METU NORTHERN CYPRUS CAMPUS

MAT 101
MATHEMATICS FOR SOCIAL SCIENCES

DIFFERENTIATION 2
Solutions of Selected Problems from Sections 12.1, 12.2, 12.4, 12.5 and 12.7

Problem 12.1 (page 533)
Suggested Problems: 1-47

2. $\frac{dy}{dx}=\frac{5}{9}\left(\frac{1}{x}\right)=\frac{5}{9x}$

4. $\frac{dy}{dx}=\frac{1}{5x-6}(5)=\frac{5}{5x-6}$

8. $\frac{dy}{dx}=\frac{1}{-x^2+6x}(-2x+6)=\frac{-2x+6}{-x^2+6x}$

$=\frac{-2(x-3)}{-x(x-6)}=\frac{2(x-3)}{x(x-6)}$

12. $\frac{dy}{dx}=x^2\left(\frac{1}{x}\right)+(\ln x)(2x)=x+2x\ln x$

$=x(1+2\ln x)$

16. $f(w)=\log\left(w^2+w\right)=\log_{10}\left(w^2+w\right)$

$=\frac{\ln\left(w^2+w\right)}{\ln 10}$

$f'(w)=\frac{1}{\ln 10}\cdot\frac{1}{w^2+w}(2w+1)$

$=\frac{2w+1}{(\ln 10)\left(w^2+w\right)}$

18. $y=x^2\log_2 x=x^2\cdot\frac{\ln x}{\ln 2}=\frac{1}{\ln 2}(x^2\ln x)$

$\frac{dy}{dx}=\frac{1}{\ln 2}\left[x^2\left(\frac{1}{x}\right)+\ln x(2x)\right]$

$=\frac{x}{\ln 2}(1+2\ln x)$

20. $\frac{dy}{dx}=\frac{(\ln x)(2x)-x^2\left(\frac{1}{x}\right)}{(\ln x)^2}$

$=\frac{2x\ln x-x}{\ln^2 x}=\frac{x[2\ln x-1]}{\ln^2 x}$

24. $y=6\ln\sqrt[3]{x}=6\cdot\frac{1}{3}\ln x=2\ln x$

$\frac{dy}{dx}=2\cdot\frac{1}{x}=\frac{2}{x}$

28. $y=\ln\left(\frac{2x+3}{3x-4}\right)=\ln(2x+3)-\ln(3x-4)$

$\frac{dy}{dx}=\frac{2}{2x+3}-\frac{3}{3x-4}$

$=\frac{2(3x-4)-3(2x+3)}{(2x+3)(3x-4)}=-\frac{17}{(2x+3)(3x-4)}$

30. $y=\ln\sqrt[3]{\frac{x^3-1}{x^3+1}}=\frac{1}{3}[\ln(x^3-1)-\ln(x^3+1)]$

$\frac{dy}{dx}=\frac{1}{3}\left[\frac{3x^2}{x^3-1}-\frac{3x^2}{x^3+1}\right]$

$=\frac{1}{3}\left[\frac{3x^2(x^3+1)-3x^2(x^3-1)}{(x^3-1)(x^3+1)}\right]$

$=\frac{2x^2}{x^6-1}$

32. $y=\ln\left[(5x+2)^4(8x-3)^6\right]$

$=4\ln(5x+2)+6\ln(8x-3)$

$\frac{dy}{dx}=4\cdot\frac{1}{5x+2}(5)+6\cdot\frac{1}{8x-3}(8)$

$=\frac{20}{5x+2}+\frac{48}{8x-3}$

34. $y=6\ln\frac{x}{\sqrt{2x+1}}=6\ln x-6\ln(2x+1)^{\frac{1}{2}}$

$=6\ln x-3\ln(2x+1)$

$\frac{dy}{dx}=\frac{6}{x}-3\cdot\frac{1}{2x+1}(2)=\frac{6}{x}-\frac{6}{2x+1}$

38. $\frac{dy}{dx}=(\ln 2)x^{(\ln 2)-1}$

40. $y=\ln^2(2x+11)=[\ln(2x+11)]^2$

$\frac{dy}{dx}=2[\ln(2x+11)]\cdot\frac{1}{2x+11}(2)=\frac{4\ln(2x+11)}{2x+11}$

42. $y = \ln\left(x^3 \sqrt[4]{2x+1}\right) = 3\ln x + \frac{1}{4}\ln(2x+1)$

$\frac{dy}{dx} = 3 \cdot \frac{1}{x} + \frac{1}{4} \cdot \frac{1}{2x+1}(2) = \frac{3}{x} + \frac{1}{2(2x+1)}$

44. $\frac{dy}{dx} = \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}(2x)\right]$

$= \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}\left(x+\sqrt{1+x^2}\right)}$

$= \frac{1}{\sqrt{1+x^2}}$

46. $y = x[\ln(x) - 1]$

$y' = x\left(\frac{1}{x}\right) + [\ln(x) - 1](1) = \ln x$

When $x = e$, $y = 0$ and $y' = 1$. The equation of the tangent line is $y - 0 = 1(x - e)$, or $y = x - e$.

Problem 12.2 (p. 537)
Suggested Problems: 1-32

2. $y' = \frac{2e^x}{5}$

4. $y' = e^{2x^2+5}(4x) = 4xe^{2x^2+5}$

10. $y' = 3x^4\left[e^{-x}(-1)\right] + e^{-x}(12x^3) = 3x^3e^{-x}(4 - x)$

14. $\frac{dy}{dx} = \frac{(e^x+e^{-x})[e^x - e^{-x}(-1)] - (e^x - e^{-x})[e^x + e^{-x}(-1)]}{(e^x+e^{-x})^2}$

$= \frac{(e^x+e^{-x})^2 - (e^x-e^{-x})^2}{(e^x+e^{-x})^2} = \frac{4}{(e^x+e^{-x})^2}$

16. $y = 2^x x^2 = e^{(\ln 2)x}x^2$

$y' = e^{(\ln 2)x}(2x) + x^2\left[e^{(\ln 2)x}(\ln 2)\right]$

$= 2x\left(2^x\right) + x^2\left(2^x\right)(\ln 2) = x\left(2^x\right)(2 + x\ln 2)$

18. $y' = e^{x-\sqrt{x}}\left(1 - \frac{1}{2}x^{-\frac{1}{2}}\right) = e^{x-\sqrt{x}}\left(1 - \frac{1}{2\sqrt{x}}\right)$

20. $y' = 3(e^{2x}+1)^2(e^{2x}(2)+0) = 6e^{2x}(e^{2x}+1)^2$

24. $y' = e^{2x}[1] + (x+6)\left[e^{2x}(2)\right] = e^{2x}(2x+13)$

26. $y' = e^{-x} \cdot \frac{1}{x} + (\ln x)\left(-e^{-x}\right) = e^{-x}\left(\frac{1}{x} - \ln x\right)$

28. $y = \ln e^{4x+1} = 4x+1$, so $\frac{dy}{dx} = 4$.

30. $f(x) = 5^{x^2 \ln x} = \left(e^{\ln 5}\right)^{x^2 \ln x} = e^{(\ln 5)x^2 \ln x}$

$f'(x) = e^{(\ln 5)x^2 \ln x}\left\{(\ln 5)\left[x^2 \cdot \frac{1}{x} + (\ln x)(2x)\right]\right\}$

$= e^{(\ln 5)x^2 \ln x}(\ln 5)[x + 2x\ln x]$

$f'(1) = e^0(\ln 5)[1+0] = \ln 5$

32. $y' = e^x$

When $x = 1$, $y = e$ and $y' = e$. Thus an equation of the tangent line is $y - e = e(x - 1)$ or $y = ex$.

Problem 12.4 (p. 548)
Suggested Problems: 1-30

2. $6x+12yy'=0$

$$y'=-\frac{x}{2y}$$

4. $4x-6yy'=0$

$$y'=\frac{2x}{3y}$$

10. $2x+xy'+y(1)-4yy'=0$

$$xy'-4yy'=-2x-y$$

$$y'=\frac{-2x-y}{x-4y}=\frac{2x+y}{-x+4y}$$

12. $3x^2-3y^2y'=3x^2y'+6xy-3x(2yy')-3y^2$

$$y'(-3y^2-3x^2+6xy)=6xy-3y^2-3x^2$$

$$y'=1$$

16. $x^3\left(3y^2y'\right)+y^3\left(3x^2\right)+1=0$

$$y'=-\frac{1+3x^2y^3}{3x^3y^2}$$

18. $2yy'+y'=\frac{1}{x}$

$$(2y+1)y'=\frac{1}{x}$$

$$y'=\frac{1}{x(2y+1)}$$

20. $\frac{xy'+y(1)}{xy}+1=0$

$$xy'+y+xy=0$$

$$xy'=-y(x+1)$$

$$y'=-\frac{y(x+1)}{x}$$

24. $e^{x+y}(1+y')=\frac{1}{x+y}(1+y')$

$$e^{x+y}+y'e^{x+y}=\frac{1}{x+y}+\frac{y'}{x+y}$$

$$y'\left(e^{x+y}-\frac{1}{x+y}\right)=\frac{1}{x+y}-e^{x+y}$$

$$y'=-1$$

26. $x\left(\frac{1}{2\sqrt{y+1}}\cdot y'\right)+\sqrt{y+1}(1)$

$$=y\left(\frac{1}{2\sqrt{x+1}}\right)+\sqrt{x+1}(y')$$

$$\frac{x}{2\sqrt{y+1}}\cdot y'-\sqrt{x+1}\cdot y'=\frac{y}{2\sqrt{x+1}}-\sqrt{y+1}$$

$$\left(\frac{x}{2\sqrt{y+1}}-\sqrt{x+1}\right)y'=\frac{y}{2\sqrt{x+1}}-\sqrt{y+1}$$

$$y'=\frac{\frac{y}{2\sqrt{x+1}}-\sqrt{y+1}}{\frac{x}{2\sqrt{y+1}}-\sqrt{x+1}}$$

At (3, 3), $\frac{dy}{dx}=\frac{\frac{3}{4}-2}{\frac{3}{4}-2}=1.$

28. $2(x^2+y^2)(2x+2yy')=8yy'$

$$(x^2+y^2)(x+yy')=2yy'$$

$$x^3+x^2yy'+xy^2+y^3y'=2yy'$$

$$(x^2y+y^3-2y)y'=-x^3-xy^2$$

$$y'=\frac{-x(x^2+y^2)}{y(x^2+y^2-2)}$$

At (0, 2), $y'=0$.

30. $2yy'+[xy'+y(1)]-2x=0$

$$y'=\frac{2x-y}{2y+x}$$

At (4, 3), $y'=\frac{1}{2}$ and the tangent line is given by

$y-3=\frac{1}{2}(x-4)$, or $y=\frac{1}{2}x+1$.

Problem 12.5 (p. 552)
Suggested Problems: 1-25

2. $\ln y = \ln\left[(3x+4)(8x-1)^2\left(3x^2+1\right)^4\right]$

$= \ln(3x+4) + 2\ln(8x-1) + 4\ln\left(3x^2+1\right)$

$\frac{y'}{y} = \frac{3}{3x+4} + 2\cdot\frac{8}{8x-1} + 4\cdot\frac{6x}{3x^2+1}$

$y' = y\left[\frac{3}{3x+4} + \frac{16}{8x-1} + \frac{24x}{3x^2+1}\right]$

$= (3x+4)(8x-1)^2\left(3x^2+1\right)^4\left[\frac{3}{3x+4} + \frac{16}{8x-1} + \frac{24x}{3x^2+1}\right]$

4. $y = (2x^2+1)\sqrt{8x^2-1}$

$\ln y = \ln\left[(2x^2+1)\sqrt{8x^2-1}\right]$

$= \ln(2x^2+1) + \frac{1}{2}\ln(8x^2-1)$

$\frac{y'}{y} = \frac{4x}{2x^2+1} + \frac{1}{2}\cdot\frac{16x}{8x^2-1}$

$y' = y\left[\frac{4x}{2x^2+1} + \frac{8x}{8x^2-1}\right]$

$= (2x^2+1)\sqrt{8x^2-1}\left[\frac{4x}{2x^2+1} + \frac{8x}{8x^2-1}\right]$

8. $\ln y = \ln\sqrt{\frac{x^2+5}{x+9}} = \frac{1}{2}\left[\ln\left(x^2+5\right) - \ln(x+9)\right]$

$\frac{y'}{y} = \frac{1}{2}\left[\frac{2x}{x^2+5} - \frac{1}{x+9}\right]$

$y' = \frac{y}{2}\left[\frac{2x}{x^2+5} - \frac{1}{x+9}\right]$

$y' = \frac{1}{2}\sqrt{\frac{x^2+5}{x+9}}\left[\frac{2x}{x^2+5} - \frac{1}{x+9}\right]$

10. $\ln y = \ln\frac{x\left(1+x^2\right)^2}{\sqrt{2+x^2}}$

$= \ln x + 2\ln\left(1+x^2\right) - \frac{1}{2}\ln\left(2+x^2\right)$

$\frac{y'}{y} = \frac{1}{x} + 2\cdot\frac{2x}{1+x^2} - \frac{1}{2}\cdot\frac{2x}{2+x^2}$

$y' = y\left[\frac{1}{x} + \frac{4x}{1+x^2} - \frac{x}{2+x^2}\right]$

$y' = \frac{x\left(1+x^2\right)^2}{\sqrt{2+x^2}}\left[\frac{1}{x} + \frac{4x}{1+x^2} - \frac{x}{2+x^2}\right]$

12. $\ln y = \ln\sqrt[3]{\frac{6\left(x^3+1\right)^2}{x^6e^{-4x}}}$

$= \frac{1}{3}\left[\ln(6) + 2\ln\left(x^3+1\right) - 6\ln(x) - (-4x)\ln e\right]$

$= \frac{1}{3}\left[\ln(6) + 2\ln\left(x^3+1\right) - 6\ln(x) + 4x\right]$

$\frac{y'}{y} = \frac{1}{3}\left[2\cdot\frac{3x^2}{x^3+1} - \frac{6}{x} + 4\right]$

$y' = \frac{y}{3}\left[\frac{6x^2}{x^3+1} - \frac{6}{x} + 4\right]$

$y = \frac{1}{3}\sqrt[3]{\frac{6\left(x^3+1\right)^2}{x^6e^{-4x}}}\left[\frac{6x^2}{x^3+1} - \frac{6}{x} + 4\right]$

14. $y = (2x)^{\sqrt{x}}$. Thus

$\ln y = \ln(2x)^{\sqrt{x}} = \sqrt{x}[\ln 2 + \ln x]$.

$\frac{y'}{y} = \sqrt{x}\left[\frac{1}{x}\right] + [\ln 2 + \ln x]\cdot\frac{1}{2\sqrt{x}}$

$y' = y\left[\frac{1}{\sqrt{x}} + \frac{\ln(2x)}{2\sqrt{x}}\right]$

$y' = (2x)^{\sqrt{x}}\left[\frac{2+\ln(2x)}{2\sqrt{x}}\right]$

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16. $y=\left(\frac{3}{x^2}\right)^x$. Thus

$$\ln y = x\ln\left(\frac{3}{x^2}\right) = x[\ln 3 - 2\ln x].$$

$$\frac{y'}{y} = x\left(-\frac{2}{x}\right) + (\ln 3 - 2\ln x)(1)$$

$$= -2 + \ln\left(\frac{3}{x^2}\right)$$

$$y' = y\left[-2+\ln\left(\frac{3}{x^2}\right)\right] = \left(\frac{3}{x^2}\right)^x\left[-2+\ln\left(\frac{3}{x^2}\right)\right]$$

18. $y=(x^2+1)^{x+1}$, thus

$$\ln y = \ln(x^2+1)^{x+1} = (x+1)\ln(x^2+1).$$

$$\frac{y'}{y} = x+1\cdot\frac{2x}{x^2+1} + \ln(x^2+1)\cdot 1$$

$$y' = y\left[\frac{2x(x+1)}{x^2+1} + \ln(x^2+1)\right]$$

$$= (x^2+1)^{x+1}\left[\frac{2x(x+1)}{x^2+1} + \ln(x^2+1)\right]$$

20. $y=(\ln x)^{e^x}$. Thus $\ln y = e^x \ln(\ln x)$.

$$\frac{y'}{y} = e^x\left[\frac{1}{x\ln x}\right] + [\ln(\ln x)]e^x$$

$$y' = y\left[\frac{1}{x\ln x} + \ln(\ln x)\right]e^x$$

$$= (\ln x)^{e^x}\left[\frac{1}{x\ln x} + \ln(\ln x)\right]e^x$$

22. $y=(\ln x)^{\ln x}$

$$\ln y = \ln(\ln x)^{\ln x} = (\ln x)\ln(\ln x)$$

$$\frac{y'}{y} = (\ln x)\left[\frac{1}{\ln x}\cdot\frac{1}{x}\right] + [\ln(\ln x)]\left(\frac{1}{x}\right)$$

$$y' = y\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]$$

$$y' = (\ln x)^{\ln x}\left[\frac{1+\ln(\ln x)}{x}\right]$$

When $x = e$, $\frac{dy}{dx} = 1^1\left[\frac{1+\ln(1)}{e}\right] = e^{-1}$.

24. $y = x^x$

$$\ln y = x\ln x$$

$$\frac{y'}{y} = x\cdot\frac{1}{x} + (\ln x)(1) = 1+\ln x$$

$$y' = y(1+\ln x) = x^x(1+\ln x)$$

When $x = 1$, then $y = 1$ and

$y' = 1^1(1+\ln 1) = 1(1+0) = 1$. An equation of the tangent line is $y - 1 = 1(x - 1)$ or $y = x$.

Problem 12.7 (p. 560)
Suggested Problems: 1-33

2. $y' = 5x^4 - 8x^3 + 14x$

$$y'' = 20x^3 - 24x^2 + 14$$

$$y''' = 60x^2 - 48x$$

4. $\frac{dy}{dx} = -1-2x$

$$\frac{d^2y}{dx^2} = -2$$

6. $\frac{dF}{dq} = \frac{1}{q+1}$

$$\frac{d^2F}{dq^2} = -\frac{1}{(q+1)^2}$$

$$\frac{d^3F}{dq^3} = \frac{2}{(q+1)^3}$$

8. $y=\frac{1}{x}=x^{-1}$

$y'=-x^{-2}$

$y''=2x^{-3}$

$y'''=-6x^{-4}=-\frac{6}{x^4}$

10. $f(x)=\sqrt{x}=x^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$

$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}=-\frac{1}{4x^{\frac{3}{2}}}$

12. $y=e^{-4x^2}$

$y'=-8xe^{-4x^2}$

$y''=-8\left[x\left(-8xe^{-4x^2}\right)+e^{-4x^2}(1)\right]$

$=8e^{-4x^2}\left(8x^2-1\right)$

14. $y=(3x+7)^5$

$y'=15(3x+7)^4$

$y''=180(3x+7)^3$

16. $y=2x^{\frac{1}{2}}+(2x)^{\frac{1}{2}}$

$y'=x^{-\frac{1}{2}}+\frac{1}{2}(2x)^{-\frac{1}{2}}(2)=x^{-\frac{1}{2}}+(2x)^{-\frac{1}{2}}$

$y''=-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{2}(2x)^{-\frac{3}{2}}(2)=-\left[\frac{1}{2x^{\frac{3}{2}}}+\frac{1}{(2x)^{\frac{3}{2}}}\right]$

18. $y=\ln\frac{(2x+5)(5x-2)}{x+1}$

$=\ln(2x+5)+\ln(5x-2)-\ln(x+1)$

$y'=\frac{2}{2x+5}+\frac{5}{5x-2}-\frac{1}{x+1}$

$y''=-\frac{4}{(2x+5)^2}-\frac{25}{(5x-2)^2}+\frac{1}{(x+1)^2}$

20. $y=\frac{x}{e^x}$

$\frac{dy}{dx}=\frac{e^x(1)-x\left(e^x\right)}{\left(e^x\right)^2}=\frac{1-x}{e^x}$

$\frac{d^2y}{dx^2}=\frac{e^x(-1)-(1-x)e^x}{\left(e^x\right)^2}=\frac{x-2}{e^x}$

22. $y=e^{2\ln\left(x^3+1\right)}=e^{\ln\left(x^3+1\right)^2}=\left(x^3+1\right)^2$

$y'=6x^2\left(x^3+1\right)=6x^5+6x^2$

$y''=30x^4+12x$

When $x=1$, then $y''=30+12=42$.

24. $x^2-y^2=16$

$2x-2yy'=0$

$y'=\frac{x}{y}$

$y''=\frac{y(1)-x(y')}{y^2}=\frac{y-x\left(\frac{x}{y}\right)}{y^2}$

$=\frac{y^2-x^2}{y^3}=\frac{-16}{y^3}=-\frac{16}{y^3}$

26. $9x^2+16y^2=25$

$18x+32yy'=0$

$y'=-\frac{9x}{16y}$

$y''=-\frac{9}{16}\cdot\frac{y(1)-xy'}{y^2}=-\frac{9}{16}\cdot\frac{y-x\left(-\frac{9x}{16y}\right)}{y^2}$

$=-\frac{9}{16}\cdot\frac{16y^2+9x^2}{16y^3}=-\frac{225}{256y^3}$

28. $y^2-6xy=4$

$2yy'-6[xy'+y(1)]=0$

$2yy'-6xy'=6y$

$(2y-6x)y'=6y$

$y'=\frac{6y}{2y-6x}=\frac{3y}{y-3x}$

$y''=3\cdot\frac{(y-3x)y'-y(y'-3)}{(y-3x)^2}=9\cdot\frac{y-xy'}{(y-3x)^2}$

$=9\cdot\frac{y-x\left[\frac{3y}{y-3x}\right]}{(y-3x)^2}=9\cdot\frac{y(y-3x)-3xy}{(y-3x)^3}$

$=9\cdot\frac{y^2-6xy}{(y-3x)^3}=9\cdot\frac{4}{(y-3x)^3}=\frac{36}{(y-3x)^3}$

30. $x^2+2xy+y^2=1$

$2x+2y+2xy'+2yy'=0$

$(x+y)y'=-(x+y)$

$y'=-1$

$y''=0$

32. $e^x - e^y = x^2 + y^2$

$e^x - e^y y' = 2x + 2yy'$

$$y' = \frac{e^x - 2x}{e^y + 2y}$$

$$y'' = \frac{\left(e^y + 2y\right)\left(e^x - 2\right) - \left(e^x - 2x\right)\left(e^y y' + 2y'\right)}{\left(e^y + 2y\right)^2}$$

$$= \frac{\left(e^y + 2y\right)\left(e^x - 2\right) - \left(e^x - 2x\right)\left(e^y + 2\right) y'}{\left(e^y + 2y\right)^2}$$

$$= \frac{\left(e^y + 2y\right)^2 \left(e^x - 2\right) - \left(e^x - 2x\right)^2 \left(e^y + 2\right)}{\left(e^y + 2y\right)^2}$$
