LINEAR VECTOR SPACES

GENERAL DEFINITIONS AND CONSIDERATIONS

We know that a vector **u** in E_3 (3D Euclidean space) can be expressed as an ordered triplet of numbers : **u=(**u_1 , u_2 , u_3) . E_3 is a 3D vector space (R^3) in which the inner (scalar) product is defined. We now generalize the real vector space to n dimensions and designate it by R^n . An element **u** of R^n has the form **u** = (u_1 , u_2 , …, u_n) with the following properties:

i) If **u** and **v** are any two vectors in R^n , then **u**+**v** would be in R^n ; in other words, R^n is closed under vector addition. Further, the vector addition obeys the rules :

1) **u+v** = **v+u** for all **u** and **v** in R^n
2) **u+(v+w)**=**(u+v)+w** for all **u**,**v** and **w** in R^n
3) there exists a a unique zero vector "**0**" in R^n so that **u**+**0**=**0**+**u**=**u** for all **u** in R^n
4) for each **u** in R^n there exists a unique vector **"-u"** so that **u**+(-**u**)=(-**u**)+**u**=**0**

ii) If **u** is any vector in R^n and c is any real number, then "c**u**" would be in R^n ; in other words, R^n is closed under scalar multiplication. Further, scalar multiplication obeys the rules :

1) c**u**=**u**c for any **u** in R^n and any real number "c"
2) c(**u**+**v**)=c**u**+c**v** for any **u** and **v** in R^n , and any real number "c"
3) (c+d)**u**=c**u**+d**u** for any **u** in R^n , and any real numbers "c" and "d"
4) (cd)**u**=c(d**u**) for any **u** in R^n, and any real numbers "c" and "d"
5) (1)**u**=**u** and (0)**u**=**0** for any **u** in R^n

A complex vector space is a collection of elements satisfying the conditions in (i) and (ii) stated above; but, in this case, the multiplication by a complex number is allowed in (ii).

Even though the properties in (i) and (ii) are given to define R^n (n-dimensional real vector space), they also define an abstract vector space (V); in other words, any space with the elements satisfying the conditions in(i) and (ii) is called a vector space.

Example : A space with the elements of the form (a,a^2) is not a vector space because it is not closed under addition. In fact consider two elements , say **A** and **B**, of that space :

$$\mathbf{A} = (a, a^2) \ ; \ \mathbf{B} = (b, b^2)$$

whose addition gives : **A** + **B** = $(a+b, a^2 + b^2)$ which is not an element of the space under consideration.

Example :Consider the space P_2 of second order polynomials: $P_2(x)=c_0 + c_1*x + c_2*x^2$ (where c_i are some constants). This space is a vector space since it satisfies all of the conditions in (i) and (ii) [you verify this as home exercise].

Example : The space of (2x2) matrices with the elements of the form : $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is a vector space because the elements of that space satisfy all of the conditions in (i) and (ii).

Subspace of a Vector Space : Let V be a vector space and W be a subset of V. Then, W would be a subspace of V if only if W is closed under vector addition and scalar multiplication.

We note that not all subsets of a vector space are subspaces. For example, the subset of R^3 containing all the vectors of the form $(u_1 , u_2 , 1)$ is not a subspace of R^3 since it is not closed under vector addition, as well as, under multiplication by scalar. On the other hand, the subset with the elements of the form $(u_1 , u_2 , 0)$ would be a subpspace of R^3 [verify this as home exercise].

Linear Combination of Vectors : Let S={$\mathbf{u_1} , \mathbf{u_2} , \ldots, \mathbf{u_n}$ } be a set of vectors in a vector space V. The vector

$$\mathbf{v}=c_1 \mathbf{u_1} + c_2 \mathbf{u_2} + \ldots c_n \mathbf{u_n} \text{ with } c_i \text{ 's being real numbers}$$

is called a linear combinations of the vectors in S.

Vectors Spanning a Vector Space : Let S={$\mathbf{u_1} , \mathbf{u_2} , \ldots, \mathbf{u_n}$ } be a set of vectors in a vector space V. The set S spans V if every vector in V can be expressed as a linear combination of the vectors in S. We say that V is spanned by S, and S is the spanning set of V whenever S spans V.

Example : Show that the set {(1,0,0) , (1,1,0) , (1,1,1)} spans R^3 .
Solution : We take an arbitrary vector in R^3 , say $\mathbf{u}=(u_1 , u_2 , u_3)$ and we show that it can be expressed as a linear combination of the vectors in S. To this end, we write

$$c_1 (1,0,0)+c_2 (1,1,0)+c_3 (1,1,1) = (u_1 , u_2 , u_3)$$

which leads to a system of linear equations : $c_1 + c_2 + c_3 = u_1$

$$c_1 + c_2 = u_2$$

$$c_3 = u_3$$

whose solution gives : $c_1 = u_1 - u_2$; $c_2 = u_2 - u_3$; $c_3 = u_3$. This completes the verification. For example, we can use the above solution for :

$$(-2,5,7)= -7(1,0,0) -2(1,1,0)+7(1,1,1)$$

Example : Show that the set {(1,0,0) , (1,1,1)} does not span R^3 .
Solution : We take an arbitrary vector in R^3 , say $\mathbf{u}=(u_1 , u_2 , u_3)$, and we show that it can not be expressed as a linear combination of the vectors in S. For that, we write

$$c_1 (1,0,0)+c_2 (1,1,1) = (u_1 , u_2 , u_3)$$

which leads to a system of linear equations : $c_1 + c_2 = u_1$

$$c_2 = u_2$$

$$c_2 = u_3$$

which has no solution since (u_1 , u_2 , u_3) is a triplet of arbitrary numbers. This completes the verification.

Linear Independence of Vectors : Let S={$\mathbf{u_1} , \mathbf{u_2} , \ldots , \mathbf{u_n}$ } be a set of vectors in a vector space V. The vectors in the set S are said to be linearly independent if the equation

$$c_1 \mathbf{u_1} + c_2 \mathbf{u_2} + \ldots c_n \mathbf{u_n} = \mathbf{0}$$

is satisfied if only if $c_i = 0$ (i= 1-n), where c_i's are real numbers. On the other hand, if the above equation is satisfied by some nontrivial (nonzero) values of c_i's , then the vectors in the set S would be linearly dependent. We note that for linearly independent case none of the vectors in S can be expressed as a linear combination of the others. An immediate consequence of the above definition is that : any set of vectors containing zero vector is linearly dependent.

Basis of a Vector Space : Let S be a set of vectors in V. Then, we say that the vectors in S form a basis for V if the vectors in S **i)** are linearly independent **ii)** span V.

Example : The set S={(1,0,0) , (1,1,0), (1,1,1)} spans R^3 and linearly independent; hence it can be used as a basis for R^3.

Example : For a space M of 2x2 matrices (which is a vector space), the basis set may be chosen as

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which are independent and span V (you may show these as home exercise).

Example : For a space P_2 of second order polynomials (which is a vector space) the basis set may be chosen as

$$S = \{1 , x , x^2 \}$$

which are independent and span P_2 (you may show these as home exercise)

The selection of the basis set for given vector space is not unique; but, the number of the base vectors contained in the set is invariant. For example, for R^3 the basis set may be chosen as S={(1,0,0) , (1,1,0), (1,1,1)} or S={(1,0,0) , (0,1,0), (0,0,1)}, etc., where we note that each basis set contains 3 base vectors.

Dimension of a Vector Space : The number of base vectors in the basis set of a given vector space V is called dimension of V and designated by "dim(V)". For example, dim(R^n)=n since R^n has "n" base vectors which may be chosen as

$$\mathbf{e_1} = (1,0,\ldots,0) \; ; \; \mathbf{e_2} = (0,1,0,..,0) \; ; \; \ldots. \; ; \; \mathbf{e_n} = (0,\ldots,0,1)$$

The space M_2 of 2x2 matrices has the dimension of 4 since it has four base vectors. The dimension of P_2 (which is the space of second order polynomials) is 3 because it has three base vectors.

INNER PRODUCT VECTOR SPACES

In this space, the inner product ($\mathbf{u}$,$\mathbf{v}$) of two vectors in R^n is defined with the following properties

1) $(\mathbf{u},\mathbf{v})=(\mathbf{v},\mathbf{u})$ (Symmetry).
2) $(c\mathbf{u},\mathbf{v})=c(\mathbf{u},\mathbf{v})$, where c is a constant.
3) $(c_1 \mathbf{u_1} + c_2 \mathbf{u_2} , \mathbf{v})=c_1 (\mathbf{u_1} ,\mathbf{v}) +c_2 (\mathbf{u_2} ,\mathbf{v})$, where c_1 and c_2 are constants. (Linearity)
4) $(\mathbf{u},\mathbf{u}) \geq 0$, where equality holds if only if $\mathbf{u}=\mathbf{0}$. (Positive-definiteness)

Orthogonality Condition : Two vectors **u** and **v** are said to be orthogonal if (**u,v**)=0. A set of vectors in which each pair is orthogonal is called orthogonal set. An orthogonal set of nonzero vectors is always linearly independent.

Definition of Inner Product in E_n: The inner product of the vectors **u**=(u_1 , u_2 ,.., u_n) and **v**=(v_1 , v_2 , .. , v_n) in n-dimensional Euclidean space (E_n) is defined by

$$(\mathbf{u},\mathbf{v}) = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

which implies

$$(\mathbf{u},\mathbf{u})=u_1^2 + u_2^2 + \dots + u_n^2 = |\mathbf{u}|^2$$

or

$$|\mathbf{u}| = (\mathbf{u},\mathbf{u})^{1/2} = \sqrt{u_1^2 + u_1^2 + ... + u_n^2}$$

which is the length of the vector **u** can be used as a **norm** in E_n.

Norm of a Vector : The norm $\|\mathbf{u}\|$ of a vector **u** is a nonzero scalar with the properties

1) $\|\mathbf{u}\| \geq 0$ where the equality holds if only if **u=0 ,**
2) $\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\|$ **,** where c is a constant,
3) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ **.**

We can define various types of norms. Among them, the most popular one is p-norm which is defined by

$$\|\mathbf{u}\|_p = (|u_1|^p + |u_2|^p + \dots + |u_n|^p)^{1/p} = \left(\sum_{i=1}^{n} |u_i|^p \right)^{1/p}$$

where “p” is a fixed number. In practice, we usually take p=1 or 2, or we use (as third norm) $\|\mathbf{u}\|_\infty$ (called maximum norm, defined below), that is,

$$\|\mathbf{u}\|_1 = (|u_1| + |u_2| + \dots + |u_n|) = \sum_{i=1}^{n} |u_i| \qquad (l_1\text{-norm}),$$

$$\|\mathbf{u}\|_2 = (|u_1|^2 + |u_2|^2 + \dots + |u_n|^2)^{1/2} = \left(\sum_{i=1}^{n} |u_i|^2 \right)^{1/2} \qquad (\text{Euclidean or } l_2\text{-norm}),$$

$$\|\mathbf{u}\|_\infty = \max_j |u_j| \qquad (\text{maximum or } l_\infty \text{ - norm}).$$

Example : Compute l_1 - , l_2 - and l_∞ - norms of the vector **u**=(2,-3,0,1,-4).

Answer : $\|\mathbf{u}\|_1$ = 2+3+0+1+4 = 10 ; $\|\mathbf{u}\|_2$ =(4+9+0+1+16)$^{1/2}$ = (30)$^{1/2}$; $\|\mathbf{u}\|_\infty$ = 4

INFINITE DIMENSIONAL EUCLIDEAN (FUNCTION) SPACE

Consider a function f(x) given in discrete form .

x	x_1	x_2	….	x_n
f	f_1	f_2	….	f_n

This discrete valued function defines an n-dimensional Euclidean space (E_n) with the elements $f=(f_1 , f_2 , \ldots, f_n)$ at the data points x (i=1-n), where the inner product of two discrete functions f and g are defined by

$$(f,g)=f_1 g_1 + f_2 g_2 + \ldots + f_n g_n = \sum_{i=1}^{n} f_i g_i \ .$$

Now suppose we squeeze the data points x_i ; thus, n goes to infinity and "f "becomes continuously defined for all x values. Then, "f" may be considered as defined in infinite dimensional Euclidean space (E_∞) , where the inner product of two continuous functions f(x) and g(x) is given by

$$(f,g) = (f,g) = \int_a^b f\ g\, dx$$

Here, we assume that f and and g are defined in the interval $a \le x \le b$. We note that the summation symbol appearing in the definition of inner product in E_n is replaced by integral in infinite dimensional Euclidean space (E_∞).

The inner product in E_∞ obeys the usual rules :

1)(f,g)=(g,f) (Symmetry)
2) (cf,g)=c(f,g) , where c is a constant
3) $(c_1 f_1+ c_2 f_2 , g)=c_1 (f_1 ,g) +c_2 (f_2 ,g)$, where c_1 and c_2 are constants. (Linearity)
4) $(f,f) = \int_a^b f^2\, dx \ge 0$, where equality holds if only if f=0. (Positive-definiteness)

The functions f and g would be orthogonal if

$$(f,g) = \int_a^b f\ g\, dx = 0$$

Example : Consider the space P_2 of second order polynomials of the form

$$P_2 (x) = c_0 + c_1 x + c_2 x^2$$

Choose the basis set as : $S=\{1, x , x^2\}$. We can express all the second order polynomials as a linear combination of these base functions. The base functions $S=\{1, x , x^2\}$ are defined in E_∞ ; but, the space spanned by $S=\{1, x , x^2\}$, P_2 , is a subspace of E_∞ . Note that the subspace P_2 containing the second order polynomials has the dimension of 3 since the basis set S contains three base functions.

Linear Independence of Functions : Let S={ v_1 , v_2 **, … ,** v_n } be a set of functions in E_∞ defined in the interval $a \le x \le b$. The functions in the set S are said to be linearly independent if the equation

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$$

is satisfied if only if $c_i = 0$ (i= 1-n), where c_i's are real numbers. On the other hand, if the above equation is satisfied by some nontrivial (nonzero) values of c_i's , then the functions in the set S would be linearly dependent. We note that for linearly independent case none of the functions in S can be expressed as a linear combination of the others. An immediate consequence of the above definition is that : any set of functions containing zero function is linearly dependent.

A Test for Linear Independence of Functions : The set of functions S={v_1 , v_2 , .., v_n } defined in the interval $a \le x \le b$ would be linearly independent if the Wronskian determinant W defined by

$$W = \begin{bmatrix} v_1 & v_2 & . \; . & v_n \\ v_1' & v_2' & . \; . & v_n' \\ . & . & . \; . & . \\ v_1^{(n-1)} & v_2^{(n-1)} & . \; . & v_n^{(n-1)} \end{bmatrix} \quad \text{where } (\,)' = d/dx \; ; \; (\,)^{\mathbf{k}} = d^k/dx^k$$

is not identically equal to zero at all points of the interval $a \le x \le b$.

Example : Consider the base functions of a second order polynomial : {1 ,x , x^2 } .We wish to determine whether these base functions are independent or not. For that, we form the Wronskian determinant :

$$W = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \ne 0 \text{ for all x values.}$$

This implies that the functions {1 , x , x^2 } are linearly independent.