MAT 101
MATHEMATICS FOR SOCIAL SCIENCES

INTEGRATION 2
Solutions of Selected Problems from Sections 15.1, 15.2, 15.4, and 15.7

Problems 15.1 (page 688)
Suggested Problems: 1-29

2. $\int xe^{3x+1}dx$

If $u = x$ and $dv = e^{3x+1}dx$, then $du = dx$ and $v = \frac{1}{3}e^{3x+1}$.

$$\int xe^{3x+1}dx = \frac{x}{3}e^{3x+1} - \int \frac{1}{3}e^{3x+1}dx$$
$$= \frac{x}{3}e^{3x+1} - \frac{1}{3}\cdot\frac{1}{3}\int e^{3x+1}[3\ dx]$$
$$= \frac{x}{3}e^{3x+1} - \frac{1}{9}e^{3x+1} + C$$
$$= \frac{1}{9}e^{3x+1}(3x-1) + C$$

4. $\int xe^{-5x}\,dx$

Letting $u = x$, $dv = e^{-5x}\,dx$, then $du = dx$, $v = -\frac{1}{5}e^{-5x}$.

$$\int xe^{-5x}\,dx = -\frac{xe^{-5x}}{5} - \int -\frac{1}{5}e^{-5x}\,dx$$
$$= -\frac{xe^{-5x}}{5} + \frac{e^{-5x}}{5(-5)} + C$$
$$= -\frac{e^{-5x}}{5}\left(x + \frac{1}{5}\right) + C$$

6. $\int x^2 \ln x\,dx$

Letting $u = \ln x$, $dv = x^2dx$, then $du = \frac{1}{x}dx$, $v = \frac{x^3}{3}$

$$\int x^2 \ln x\,dx = \frac{x^3 \ln x}{3} - \int \frac{x^3}{3}\left(\frac{1}{x}dx\right)$$
$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C = \frac{x^3}{3}\left[\ln(x) - \frac{1}{3}\right] + C$$

8. $\int\left(\frac{t}{e^t}\right)dt$

Letting $u = t$, $dv = e^{-t}dt$, then $du = dt$, $v = -e^{-t}$

$$\int\left(\frac{t}{e^t}\right)dt = -te^{-t} - \int -e^{-t}dt$$
$$= -te^{-t} - e^{-t} + C = -e^{-t}(t+1) + C$$

10. $\int \frac{12x}{\sqrt{1+4x}}dx$

Letting $u = 12x$, $dv = (1+4x)^{-\frac{1}{2}}dx$, then $du = 12dx$, $v = \frac{1}{2}(1+4x)^{\frac{1}{2}}$

$$\int \frac{12x}{\sqrt{1+4x}}dx = 12x\cdot\frac{\sqrt{1+4x}}{2} - \int \frac{(1+4x)^{\frac{1}{2}}}{2}\cdot 12\,dx$$
$$= 6x\sqrt{1+4x} - (1+4x)^{\frac{3}{2}} + C$$
$$= \sqrt{4x-1}[6x - (1+4x)] + C$$
$$= (2x-1)\sqrt{4x+1} + C$$

14. $\int \frac{3x+5}{e^{2x}}dx$

Letting $u = 3x + 5$, $dv = e^{-2x}dx$, then $du = 3dx$ and $v = -\frac{1}{2}e^{-2x}$.

$$\int \frac{3x+5}{e^{2x}}dx = -\frac{3x+5}{2e^{2x}} - \int -\frac{1}{2}e^{-2x}\cdot 3\,dx$$
$$= -\frac{3x+5}{2e^{2x}} + \frac{3}{2}\int e^{-2x}dx$$
$$= -\frac{3x+5}{2e^{2x}} + \frac{3}{2}\left(-\frac{1}{2}e^{-2x}\right) + C$$
$$= -\frac{1}{4e^{2x}}[2(3x+5) + 3] + C$$
$$= -\frac{1}{4e^{2x}}(6x+13) + C$$

16. $\int_1^2 2xe^{-3x}\,dx$

Letting $u = 2x$, $dv = e^{-3x}\,dx$, then $du = 2\,dx$ and $v = -\frac{1}{3}e^{-3x}$.

$$\int_1^2 2xe^{-3x}\,dx$$
$$= \left[-\frac{2xe^{-3x}}{3} - \int -\frac{2}{3}e^{-3x}\,dx\right]_1^2$$
$$= \left[-\frac{2xe^{-3x}}{3} + \frac{2}{3}\cdot\frac{e^{-3x}}{-3}\right]_1^2$$
$$= \left[-\frac{2xe^{-3x}}{3} - \frac{2e^{-3x}}{9}\right]_1^2$$
$$= \left[-\frac{2e^{-3x}}{3}\left(x+\frac{1}{3}\right)\right]_1^2$$
$$= \left[-\frac{2e^{-6}}{3}\left(2+\frac{1}{3}\right)\right] - \left[-\frac{2e^{-3}}{3}\left(1+\frac{1}{3}\right)\right]$$
$$= -\frac{2e^{-6}}{3}\left[\frac{7}{3} - e^3\left(\frac{4}{3}\right)\right]$$
$$= -\frac{2}{9e^6}[7-4e^3]$$

20. $\int (\ln x)^2\,dx$

Letting $u = (\ln x)^2$, $dv = dx$, then

$du = \left[\frac{2\ln x}{x}\right]dx$, $v = x$.

$$\int (\ln x)^2\,dx = x(\ln x)^2 - \int x\left[\frac{2\ln x}{x}\,dx\right]$$
$$= x(\ln x)^2 - 2\int \ln(x)\,dx$$

For $\int \ln(x)\,dx$, let $u = \ln x$, $dv = dx$. Then

$du\left(\frac{1}{x}\right)dx$, $v = x$, so

$$\int \ln(x)\,dx = x\ln x - \int x\left(\frac{1}{x}\,dx\right) = x[\ln(x) - 1] + C_1 .$$

Thus $\int (\ln x)^2\,dx = x\left[(\ln x)^2 - 2\ln(x) + 2\right] + C$.

22. $\int \frac{xe^x}{(x+1)^2}\,dx$

Letting $u = xe^x$, $dv = (x+1)^{-2}\,dx$, then

$du = (x+1)e^x\,dx$, $v = -(x+1)^{-1}$.

$$\int \frac{xe^x}{(x+1)^2}\,dx = -\frac{xe^x}{x+1} + \int e^x\,dx$$
$$= -\frac{xe^x}{x+1} + e^x + C$$
$$= e^x\left(1 - \frac{x}{x+1}\right) = e^x\left(\frac{x+1-x}{x+1}\right) + C = \frac{e^x}{x+1} + C$$

28. $\int x^5 e^{x^2}\,dx$

Letting $u = x^4$ and $dv = xe^{x^2}\,dx$, then

$du = 4x^3\,dx$ and $v = \frac{1}{2}e^{x^2}$.

$$\int x^5 e^{x^2}\,dx = \frac{x^4}{2}e^{x^2} - \int \frac{1}{2}e^{x^2}\cdot 4x^3\,dx$$
$$= \frac{x^4}{2}e^{x^2} - 2\int x^3 e^{x^2}\,dx$$

Using Problem 27 for $\int x^3 e^{x^2}\,dx$,

$$\int x^5 e^{x^2}\,dx = \frac{x^4}{2}e^{x^2} - 2\cdot\left[\frac{1}{2}e^{x^2}(x^2-1)\right] + C$$
$$= \frac{x^4}{2}e^{x^2} - e^{x^2}(x^2-1) + C$$
$$= \frac{1}{2}e^{x^2}(x^4 - 2x^2 + 2) + C$$

Problem 15.2 (p. 694)
Suggested Problems: 1-28

2. $\dfrac{x+5}{x^2-1}=\dfrac{x+5}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$

$x+5=A(x+1)+B(x-1)$
If $x=-1$, then $4=-2B$, or $B=-2$. If $x=1$, then $6=2A$, or $A=3$.

Answer $\dfrac{3}{x-1}-\dfrac{2}{x+1}$

4. $\dfrac{2x^2-15}{x^2+5x}=2+\dfrac{-10x-15}{x^2+5x}$ (by long division).

$\dfrac{-10x-15}{x^2+5x}=\dfrac{-10x-15}{x(x+5)}=\dfrac{A}{x}+\dfrac{B}{x+5}$

$-10x-15=A(x+5)+Bx$. If $x=0$, then $-15=5A$, or $A=-3$. If $x=-5$, then $35=-5B$, or $B=-7$.

Answer: $2-\dfrac{3}{x}-\dfrac{7}{x+5}$

6. $\dfrac{2x+3}{x^2(x-1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x-1}$

$2x+3=Ax(x-1)+B(x-1)+Cx^2$

If $x=0$, then $3=-B$, or $B=-3$. If $x=1$, then $5=C$. If $x=-1$, then $1=2A-2B+C$, $1=2A+6+5$, or $A=-5$.

Answer: $-\dfrac{5}{x}-\dfrac{3}{x^2}+\dfrac{5}{x-1}$

12. $\dfrac{2x-1}{x^2-x-12}=\dfrac{2x-1}{(x-4)(x+3)}=\dfrac{A}{x-4}+\dfrac{B}{x+3}$

$2x-1=A(x+3)+B(x-4)$
If $x=-3$, then $-7=-7B$, or $B=1$. If $x=4$, then $7=7A$, or $A=1$.

$\displaystyle\int\frac{2x-1}{x^2-x-12}\,dx=\int\left(\frac{1}{x-4}+\frac{1}{x+3}\right)dx$

$=\ln|x-4|+\ln|x+3|+C=\ln|(x-4)(x+3)|+C$

14. $\dfrac{7(4-x^2)}{(x-4)(x-2)(x+3)}=\dfrac{7(2+x)(2-x)}{(x-4)(x-2)(x+3)}$

$=\dfrac{-7(x+2)}{(x-4)(x+3)}=\dfrac{A}{x-4}+\dfrac{B}{x+3}$

$-7(x+2)=A(x+3)+B(x-4)$
If $x=-3$, then $7=-7B$, or $B=-1$. If $x=4$, then $-42=7A$, or $A=-6$.

$\displaystyle\int\frac{7(4-x^2)}{(x-4)(x-2)(x+3)}\,dx=\int\left(\frac{-6}{x-4}+\frac{-1}{x+3}\right)dx$

$=-6\ln|x-4|-\ln|x+3|+C$

$=-\ln\left|(x-4)^6(x+3)\right|+C$

8. $\dfrac{3x^2+5}{(x^2+4)^2}=\dfrac{Ax+B}{x^2+4}+\dfrac{Cx+D}{(x^2+4)^2}$

$3x^2+5=(Ax+B)(x^2+4)+(Cx+D)$

$3x^2+5=Ax^3+Bx^2+(4A+C)x+(4B+D)$

Thus $A=0$, $B=3$, $4A+C=0$, $4B+D=5$. This gives $A=0$, $B=3$, $C=0$, $D=-7$.

Answer: $\dfrac{3}{x^2+4}-\dfrac{7}{(x^2+4)^2}$

10. $\dfrac{7x+6}{x^2+3x}=\dfrac{7x+6}{x(x+3)}=\dfrac{A}{x}+\dfrac{B}{x+3}$

$7x+6=A(x+3)+Bx$
If $x=-3$, then $-15=-3B$, or $B=5$.
If $x=0$, then $6=3A$, or $A=2$.

$\displaystyle\int\frac{7x+6}{x^2+3x}\,dx=\int\left(\frac{2}{x}+\frac{5}{x+3}\right)dx$

$=2\ln|x|+5\ln|x+3|+C$

$=\ln\left|x^2(x+3)^5\right|+C$

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20. $\dfrac{-3x^3+2x-3}{x^2(x^2-1)}=\dfrac{-3x^3+2x-3}{x^2(x+1)(x-1)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+1}+\dfrac{D}{x-1}$

$$-3x^3+2x-3=Ax(x+1)(x-1)+B(x+1)(x-1)+Cx^2(x-1)+Dx^2(x+1)$$

If $x = 0$, then $-3 = -B$, or $B = 3$. If $x = -1$, then $-2 = -2C$, or $C = 1$. If $x = 1$, then $-4 = 2D$, or $D = -2$. If $x = 2$, then $-23 = 6A + 3B + 4C + 12D$, $-23 = 6A + 9 + 4 - 24$, or $A = -2$.

$$\int\frac{-3x^3+2x-3}{x^2(x^2-1)}dx=\int\left(\frac{-2}{x}+\frac{3}{x^2}+\frac{1}{x+1}+\frac{-2}{x-1}\right)dx$$

$$=-2\ln|x|-\frac{3}{x}+\ln|x+1|-2\ln|x-1|+C=-\frac{3}{x}+\ln\left|\frac{x+1}{x^2(x-1)^2}\right|+C$$

24. $\dfrac{5x^4+9x^2+3}{x(x^2+1)^2}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$

$$\begin{aligned}5x^4+9x^2+3&=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x\\&=A(x^4+2x^2+1)+(Bx+C)(x^3+x)+Dx^2+Ex\\&=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A\end{aligned}$$

Thus, $A + B = 5$, $C = 0$, $2A + B + D = 9$, $C + E = 0$, and $A = 3$. This gives $A = 3$, $B = 2$, $C = 0$, $D = 1$, and $E = 0$.

$$\begin{aligned}\int\frac{5x^4+9x^2+3}{x(x^2+1)^2}dx&=\int\left(\frac{3}{x}+\frac{2x}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx\\&=3\ln|x|+\ln\left|x^2+1\right|-\frac{1}{2(x^2+1)}+C\\&=\ln\left|x^3(x^2+1)\right|-\frac{1}{2(x^2+1)}+C\end{aligned}$$

26. $\dfrac{12x^3+20x^2+28x+4}{3(x^2+2x+3)(x^2+1)}=\dfrac{1}{3}\left(\dfrac{Ax+B}{x^2+2x+3}+\dfrac{Cx+D}{x^2+1}\right)$

$$\begin{aligned}12x^3+20x^2+28x+4&=(Ax+B)(x^2+1)+(x^2+2x+3)(Cx+D)\\&=(A+C)x^3+(B+D+2C)x^2+(A+2D+3C)x+(B+3D)\end{aligned}$$

Thus, $A + C = 12$, $B + D + 2C = 20$, $A + 2D + 3C = 28$, $B + 3D = 4$. This gives $A = 4$, $B = 4$, $C = 8$, $D = 0$.

$$\begin{aligned}\int\frac{12x^3+20x^2+28x+4}{3\left(x^2+2x+3\right)\left(x^2+1\right)}dx&=\frac{1}{3}\int\left(\frac{4x+4}{x^2+2x+3}+\frac{8x}{x^2+1}\right)dx\\&=\frac{1}{3}\left[2\ln\left(x^2+2x+3\right)+4\ln\left(x^2+1\right)\right]+C\\&=\ln\left[\left(x^2+2x+3\right)^{\frac{2}{3}}\left(x^2+1\right)^{\frac{4}{3}}\right]+C\end{aligned}$$

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Problem 15.4 (p. 700)
Suggested Problems: 1-8

2. $\overline{f}=\frac{1}{2-1}\int_1^2(3x-1)\,dx=\left(\frac{3x^2}{2}-x\right)\Bigg|_1^2=\frac{7}{2}$

4. $\overline{f}=\frac{1}{3-1}\int_1^3\left(x^2+x+1\right)dx$

$=\frac{1}{2}\left(\frac{x^3}{3}+\frac{x^2}{2}+x\right)\Bigg|_1^3=\frac{22}{3}$

6. $\overline{f}=\frac{1}{4-0}\int_0^4 t\sqrt{t^2+9}\,dt$

$=\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\int_0^4\sqrt{t^2+9}[2t\ dt]$

$=\frac{1}{8}\left[\frac{2\left(t^2+9\right)^{\frac{3}{2}}}{3}\right]\Bigg|_0^4=\frac{49}{6}$

8. $\overline{f}=\frac{1}{3-1}\int_1^3\frac{5}{x^2}\,dx=\frac{1}{2}\cdot-\frac{5}{x}\Bigg|_1^3=\frac{1}{2}\left(-\frac{5}{3}+5\right)$

$=\frac{5}{3}$

Problem 15.7 (p. 716)
Suggested Problems: 1-12

2. $\int_1^\infty\frac{1}{(3x-1)^2}\,dx=\lim_{r\to\infty}\frac{1}{3}\int_1^r(3x-1)^{-2}[3\,dx]$

$=\lim_{r\to\infty}\left[-\frac{1}{3(3x-1)}\right]\Bigg|_1^r$

$=\lim_{r\to\infty}\left[-\frac{1}{3(3r-1)}+\frac{1}{6}\right]$

$=0+\frac{1}{6}$

$=\frac{1}{6}$

4. $\int_2^\infty\frac{1}{\sqrt[3]{(x+2)^2}}\,dx=\lim_{r\to\infty}\int_2^r(x+2)^{-\frac{2}{3}}\,dx$

$=\lim_{r\to\infty}\frac{(x+2)^{\frac{1}{3}}}{\frac{1}{3}}\Bigg|_2^r$

$=\lim_{r\to\infty}3\left[\sqrt[3]{r+2}-\sqrt[3]{(4)}\right]$

$=\infty\Rightarrow$ diverges

6. $\int_0^\infty\left(5+e^{-x}\right)dx=\lim_{r\to\infty}\int_0^r\left(5+e^{-x}\right)dx$

$=\lim_{r\to\infty}\left(5x-e^{-x}\right)\Big|_0^r=\lim_{r\to\infty}\left[\left(5r-e^{-r}\right)-(0-1)\right]$

$=\lim_{r\to\infty}\left(5r-\frac{1}{e^r}+1\right)=\infty\Rightarrow$ diverges

8. $\int_4^\infty\frac{x}{\sqrt{\left(x^2+9\right)^3}}\,dx=\lim_{r\to\infty}\frac{1}{2}\int_4^r\left(x^2+9\right)^{-\frac{3}{2}}[2x\ dx]$

$=\lim_{r\to\infty}\left[-\left(x^2+9\right)^{-\frac{1}{2}}\right]\Bigg|_4^r=\lim_{r\to\infty}\left[-\frac{1}{\sqrt{r^2+9}}+\frac{1}{5}\right]$

$=0+\frac{1}{5}=\frac{1}{5}$

10. $\int_{-\infty}^3\frac{1}{\sqrt{7-x}}\,dx=\lim_{r\to-\infty}-\int_r^3(7-x)^{-\frac{1}{2}}[-dx]$

$=\lim_{r\to-\infty}-2(7-x)^{\frac{1}{2}}\Big|_r^3$

$=\lim_{r\to-\infty}\left(-4+2\sqrt{7-r}\right)=\infty\Rightarrow$ diverges

12. $\int_{-\infty}^\infty(5-3x)\,dx=\int_{-\infty}^0(5-3x)\,dx+\int_0^\infty(5-3x)\,dx$

$\int_{-\infty}^0(5-3x)\,dx=\lim_{r\to-\infty}\int_r^0(5-3x)\,dx$

$=\lim_{r\to-\infty}\left(5x-\frac{3}{2}x^2\right)\Bigg|_r^0$

$=\lim_{r\to-\infty}\left[(0-0)-\left(5r-\frac{3}{2}r^2\right)\right]$

$=\lim_{r\to-\infty}\left(-5r+\frac{3}{2}r^2\right)=\infty$

Thus $\int_{-\infty}^\infty(5-3x)\,dx$ diverges
