MAT 101
MATHEMATICS FOR SOCIAL SCIENCES

MATRIX ALGEBRA
Solutions of Selected Problems from Sections 6.1, 6.2, 6.3, 6.4, 6.5 and 6.6

Problems 6.1 (page 231)
Suggested Problems: 1-27

14. The main diagonal is the diagonal extending from the upper left corner to the lower right corner.

a. 1, 0, –5, 2

b. x, y, z

16. If **A** is 7×9, then $\mathbf{A}^T$ is 9×7.

18. $\mathbf{A}^T = \begin{bmatrix} 2 & 4 & 6 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$

20. $\mathbf{A}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

24. Equating corresponding entries gives $2x = 4$, $y = 6$, $z = 0$, and $3w = 7$. Thus $x = 2$, $y = 6$, $z = 0$, $w = \frac{7}{3}$.

26. Equating entries in the 3rd row and 3rd column gives 7 = 8, which is never true, so there is no solution.

Problem 6.2 (p. 237)
Suggested Problems: 1-40

2. $\begin{bmatrix} 2 & -7 \\ -6 & 4 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 2+7+2 & -7+(-4)+7 \\ -6+(-2)+7 & 4+1+2 \end{bmatrix} = \begin{bmatrix} 11 & -4 \\ -1 & 7 \end{bmatrix}$

4. $\frac{1}{2}\begin{bmatrix} 4 & -2 & 6 \\ 2 & 10 & -12 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\cdot 4 & \frac{1}{2}(-2) & \frac{1}{2}\cdot 6 \\ \frac{1}{2}\cdot 2 & \frac{1}{2}\cdot 10 & \frac{1}{2}(-12) \\ \frac{1}{2}\cdot 0 & \frac{1}{2}\cdot 0 & \frac{1}{2}\cdot 8 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 5 & -6 \\ 0 & 0 & 4 \end{bmatrix}$

6. $\begin{bmatrix} 7 & 7 \end{bmatrix}$ is a matrix and 66 is a number, so the sum is not defined.

8. $\begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix} + 3\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 7 & 4 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -6 \\ 4 & 9 \end{bmatrix} - 3\begin{bmatrix} -6 & 9 \\ 2 & 6 \\ 1 & -2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -6 \\ 4 & 9 \end{bmatrix} - \begin{bmatrix} -18 & 27 \\ 6 & 18 \\ 3 & -6 \\ 12 & 15 \end{bmatrix} = \begin{bmatrix} 19 & -28 \\ -4 & -18 \\ 0 & 0 \\ -8 & -6 \end{bmatrix}$

12.
$$3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3\left(\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 2 \\ -3 & 21 & -9 \\ 0 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} -3 & 4 & -2 \\ 3 & -23 & 10 \\ 0 & -1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} -9 & 12 & -6 \\ 9 & -69 & 30 \\ 0 & -3 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & -12 & 6 \\ -9 & 72 & -30 \\ 0 & 3 & 0 \end{bmatrix}$$

16. $\mathbf{A} - \mathbf{B} + \mathbf{C} = \begin{bmatrix} 2-(-6)+(-2) & 1-(-5)+(-1) \\ 3-2+(-3) & -3-(-3)+3 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -2 & 3 \end{bmatrix}$

18. $0(\mathbf{A} + \mathbf{B}) = 0\begin{bmatrix} -4 & -4 \\ 5 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$

22. $3\mathbf{C} - 2\mathbf{B} = \begin{bmatrix} -6 & -3 \\ -9 & 9 \end{bmatrix} - \begin{bmatrix} -12 & -10 \\ 4 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ -13 & 15 \end{bmatrix}$

24. $\frac{1}{2}\mathbf{A} - 5(\mathbf{B} + \mathbf{C}) = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix} - 5\begin{bmatrix} -8 & -6 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix} + \begin{bmatrix} 40 & 30 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 41 & \frac{61}{2} \\ \frac{13}{2} & -\frac{3}{2} \end{bmatrix}$

30. $(\mathbf{B} - \mathbf{C})^{\mathrm{T}} = \left\{\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\right\}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 \\ 3 & -3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 3 \\ 3 & -3 \end{bmatrix}$

32. $2\mathbf{B} + \mathbf{B}^{\mathrm{T}} = 2\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 8 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 11 & -3 \end{bmatrix}$

34. $\left(\mathbf{D}-2\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \left\{\begin{bmatrix}1 & 2 & -1\\1 & 0 & 2\end{bmatrix} - 2\begin{bmatrix}1 & 0 & 7\\2 & -1 & 0\end{bmatrix}\right\}^{\mathrm{T}}$

$= \left\{\begin{bmatrix}1 & 2 & -1\\1 & 0 & 2\end{bmatrix} - \begin{bmatrix}2 & 0 & 14\\4 & -2 & 0\end{bmatrix}\right\}^{\mathrm{T}} = \begin{bmatrix}-1 & 2 & -15\\-3 & 2 & 2\end{bmatrix}^{\mathrm{T}}$

$= \begin{bmatrix}-1 & -3\\2 & 2\\-15 & 2\end{bmatrix}$

36. $\begin{bmatrix}2x-4y\\5x+7y\end{bmatrix} = \begin{bmatrix}16\\-3\end{bmatrix}$

$\begin{bmatrix}2x\\5x\end{bmatrix} + \begin{bmatrix}-4y\\7y\end{bmatrix} = \begin{bmatrix}16\\-3\end{bmatrix}$

$x\begin{bmatrix}2\\5\end{bmatrix} + y\begin{bmatrix}-4\\7\end{bmatrix} = \begin{bmatrix}16\\-3\end{bmatrix}$

38. $3\begin{bmatrix}x\\2\end{bmatrix} - 4\begin{bmatrix}7\\-y\end{bmatrix} = \begin{bmatrix}-x\\2y\end{bmatrix}$

$\begin{bmatrix}3x-28\\6+4y\end{bmatrix} = \begin{bmatrix}-x\\2y\end{bmatrix}$

$3x - 28 = -x$, $4x = 28$, or $x = 7$.
$6 + 4y = 2y$, $2y = -6$, or $y = -3$.
Thus $x = 7$, $y = -3$.

40. $x\begin{bmatrix}2\\0\\2\end{bmatrix} + 2\begin{bmatrix}-1\\0\\6\end{bmatrix} + y\begin{bmatrix}0\\2\\-5\end{bmatrix} = \begin{bmatrix}10\\6\\2x+12-5y\end{bmatrix}$

$\begin{bmatrix}2x-2\\2y\\2x+12-5y\end{bmatrix} = \begin{bmatrix}10\\6\\2x+12-5y\end{bmatrix}$

$2x - 2 = 10$, $2x = 12$, or $x = 6$.
$2y = 6$ or $y = 3$.
$2x + 12 - 5y = 2x + 12 - 5y$, which is true for all values of x and y. Thus $x = 6$, $y = 3$.

Problem 6.3 (p. 248)
Suggested Problems: 1-62

20. $\begin{bmatrix} -1 & 1 \\ 0 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1(1)+1(3) & -1(-2)+1(4) \\ 0(1)+4(3) & 0(-2)+4(4) \\ 2(1)+1(3) & 2(-2)+1(4) \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 12 & 16 \\ 5 & 0 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 6 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = [1(0)+0(1)+6(2)+2(3)] = [18]$

24. $\begin{bmatrix} 4 & 2 & -2 \\ 3 & 10 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4(3)+2(0)+(-2)(0) & 4(1)+2(0)+(-2)(1) & 4(1)+2(0)+(-2)(0) & 4(0)+2(0)+(-2)(1) \\ 3(3)+10(0)+0(0) & 3(1)+10(0)+0(1) & 3(1)+10(0)+0(0) & 3(0)+10(0)+0(1) \\ 1(3)+0(0)+2(0) & 1(1)+0(0)+2(1) & 1(1)+0(0)+2(0) & 1(0)+0(0)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 2 & 4 & -2 \\ 9 & 3 & 3 & 0 \\ 3 & 3 & 1 & 2 \end{bmatrix}$$

26. The first matrix is 1×2 and the second is 3×2, so the product is not defined.

28. $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0(1)+1(1) & 0(1)+1(1) & 0(1)+1(1) \\ 2(1)+3(1) & 2(1)+3(1) & 2(1)+3(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 5 & 5 \end{bmatrix}$$

34. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}$

36. $\begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 \\ x_2 \\ 2x_1 + x_2 \end{bmatrix}$

42. $\mathbf{FE(D-I)} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

44. $$\mathbf{A}(\mathbf{BC}) = \mathbf{A}\left\{\begin{bmatrix} -2 & 3 & 0 \\ 1 & -4 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 3 \\ 2 & 4 \end{bmatrix}\right\} = \mathbf{A}\begin{bmatrix} 2+0+0 & -2+9+0 \\ -1+0+2 & 1-12+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 1 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 7+14 \\ 0+3 & 0-21 \end{bmatrix} = \begin{bmatrix} 0 & 21 \\ 3 & -21 \end{bmatrix}$$

46. $$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

48. $$\mathbf{A}(\mathbf{B}^{\mathrm{T}})^2\mathbf{C} = \mathbf{A}\begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}\mathbf{C}$$

$$= \mathbf{A}\begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 4 \end{bmatrix}\mathbf{C}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & 4 \end{bmatrix}\mathbf{C}$$

$$= \begin{bmatrix} 0 & -3 & 0 \\ -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 3 \\ -4 & 5 \end{bmatrix}$$

50. $$\mathbf{A}^{\mathrm{T}}\left(2\mathbf{C}^{\mathrm{T}}\right) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 & 0 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 0 \\ -2 & -6 & 2 \\ 0 & -2 & 2 \end{bmatrix}$$

52. $$(2\mathbf{B})^{\mathrm{T}} = \left\{2\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right\}^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & -2 \\ 4 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & -2 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

58. $\mathbf{B}^2 - 3\mathbf{B} + 2\mathbf{I}$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -2 \\ -2 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -3 \\ 6 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ -8 & 4 & -2 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -8 & 6 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

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Problem 6.4 (p. 257)
Suggested Problems: 1-26

2. Reduced.

4. In row 2, the first nonzero entry is in column 2, but not all other entries in column 2 are zeros, hence not reduced.

6. The first nonzero entry of row 2 is to the left of the first nonzero entry of row 1, hence not reduced.

8.
$$\begin{bmatrix} 0 & -3 & 0 & 2 \\ 1 & 5 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & -3 & 0 & 2 \end{bmatrix}$$
$$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 5 & 0 & 2 \\ 0 & 1 & 0 & -\frac{2}{3} \end{bmatrix}$$
$$\xrightarrow{-5R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{16}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \end{bmatrix}$$

10.
$$\begin{bmatrix} 2 & 3 \\ 1 & -6 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -6 \\ 2 & 3 \\ 4 & 8 \\ 1 & 7 \end{bmatrix} \xrightarrow[\substack{-4R_1 + R_3 \\ -R_1 + R_4}]{-2R_1 + R_2} \begin{bmatrix} 1 & -6 \\ 0 & 15 \\ 0 & 32 \\ 0 & 13 \end{bmatrix} \xrightarrow{\frac{1}{15}R_2} \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 0 & 32 \\ 0 & 13 \end{bmatrix} \xrightarrow[\substack{-32R_2 + R_3 \\ -13R_2 + R_4}]{6R_2 + R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

12.
$$\begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix}$$
$$\xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{-4R_2 + R_4} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_3 + R_4]{-\frac{3}{2}R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

14.
$$\left[\begin{array}{cc|c} 1 & -3 & -11 \\ 4 & 3 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -11 \\ 0 & 15 & 53 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -11 \\ 0 & 1 & \frac{53}{15} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{53}{15} \end{array}\right]$$

Thus $x = -\dfrac{2}{5}$, $y = \dfrac{53}{15}$.

16.
$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row indicates that 0 = 1, which is never true. There is no solution.

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18. $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 1 & 1 & 5 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -2 & 3 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{9}{2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{13}{2} & \frac{29}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{9}{2} \end{array}\right]$

Thus $x = -\frac{13}{2}r + \frac{29}{2}$, $y = \frac{3}{2}r - \frac{9}{2}$, $z = r$, where r is any real number.

20. $\left[\begin{array}{cc|c} 1 & 4 & 9 \\ 3 & -1 & 6 \\ 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & -13 & -21 \\ 0 & -5 & -7 \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 4 & 9 \\ 0 & 1 & \frac{21}{13} \\ 0 & -5 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{33}{13} \\ 0 & 1 & \frac{21}{13} \\ 0 & 0 & \frac{14}{13} \end{array}\right]$

$\to \left[\begin{array}{cc|c} 1 & 0 & \frac{33}{13} \\ 0 & 1 & \frac{21}{13} \\ 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

The last row indicates that 0 = 1, which is never true. There is no solution.

22. $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 2 & -3 & -2 & 4 \\ 1 & -1 & -5 & 23 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & -5 & 0 & -10 \\ 0 & -2 & -4 & 16 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & -2 & -4 & 16 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & 20 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right]$

Thus $x = 0$, $y = 2$, $z = -5$.

24. $\left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 3 & 2 & 11 & 1 \\ 1 & 1 & 4 & 1 \\ 2 & -3 & 3 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & -3 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -3 & -3 & -6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Thus $x = -3r - 1$, $y = -r + 2$, $z = r$, where r is any real number.

26. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & -2 & -2 & 2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

Thus $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$.

Problem 6.5 (p. 263)
Suggested Problems: 1-24

2. $$\left[\begin{array}{cccc|c} 2 & 1 & 10 & 15 & -5 \\ 1 & -5 & 2 & 15 & -10 \\ 1 & 1 & 6 & 12 & 9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 2 & 1 & 10 & 15 & -5 \\ 1 & 1 & 6 & 12 & 9 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 0 & 11 & 6 & -15 & 15 \\ 0 & 6 & 4 & -3 & 19 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -5 & 2 & 15 & -10 \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 6 & 4 & -3 & 19 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & \frac{52}{11} & \frac{90}{11} & -\frac{35}{11} \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 0 & \frac{8}{11} & \frac{57}{11} & \frac{119}{11} \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & \frac{52}{11} & \frac{90}{11} & -\frac{35}{11} \\ 0 & 1 & \frac{6}{11} & -\frac{15}{11} & \frac{15}{11} \\ 0 & 0 & 1 & \frac{57}{8} & \frac{119}{8} \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{51}{2} & -\frac{147}{2} \\ 0 & 1 & 0 & -\frac{21}{4} & -\frac{27}{4} \\ 0 & 0 & 1 & \frac{57}{8} & \frac{119}{8} \end{array}\right]$$

Thus, $w = \frac{51}{2}r - \frac{147}{2}$, $x = \frac{21}{4}r - \frac{27}{4}$, $y = -\frac{57}{8}r + \frac{119}{8}$, $z = r$ (where r is any real number).

4. $$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & -3 & 4 & -7 & 1 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & -4 & 4 & -12 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 0 & 5 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -4 & 4 & -12 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $w = -r - 2s + 1$, $x = r - 3s$, $y = r$, $z = s$ (where r and s are any real numbers).

6. $$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 2 & 1 & 2 & 2 & 7 \\ 1 & 2 & 1 & 4 & 5 \\ 3 & -2 & 3 & -4 & 7 \\ 4 & -3 & 4 & -6 & 9 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 0 & -1 & 0 & -2 & -1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -5 & 0 & -10 & -5 \\ 0 & -7 & 0 & -14 & -7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & -5 & 0 & -10 & -5 \\ 0 & -7 & 0 & -14 & -7 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $w = -r + 3$, $x = -2s + 1$, $y = r$, $z = s$ (where r and s are any real numbers).

8. $$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 2 & -2 & 3 & 10 & 15 & 10 \\ 1 & 2 & 3 & -2 & 2 & -2 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & -2 & -3 & 8 & 7 & 8 \\ 0 & 2 & 0 & -3 & -2 & -3 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 4 & 7 & 8 \\ 0 & 0 & -2 & 1 & -2 & -3 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 1 & 4 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -4 & -7 & -8 \\ 0 & 0 & 0 & -7 & -16 & -19 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & \frac{12}{7} & -\frac{12}{7} \\ 0 & 1 & 1 & 0 & \frac{32}{7} & \frac{38}{7} \\ 0 & 0 & 1 & 0 & \frac{15}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 1 & \frac{16}{7} & \frac{19}{7} \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -\frac{33}{7} & -\frac{72}{7} \\ 0 & 1 & 0 & 0 & \frac{17}{7} & \frac{18}{7} \\ 0 & 0 & 1 & 0 & \frac{15}{7} & \frac{20}{7} \\ 0 & 0 & 0 & 1 & \frac{16}{7} & \frac{19}{7} \end{array}\right]$$

Thus $x_1 = -\frac{72}{7} + \frac{33}{7}r$, $x_2 = \frac{18}{7} - \frac{17}{7}r$, $x_3 = \frac{20}{7} - \frac{15}{7}r$, $x_4 = \frac{19}{7} - \frac{16}{7}r$, and $x_5 = r$, where r is any real number.

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Problem 6.6 (p. 269)
Suggested Problems: 1-34

2. $\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 1 & 2 & 0 & \frac{1}{3} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{3} \end{array}\right]$

The given matrix is not invertible.

4. $\left[\begin{array}{cc|cc} \frac{1}{4} & \frac{3}{8} & 1 & 0 \\ 0 & -\frac{1}{6} & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{3}{2} & 4 & 0 \\ 0 & 1 & 0 & -6 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 4 & 9 \\ 0 & 1 & 0 & -6 \end{array}\right]$

The inverse is $\begin{bmatrix} 4 & 9 \\ 0 & -6 \end{bmatrix}$.

6. $\left[\begin{array}{ccc|ccc} 2 & 0 & 8 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 4 & 4 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -8 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 1 & -8 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & -9 & -\frac{9}{8} & -\frac{1}{4} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & 1 & 0 & 0 & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right]$

The inverse is $\begin{bmatrix} 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{bmatrix}$.

12. $\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -3 & 3 & -1 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 15 & -1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -9 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \\ 0 & 1 & 0 & \frac{4}{15} & \frac{1}{5} & -\frac{4}{15} \\ 0 & 0 & 1 & -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{array}\right]$

The inverse is $\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & \frac{3}{5} \\ \frac{4}{15} & \frac{1}{5} & -\frac{4}{15} \\ -\frac{1}{15} & \frac{1}{5} & \frac{1}{15} \end{bmatrix}$.

14. $\left[\begin{array}{ccc|ccc} 2 & 3 & -1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -3 & 0 & 0 & -1 \\ 2 & 3 & -1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -4 & 0 & -1 & -1 \\ 0 & -1 & -3 & 1 & -2 & 0 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & -1 & -3 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & -4 & 5 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -8 & 5 \\ 0 & 1 & 0 & -4 & 5 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$

The inverse is $\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}$.

20. $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 16 \end{bmatrix} \Rightarrow x_1 = 9,\ x_2 = 6,\ x_3 = 16$

22. $\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 5 & \frac{1}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{10} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{10} & \frac{1}{5} \end{array}\right]$

$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} \frac{3}{10} & -\frac{2}{5} \\ \frac{1}{10} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{23}{10} \\ \frac{1}{10} \end{bmatrix} \Rightarrow x = \frac{23}{10},\ y = \frac{1}{10}$

24. $\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 4 & 3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{4}{3} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & \frac{1}{3} & -\frac{4}{3} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -4 & 3 \end{array}\right]$

$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 26 \\ 37 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \Rightarrow x = 4,\ y = 7$

26. The coefficient matrix is not invertible. The method of reduction yields

$\left[\begin{array}{cc|c} 2 & 6 & 8 \\ 3 & 9 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 3 & 9 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 4 \\ 0 & 0 & -5 \end{array}\right]$.

Second row indicates $0 = -5$, which is never true, so there is no solution.

30. $\left[\begin{array}{ccc|ccc} 2 & 0 & 8 & 1 & 0 & 0 \\ -1 & 4 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 0 & -1 & 0 \\ 2 & 0 & 8 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 0 & -1 & 0 \\ 0 & 8 & 8 & 1 & 2 & 0 \\ 0 & 9 & 0 & 0 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -4 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 9 & 0 & 0 & 2 & 1 \end{array}\right]$

$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & -9 & -\frac{9}{8} & -\frac{1}{4} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{8} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & 1 & 0 & 0 & \frac{2}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{array}\right]$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 0 & -\frac{1}{9} & \frac{4}{9} \\ 0 & \frac{2}{9} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{36} & -\frac{1}{9} \end{bmatrix}\begin{bmatrix} 8 \\ 36 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}$

Thus, $x = 0$, $y = 9$, $z = 1$.

32. The coefficient matrix is not invertible. The method of reduction yields

$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & -5 & -5 & -10 \\ 0 & -2 & -2 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 3 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -2 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Thus, $x = 1$, $y = -r + 2$, $z = r$.