$ if we identify operators with nets as follows $T = \mathbf{T} = (T)_{\lambda\in\Lambda}$. The family $\{\mathcal{U} \in \mathcal{NL}(X) : \mathcal{U} \overset{sa}{\approx} \mathbf{0}\}$ of all operator nets converging to 0 strongly is a vector subspace of $\mathcal{NL}(X)$ but it is not an algebraic ideal in general.

Definition 1. *Let $\mathcal{U} = (U_\lambda)_{\lambda\in\Lambda}$ and $\mathcal{V} = (V_\lambda)_{\lambda\in\Lambda}$ be operator nets on X. We say that:*

(*i*) $\mathcal{U}$ *is* strongly asymptotically equivalent *to $\mathcal{V}$, and write $\mathcal{U} \overset{sa}{\approx} \mathcal{V}$, if*

$$\lim_{\lambda\to\infty} \|U_\lambda x - V_\lambda x\| = 0 \quad (\forall x \in X)\,; \tag{5}$$

(*ii*) $\mathcal{U}$ *is* uniformly asymptotically equivalent *to $\mathcal{V}$, and write $\mathcal{U} \overset{ua}{\approx} \mathcal{V}$, if*

$$\lim_{\lambda\to\infty} \|U_\lambda - V_\lambda\| = 0\,. \tag{6}$$

Definition 1 agrees with the definition of asymptotic equivalence given in the introduction. Indeed, in the case of the strong asymptotic equivalence, they even coincide. For the uniform equivalence, we may regard the nets $\mathcal{U}$ and $\mathcal{V}$ as nets on the normed space $(L(X), \|\cdot\|)$ by identifying an operator $T \in L(X)$ with the operator $\tilde{T} \in L(L(X))$ acting on $L(X)$ as the left multiplication

$$\tilde{T}(S) = T \circ S \qquad (S \in L(X))\,.$$

It is easy to see that $(U_\lambda)_{\lambda\in\Lambda} \overset{ua}{\approx} (V_\lambda)_{\lambda\in\Lambda}$ if and only if $(\tilde{U_\lambda})_{\lambda\in\Lambda} \overset{a}{\approx} (\tilde{V_\lambda})_{\lambda\in\Lambda}$.

2.2 Lotz – Räbiger nets and martingale nets

The conditions (3) and (4) adopted to operator nets (we additionally require the uniform boundedness) give us the following definition.

Definition 2. *A uniformly bounded operator net $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda} \subseteq L(X)$ is called*

(a) *a* martingale net (*an* M-net *for the short*) *if*

$$\mathbf{T}_\mu \bullet \mathcal{T} \overset{sa}{\approx} \mathbf{T}_\mu \overset{sa}{\approx} \mathcal{T} \bullet \mathbf{T}_\mu \qquad (\forall \mu \in \Lambda)\,; \tag{7}$$

(b) *a* Lotz – Räbiger net (*an* LR-net *for the short*) *if*

$$\mathbf{T}_\mu \bullet \mathcal{T} \overset{sa}{\approx} \mathcal{T} \overset{sa}{\approx} \mathcal{T} \bullet \mathbf{T}_\mu \qquad (\forall \mu \in \Lambda)\,. \tag{8}$$

This definition goes back to H.P. Lotz, who invented martingale nets in the paper [11] for the special case $\Lambda = \mathbb{N}$ under the name *M-sequence*, and to F. Räbiger, who called LR-nets by *M-nets* in the paper [12]. In the present paper we use the name M-net for the generalization of M-sequence in the original sense of Lotz. An obvious example of a Lotz – Räbiger net is given by any operator net $\mathcal{T} \overset{sa}{\approx} \mathbf{0}$. More examples of LR-nets are presented below. The following proposition is obvious.

Proposition 2. *An operator net $(T_\lambda)_{\lambda\in\Lambda} \subseteq L(X)$ is an LR-net if and only if the net $(I - T_\lambda)_{\lambda\in\Lambda}$ is an M-net.*

In view of this proposition, we may and do choose one of these notions for the investigation. In this paper, we prefer to deal with LR-nets.

For a given LR-net $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda}$, denote by $\hat{\Lambda}$ the set $\Lambda \cup \{\lambda_0\}$ for $\lambda_0 \notin \Lambda$, and extend the partial order from Λ to $\hat{\Lambda}$ by setting $\lambda_0 \prec \lambda$ for all $\lambda \in \Lambda$. Put $T_{\lambda_0} := I$. The family $\hat{\mathcal{T}} = (T_\lambda)_{\lambda\in\hat{\Lambda}}$ is an LR-net containing the identity operator. Since the only interesting questions concerning LR-nets are those of asymptotic nature (e.g., whenever $\lambda \to \infty$), we may suppose that any LR-net contains the identity operator.

In general, LR-nets are far from operator semigroups, though some operator semigroups are LR-nets and some LR-nets are operator semigroups.

The next useful proposition follows immediately from the definition of an LR-net.

Proposition 3. *Let $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda}$ be an LR-net on X and let $\bar{\text{co}}(\text{semigroup}(\mathcal{T}))$ be the norm-closure of the convex hull of the multiplicative semigroup generated by $\mathcal{T}$ in $L(X)$, then*

$$(C \circ T_\lambda \circ D)_{\lambda\in\Lambda} \overset{sa}{\approx} (T_\lambda)_{\lambda\in\Lambda} \qquad (\forall\ C, D \in \bar{\text{co}}(\text{semigroup}(\mathcal{T})))\,.$$

In particular,

$$(T_\lambda \circ T_\mu^k)_{\lambda\in\Lambda} \overset{sa}{\approx} (T_\lambda)_{\lambda\in\Lambda} \overset{sa}{\approx} (T_\mu^k \circ T_\lambda)_{\lambda\in\Lambda} \qquad (\forall\ k \in \mathbb{N}\,, \mu \in \Lambda)\,.$$

In general, an operator net Υ which is strongly asymptotically equivalent to an LR-net $\mathcal{T}$ is not necessarily an LR-net though Υ may share many asymptotic properties with $\mathcal{T}$. This remark motivates the following definition.

Definition 3. *An operator net $\Upsilon \subseteq L(X)$ is called an* asymptotic LR-net *(an* ALR-net *for the short) if there exists an LR-net $\mathcal{T}$ such that $\Upsilon \overset{sa}{\approx} \mathcal{T}$.*

2.3 Examples of LR-nets and M-nets

In the next section, we investigate some rather involved questions on convergence of LR-nets. Before doing this, let us present several examples of martingale nets and of Lotz – Räbiger nets which indicate importance of these concepts.

Example 1. A rather trivial example arises if we consider an arbitrary strongly continuous semigroup $(T_t)_{t\in(0,\infty)} \subseteq L(X)$ which is bounded in a nonempty interval $(0, a)$ of values of t. As the directed set Λ, we take the interval $(0, a)$ ordered with the ordering $\leq^*$, that is inverse to the usual one on $\mathbb{R}$, and put $\mathcal{T} := (T_t)_{t\in(0,a)_{\leq^*}}$. Then the strong continuity of the semigroup $(T_t)_{t\in(0,\infty)}$ gives us exactly the condition (7) of Definition 2 for the operator net $\mathcal{T}$. Thus $\mathcal{T}$ is an M-net.

Example 2 ([11, 12])**.** Let $T \in L(X)$ be a contraction (i.e. $\|T\| \leq 1$). Then the sequence $(A_n^T)_{n=1}^\infty$ of *Cesàro means* $A_n^T := \frac{1}{n}\sum\limits_{k=0}^{n-1} T^k$ is an LR-net.

More generally, let $(A_\lambda)_{\lambda\in\Lambda}$ be a *system of almost invariant integrals* in the sense of W. Eberlein [2] for an operator semigroup $\mathcal{S}$ ($\mathcal{S}$-*ergodic net* in the modern terminology [8, p.75]). Then $(A_\lambda)_{\lambda\in\Lambda}$ is an LR-net.

Example 3. Let $(H_t)_{t\geq 0}$ be a C_0-semigroup of kernel operators acting on the space $L_1(\mathbb{R})$ as follows:

$$(H_t f)[x] = (4\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp\left[\frac{-(x-y)^2}{4t}\right] f(y)\,dy \quad (\forall x \in X\,,\, t \in \mathbb{R}_+)\,.$$

These operators deliver solution $u(t, x) = (H_t f)[x]$ to the *heat equation* on the real line

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{with the initial condition } \; u(0, x) \; = \; f(x)\,.$$

Then $(H_t)_{t\geq 0}$ is an LR-net.

More generally, any one-parameter semigroup of *completely mixing* Markov operators on an L_1-space (cf. [8, p.254]) is an LR-net.

Example 4 ([11])**.** Let (Ω, Σ, μ) be a probability space, let $1 \leq p \leq \infty$, and let $X = L_p(\Omega, \Sigma, \mu)$. Let $(\Sigma_n)_{n=1}^\infty$ be an increasing sequence of σ-subalgebras of Σ. For every $f \in X$, let $C_n f \in X$ be the conditional expectation of f with respect to Σ_n. Then the sequence $(C_n)_{n=1}^\infty \subseteq L(A)$ is a commutative M-net.

Example 5 ([7])**.** Denote by $\Lambda = \mathcal{P}_{fin}(\mathbb{N})$ the directed partially ordered set of all nonempty finite subsets of $\mathbb{N}$. Let $\pi \in \Lambda$ be given. Define the local averaging operator $A_\pi : \ell_\infty \to \ell_\infty$ by

$$(A_\pi \mathbf{x})_k = (A_\pi \mathbf{x})(k) := \begin{cases} x_k & k \notin \pi \\ \mathrm{card}(\pi)^{-1} \cdot \sum_{i\in\pi} x_i & k \in \pi \end{cases}$$

Then the operator net $(A_\pi)_{\pi\in\Lambda}$ on ℓ_∞ is an LR-net.

Example 6. Let $A = (A, *, \|\cdot\|)$ be a Banach algebra. We embed A to $L(A)$ isometrically as follows

$$\pi(a)(x) := a * x \quad (\forall x \in A).$$

Then, for any approximate unit $(e_\lambda)_{\lambda\in\Lambda}$ in $(A, *, \|\cdot\|)$, that is a uniformly bounded net satisfying $\lim\limits_{\lambda\to\infty} e_\lambda * a = \lim\limits_{\lambda\to\infty} a * e_\lambda = a$ for all $a \in A$, the operator net $(\pi(e_\lambda))_{\lambda\in\Lambda} \subseteq L(X)$ is an M-net.

For further examples of LR- and martingale nets, we refer the reader to [11, 12].

3 Convergence theorem for Lotz – Räbiger nets

3.1 Convergence theorem

The following *convergence theorem* goes back to Lotz [11], who proved it for martingale sequences, and to Räbiger [12, Prop.2.3], who announced without a proof the equivalence of conditions $(i)-(iii)$ of Theorem 1 for arbitrary LR-nets. The complete proof of this theorem was published in [4, Prop.1.2, Thm.2.1, Thm.3.1] by N. Erkursun and the author.

Theorem 1. *Let $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda}$ be an LR-net on a Banach space X. Then the following conditions are equivalent:*

(i) *$\mathcal{T}$ converges strongly. In this case, the strong limit of the net $\mathcal{T}$ is a projection onto the fixed space* $\mathrm{Fix}(\mathcal{T})$ *of $\mathcal{T}$;*

(ii) *for every $x \in X$, the net $(T_\lambda x)_{\lambda\in\Lambda}$ has a weak cluster point;*

(iii)

$$X = \mathrm{Fix}(\mathcal{T}) \bigoplus \overline{\bigcup_{\lambda\in\Lambda} (I - T_\lambda)X}\,; \tag{9}$$

(iv) $\mathrm{Fix}(\mathcal{T})$ *separates the fixed space* $\mathrm{Fix}(\mathcal{T}^*)$ *of the adjoint net* $\mathcal{T}^* := (T^*_\lambda)_{\lambda\in\Lambda}$.

Actually the formula, which is a bit weaker than (9), had been proved in [4, Prop.1.2], namely

$$X = \mathrm{Fix}(\Theta) \bigoplus \overline{\mathrm{span} \bigcup_{\lambda\in\Lambda} (I - T_\lambda)X}\,. \tag{10}$$

But (10) implies (9) immediately. Indeed, let $x = (I - T_{\lambda_1})u$ and $y = (I - T_{\lambda_2})w$ be in $\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X$. Since $(T_\lambda \circ (I - T_\mu))_{\lambda\in\Lambda} \overset{sa}{\approx} 0$ for every $\mu \in \Lambda$, by (8), we obtain

$$\lim_{\lambda\to\infty} \|x - (I - T_\lambda)x\| \;=\; \lim_{\lambda\to\infty} \|T_\lambda x\| \;=\; 0\,,$$
$$\lim_{\lambda\to\infty} \|y - (I - T_\lambda)y\| \;=\; \lim_{\lambda\to\infty} \|T_\lambda y\| \;=\; 0\,.$$

Therefore

$$x + y = \lim_{\lambda\to\infty} (I - T_\lambda)(x + y) \in \overline{\bigcup_{\lambda\in\Lambda} (I - T_\lambda)X} \,.$$

By the approximation, this is true for arbitrary $x, y \in \overline{\operatorname{span} \bigcup_{\lambda\in\Lambda}(I - T_\lambda)X}$. Hence $\overline{\operatorname{span} \bigcup_{\lambda\in\Lambda}(I - T_\lambda)X} = \overline{\bigcup_{\lambda\in\Lambda}(I - T_\lambda)X}$.

We remark that the equivalence of conditions $(i) - (iii)$ for an $\mathcal{S}$-ergodic net goes back to W. Eberlein [2]. The equivalence of condition (iv) to the other conditions is essentially due to R. Sine [13], who discovered it for the sequence of Cesàro means of a single operator.

The following corollary for ALR-nets is obvious.

Corollary 1. *Let $(U_\lambda)_{\lambda\in\Lambda}$ be an ALR-net on a Banach space X. Then the following conditions are equivalent*:

(i) *$(U_\lambda)_{\lambda\in\Lambda}$ converges strongly to a projection*;

(ii) *for every $x \in X$, the net $(U_\lambda x)_{\lambda\in\Lambda}$ has a weak cluster point.*

3.2 Inheritance of the strong convergence under domination

As a first application of the convergence theorem, we consider the inheritance of the strong convergence of LR-nets under domination. We suppose the positive cone of an ordered Banach space X to be generating, i.e. $X = X_+ - X_+$.

Definition 4 ([6, Def.10]). *An ordered Banach space X is called* ideally ordered *if any* order interval $[x, y] \subseteq X$ *is weakly compact, and for any net $(x_\lambda)_{\lambda\in\Lambda} \subseteq X_+$ such that $\lim_{\lambda\to\infty} \|x_\lambda - y\| = 0$, the following*

$$\lim_{\lambda\to\infty} \operatorname{dist}([0, x_\lambda], [0, y]) = 0 \tag{11}$$

holds.

Many of well known Banach spaces, such as Banach lattices with order continuous norm and preduals of von Neumann algebras, are ideally ordered (cf. [3, Sec.2.1, 3.1]).

Theorem 2. *Let X be an ideally ordered Banach space, and let $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda}$ and $\mathcal{S} = (S_\lambda)_{\lambda\in\Lambda}$ be operator nets satisfying*

$$0 \le S_\lambda x \le T_\lambda x \qquad (\forall x \in X_+) \,.$$

If $\mathcal{S}$ is an LR-net and $\mathcal{T}$ converges strongly then $\mathcal{S}$ converges strongly.

Proof. Let $z \in X_+$, and let $T_\lambda z \overset{\|\cdot\|}{\to} y$. Then, by the assumption (11), $\lim_{\lambda\to\infty} \operatorname{dist}([0, T_\lambda z], [0, y]) = 0$ and hence $\lim_{\lambda\to\infty} \operatorname{dist}(S_\lambda z, [0, y]) = 0$. The net $(S_\lambda z)_{\lambda\in\Lambda}$ has a weak cluster point, since the order interval $[0, y]$ is weakly compact. Thus, for an arbitrary $x \in X$, which is always a difference of two positive elements, the net $(T_\lambda x)_{\lambda\in\Lambda}$ has a weak cluster point. By Theorem 1, this implies the strong convergence of $\mathcal{S}$. □

Theorem 2 goes back to W. Arendt and C.J.K. Batty [1, Thm.1.1], who proved it for LR-net of Cesàro averages on a Banach lattice with order continuous norm. Then Räbiger in [12, Prop.2.4] extended the Arendt – Batty result to an arbitrary LR-net on a Banach lattice with order continuous norm. Later, in the paper [6, Thm.13] of M.P.H. Wolff and the author, the assumption of order continuity of norm in the Arendt – Batty theorem had been relaxed to the assumption of ideally orderedness. We remark that Theorem 2 remains true if we require $\mathcal{S}$ to be a positive ALR-net only. Moreover, it admits an immediate extension to the case, when a net $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda}$ asymptotically dominates (in the sense of [5, p.2636]) a positive ALR-net $\mathcal{S} = (S_\lambda)_{\lambda\in\Lambda}$, i.e.

$$\limsup_{\lambda\to\infty} \operatorname{dist}(T_\lambda x - S_\lambda x, X_+) = 0 \quad (\forall x \in X_+)\,.$$

3.3 Constrictors of operator nets

For one more application of the convergence theorem, we need the following definition.

Definition 5. *Let $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda}$ be an operator net on a Banach space X. A subset $A \subseteq X$ is called a* constrictor *of $\mathcal{T}$ if*

$$\lim_{\lambda\to\infty} \operatorname{dist}(T_\lambda x, A) = 0 \quad (\forall x \in B_X)\,, \tag{12}$$

where B_X is the unit ball of X. We denote the family of all constrictors of $\mathcal{T}$ by Con($\mathcal{T}$).

This definition goes back to the pioneered paper [9] of A. Lasota, T.Y. Li, and J.A. Yorke, where the concept of constrictor was invented for a one parameter discrete Markov semigroup. In last two decades, the notion of constrictor of an abelian semigroup of linear operators on a Banach space was investigated by many authors (see [3, 10] for related results and references therein). Remark that the notion of constrictor is an asymptotic notion in the sense that $\mathrm{Con}(\mathcal{U}) = \mathrm{Con}(\mathcal{T})$ when $\mathcal{U} \overset{sa}{\approx} \mathcal{T}$.

Theorem 3. *Every LR-net containing a weakly compact operator is strongly convergent.*

Proof. Let $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda}$ be an LR-net on a Banach space X and let T_{λ_0} be weakly compact. Denote $M := \sup_{\lambda\in\Lambda} \|T_\lambda\|$. Let $x \in B_X$. Then

$$\lim_{\lambda\to\infty} \operatorname{dist}(T_\lambda x, T_{\lambda_0}(M \cdot B_X)) = 0\,,$$

since $\|T_\lambda x - T_{\lambda_0} \circ T_\lambda\, x\| \to 0$, as $\lambda \to \infty$, and $T_\lambda\, x \in M \cdot B_X$ for all λ. Hence the norm closure $\overline{T_{\lambda_0}(M \cdot B_X)}$ of $T_{\lambda_0}(M \cdot B_X)$ is a weakly compact constrictor of $\mathcal{T}$. Hence, for every $x \in X$, the net $(T_\lambda x)_{\lambda\in\Lambda}$ possesses a weak cluster point. The strong convergence of $\mathcal{T}$ follows from Theorem 1. $\square$

3.4 An axillary lemma

The following elementary lemma will be used in the proof of Theorem 4. The author does not know whether it possible to replace the compactness of the sets A_ε in the lemma by the weak compactness.

Lemma 3.1. *Any ALR-net $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda}$ on a Banach space X possessing constrictors of the form $A_\varepsilon + \varepsilon \cdot B_X$ for all $\varepsilon > 0$, where sets A_ε are compact, is strongly convergent.*

Proof. In view of Corollary 1 of Theorem 1, it is enough to show that, for every $x \in B_X$, the net $(T_\lambda x)_{\lambda\in\Lambda}$ possesses a weak-cluster point. We will prove that this net possesses even a norm-cluster point.

Let $x \in B_X$ be arbitrary. By the assumption, there exist a sequence $(C_n)_{n=1}^\infty$ of compact subsets of X and an increasing sequence $(\lambda_n)_{n=1}^\infty \subseteq \Lambda$ of indexes such that $\text{dist}(T_\lambda x, C_n) \leq \frac{1}{2^n}$ for all $\lambda \succ \lambda_n$. For every $m \in \mathbb{N}$, we may choose a net $(c_\lambda^m)_{\lambda\in\Lambda} \subseteq C_m$ satisfying

$$\|T_\lambda x - c_\lambda^m\| \leq \frac{1}{2^m} \quad (\forall \lambda \succ \lambda_m)\,. \tag{13}$$

Let $c^1 \in C_1$ be a cluster point of the net $(c_\lambda^1)_{\lambda\in\Lambda} \subseteq C_1$, which exists because of compactness of C_1. Then there exists a cofinal subset $\Lambda^{(1)}$ of $\Lambda^{(0)} := \Lambda$ such that the subnet $(c_\lambda^1)_{\lambda\in\Lambda^{(1)}}$ of the net $(c_\lambda^1)_{\lambda\in\Lambda^{(0)}}$ converges to c^1. By the induction, we may construct a sequence $(\Lambda^{(m)})_{m=1}^\infty$ of subsets of Λ and a sequence $(c^m)_{m=1}^\infty$ of elements of X satisfying:

I) for every $m \in \mathbb{N}$, the set $\Lambda^{(m+1)}$ is cofinal in $\Lambda^{(m)}$ and therefore in Λ;

II) for every $m \in \mathbb{N}$, the net $(c_\lambda^m)_{\lambda\in\Lambda^{(m)}}$ converges to c^m.

Then $(c^m)_{m=1}^\infty$ is a Cauchy sequence. Indeed, by using (13), for arbitrary m_2, m_1 $(m_2 \geq m_1 \geq m)$, we obtain

$$\|c^{m_1} - c^{m_2}\| = \lim_{\lambda\in\Lambda^{(m_2)}} \|c_\lambda^{m_1} - c_\lambda^{m_2}\| = \lim_{\lambda\in\Lambda^{(m_2)}} \|c_\lambda^{m_1} - T_\lambda x + T_\lambda x - c_\lambda^{m_2}\| \leq$$

$$\lim\sup_{\lambda\in\Lambda^{(m_2)}} \|T_\lambda x - c_\lambda^{m_1}\| + \lim\sup_{\lambda\in\Lambda^{(m_2)}} \|T_\lambda x - c_\lambda^{m_2}\| \leq \frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} \leq \frac{2}{2^m}\,. \tag{14}$$

Let c be the norm-limit of the sequence $(c^m)_{m=1}^\infty$, then (14) implies

$$\|c^m - c\| \leq \frac{2}{2^m} \quad (\forall m \in \mathbb{N})\,. \tag{15}$$

Take an arbitrary $\varepsilon > 0$. Then the inequality $\frac{4}{2^m} \leq \varepsilon$ holds for some $m \in \mathbb{N}$. It follows from (15) that

$$\|T_\lambda x - c\| \leq \|T_\lambda x - c^m\| + \|c^m - c\| \leq \|T_\lambda x - c^m\| + \frac{2}{2^m} \quad (\forall \lambda \in \Lambda)\,. \tag{16}$$

By using (13) and *II*), we may find for every $\lambda_* \in \Lambda$ an index $\zeta_m \in \Lambda^{(m)}$ such that $\zeta_m \succ \lambda_*, \lambda_m$ and

$$\|T_{\zeta_m} x - c^m\| \leq \|T_{\zeta_m} x - c_{\zeta_m}^m\| + \|c_{\zeta_m}^m - c^m\| \leq \frac{1}{2^m} + \frac{1}{2^m}\,. \tag{17}$$

Formulas (16) and (17) give us that, for every $\lambda_* \in \Lambda$, there exists $\lambda_{**} := \zeta_m$ such that $\lambda_{**} \succ \lambda_*$ and

$$\|T_{\lambda_{**}}x - c\| \leq \frac{4}{2^m} \leq \varepsilon\,.$$

Since $\varepsilon > 0$ were chosen arbitrarily, c is a norm-cluster point of $(T_\lambda x)_{\lambda\in\Lambda}$. □

3.5 The splitting theorem for ALR-nets possessing a compact constrictor

The following result might be considered as a version of the Lasota – Li – Yorke – Phòng – Sine theorem (cf. [3, Thm.1.3.3]) for ALR-nets.

Theorem 4. *Let $\mathcal{T} = (T_\lambda)_{\lambda\in\Lambda} \subseteq L(X)$ be an ALR-net on a Banach space X possessing a compact constrictor. Then $\mathcal{T}$ converges strongly to a finite rank projection.*

Proof. By Lemma 3.1, the net $\mathcal{T}$ converges strongly. Let A be a compact constrictor of $\mathcal{T}$, and let $\mathcal{T}'$ be an LR-net such that $\mathcal{T} \overset{sa}{\approx} \mathcal{T}'$. By Corollary 1, both nets $\mathcal{T}'$ and $\mathcal{T}$ converge to a projection onto the fixed space $\mathrm{Fix}(\mathcal{T}') \subseteq X$ of $\mathcal{T}'$. Since A is compact and $B_{\mathrm{Fix}(\mathcal{T}')} = B_X \cap \mathrm{Fix}(\mathcal{T}') \subseteq A$, the closed unit ball of $\mathrm{Fix}(\mathcal{T}')$ is compact. Hence $\dim(\mathrm{Fix}(\mathcal{T}')) < \infty$, and both nets $\mathcal{T}'$ and $\mathcal{T}$ converge strongly to a finite rank projection. □

Remark that the compactness of a constrictor A of $\mathcal{T}$ in Theorem 4 cannot be replaced by the condition that $A = K + \varepsilon B_X$, where K is compact and $0 \leq \varepsilon < 1$. As an example, one may take the LR-net of Cesàro averages of the operator T_1 from [3, Example 1.3.8] on the Banach space c_0 of vanishing sequences.

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