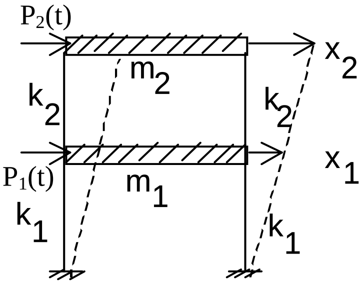
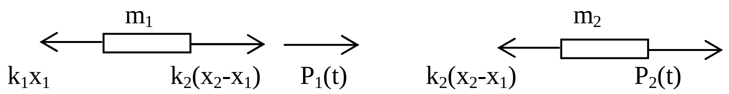
Forced Vibrations of Structural Systems:



Consider a 2 storey building under the effect of time dependent loads $P_1(t)$ and $P_2(t)$.

$$\vec{F} = m\vec{a}$$



$$m_1\ddot{x}_1 = P_1(t) - k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = P_2(t) - k_2(x_2 - x_1)$$

$$\underset{\underline{M}}{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}\underset{\underline{\ddot{X}}}{\begin{bmatrix} \ddot{x}1 \\ \ddot{x}2 \end{bmatrix}} + \underset{\underline{K}}{\begin{bmatrix} k1 + k2 & -k2 \\ -k2 & k2 \end{bmatrix}}\underset{\underline{X}}{\begin{bmatrix} x1 \\ x2 \end{bmatrix}} = \underset{\underline{P}}{\begin{bmatrix} P1 \\ P2 \end{bmatrix}}$$

$\underline{M}\,\underline{\ddot{X}} + \underline{K}\,\underline{X} = \underline{P}$, if $\underline{P} = \underline{0}$ (free vibration), $\underline{K}\underline{a} = \lambda\underline{M}\underline{a}$

$$k_1 = \frac{24E_1I_1}{h_1^3} \quad k_2 = \frac{24E_2I_2}{h_2^3}$$

$$(\underline{K} - \lambda\underline{M})\underline{a} = \underline{0}$$

Let $\underline{x} = \begin{bmatrix} x1 \\ x2 \end{bmatrix} = y_1(t)\underline{a}^{(1)} + y_2(t)\underline{a}^{(2)} + ... + y_n(t)\underline{a}^{(n)}$

$$\underline{x} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \underline{a}^{(1)} & \underline{a}^{(2)} & \underline{a}^{(3)} \\ \downarrow & \downarrow & \downarrow \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_n \end{bmatrix} = \underline{Q}\underline{y} \quad \text{and} \quad \begin{aligned} \underline{X} &= \underline{Q}\underline{Y} \\ \underline{\dot{X}} &= \underline{Q}\underline{\dot{Y}} \\ \underline{\ddot{X}} &= \underline{Q}\underline{\ddot{Y}} \end{aligned}$$

$\underline{M}\,\underline{\ddot{X}} + \underline{K}\,\underline{X} = \underline{P}$ or $\underline{M}\,\underline{Q}\underline{\ddot{Y}} + \underline{K}\,\underline{Q}\underline{Y} = \underline{P}$ and multiply both side by $\underline{Q}^T$, then,

$$\underline{Q}^T\underline{M}\,\underline{Q}\underline{\ddot{Y}} + \underline{Q}^T\underline{K}\,\underline{Q}\underline{Y} = \underline{Q}^T\underline{P}$$

$$\begin{bmatrix} \leftarrow & \underline{a}^{(1)T} & \rightarrow \\ \leftarrow & \underline{a}^{(2)T} & \rightarrow \\ \leftarrow & \underline{a}^{(n)T} & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \underline{M}\,\underline{a}^{(1)} & \underline{M}\,\underline{a}^{(2)} & \underline{M}\,\underline{a}^{(n)} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_n \end{bmatrix} + \begin{bmatrix} \leftarrow & \underline{a}^{(1)T} & \rightarrow \\ \leftarrow & \underline{a}^{(2)T} & \rightarrow \\ \leftarrow & \underline{a}^{(n)T} & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \underline{K}\,\underline{a}^{(1)} & \underline{K}\,\underline{a}^{(2)} & \underline{K}\,\underline{a}^{(n)} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} \leftarrow & \underline{a}^{(1)T} & \rightarrow \\ \leftarrow & \underline{a}^{(2)T} & \rightarrow \\ \leftarrow & \underline{a}^{(n)T} & \rightarrow \end{bmatrix} \begin{bmatrix} P_1 \\ . \\ P_n \end{bmatrix}$$

$$\begin{bmatrix} \underline{a}^{(1)T}\underline{M}\,\underline{a}^{(1)} & \underline{a}^{(1)T}\underline{M}\,\underline{a}^{(2)} & \cdots & \underline{a}^{(1)T}\underline{M}\,\underline{a}^{(n)} \\ \underline{a}^{(2)T}\underline{M}\,\underline{a}^{(1)} & \underline{a}^{(2)T}\underline{M}\,\underline{a}^{(2)} & \cdots & \underline{a}^{(2)T}\underline{M}\,\underline{a}^{(n)} \\ . & & & . \\ \underline{a}^{(n)T}\underline{M}\,\underline{a}^{(1)} & & & \underline{a}^{(n)T}\underline{M}\,\underline{a}^{(n)} \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ . \\ \ddot{y}_n \end{bmatrix} +$$

$$\begin{bmatrix} \underline{a}^{(1)T}\underline{K}\,\underline{a}^{(1)} & \underline{a}^{(1)T}\underline{K}\,\underline{a}^{(2)} & \cdots & \underline{a}^{(1)T}\underline{K}\,\underline{a}^{(n)} \\ \underline{a}^{(2)T}\underline{K}\,\underline{a}^{(1)} & \underline{a}^{(2)T}\underline{K}\,\underline{a}^{(2)} & \cdots & \underline{a}^{(2)T}\underline{K}\,\underline{a}^{(n)} \\ . & & & . \\ \underline{a}^{(n)T}\underline{K}\,\underline{a}^{(1)} & & & \underline{a}^{(n)T}\underline{K}\,\underline{a}^{(n)} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} \underline{a}^{(1)T}\underline{P} \\ \underline{a}^{(2)T}\underline{P} \\ . \\ \underline{a}^{(n)T}\underline{P} \end{bmatrix}$$

if $i \neq j$ $\underline{a}^{(i)T}\underline{M}\,\underline{a}^{(j)} = \underline{0}$ and $\underline{a}^{(i)T}\underline{K}\,\underline{a}^{(j)} = \underline{0}$

$\underline{a}^{(i)T}\underline{M}\,\underline{a}^{(i)} = M_i$, $\underline{K}\underline{a}^{(i)} = \lambda_i \underline{M}\,\underline{a}^{(i)}$ and $\underline{a}^{(i)T}\underline{P} = P^*_i$, where $\lambda_i = \omega_i^2$

$$\begin{bmatrix} M^*_1 & . & 0 \\ . & M^*_2 & . \\ 0 & . & M^*_n \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_n \end{bmatrix} + \begin{bmatrix} \omega_1^2 M^*_1 & . & 0 \\ . & \omega_2^2 M^*_2 & . \\ 0 & . & \omega_n^2 M^*_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_n \end{bmatrix} = \begin{bmatrix} P^*_1 \\ P^*_2 \\ P^*_n \end{bmatrix} \quad \text{or}$$

$$\ddot{y}_i + \omega_i^2 y_i = \frac{P_i^*}{M_i^*}, \text{ for } i=1,..,n$$

Example: $m_1 = m_2 = 20$ kg $\quad k_1 = k_2 = 6.10^4$ N/m $\quad p_1(t) = 0 \quad p_2(t) = 1000$ N for $t > 0$

$= 0$ for $t \leq 0$

$$|\underline{K} - \lambda\,\underline{M}| = 0 \text{ and } \underline{K} = \begin{bmatrix} 12x10^4 & -6x10^4 \\ -6x10^4 & 6x10^4 \end{bmatrix} \quad \underline{M} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$

$$|\underline{K} - \lambda\,\underline{M}| = \begin{vmatrix} 12.10^4 - 20\lambda & -6.10^4 \\ -6.10^4 & 6.10^4 - 20\lambda \end{vmatrix} = 0 \Rightarrow \text{then}$$

$\lambda_2 = 7.85\text{x}10^3 \ \text{rad}^2/\text{s}^2 \qquad \lambda_1 = 1.15\text{x}10^3 \text{rad}^2/\text{s}^2$

$\omega_1 = 34 \text{ rad/s}$

$\omega_2 = 88 \text{ rad/s}$

$(\underline{K} - \lambda \underline{M})\underline{a}^{(i)} = \underline{0}$ ➔ Normalized eigenvectors are obtained as follows:

$$\underline{a}^{(1)} = \begin{bmatrix} 1 \\ 1.62 \end{bmatrix} \quad \underline{a}^{(2)} = \begin{bmatrix} 1 \\ -0.62 \end{bmatrix}$$

$$\underline{X} = \underline{QY} \quad \text{then} \quad \ddot{y}_i + \omega_i^2 y_i = \frac{P_i^*}{M_i^*}$$

$$\underline{M}_1^* = \underline{a}^{(1)T}\underline{M}\,\underline{a}^{(1)} = [1 \quad 1.62]\begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}\begin{bmatrix} 1 \\ 1.62 \end{bmatrix} = 72.5$$

$$\underline{M}_2^* = \underline{a}^{(2)T}\underline{M}\,\underline{a}^{(2)} = [1 \quad -0.62]\begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}\begin{bmatrix} 1 \\ -0.62 \end{bmatrix} = 27.7$$

$$\underline{P}_1^* = \underline{a}^{(1)T}\underline{P} = [1 \quad 1.62]\begin{bmatrix} 0 \\ 10^3 \end{bmatrix} = 1620 \text{ kN}$$

$$\underline{P}_2^* = \underline{a}^{(2)T}\underline{P} = [1 \quad -0.62]\begin{bmatrix} 0 \\ 10^3 \end{bmatrix} = -620 \text{ kN}$$

$$\ddot{y}_1 + 34^2 y_1 = \frac{P_1^*}{M_1^*} \qquad \ddot{y}_1 + 34^2 y_1 = 22.345$$

$$\ddot{y}_2 + 88^2 y_2 = \frac{P_2^*}{M_2^*} \qquad \ddot{y}_2 + 88^2 y_2 = -22.383$$

$$y_1(t) = A_1 \sin 34t + B_1 \cos 34t + \frac{22.3}{34^2}$$

$$y_2(t) = A_2 \sin 88t + B_2 \cos 88t - \frac{22.3}{88^2}$$

$$y_1(t) = A_1 \sin 34t + B_1 \cos 34t + 1.94.10^{-2}$$

$$y_2(t) = A_2 \sin 88t + B_2 \cos 88t - 2.84.10^{-3}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [\underline{Q}] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Initial conditions: $x(0) = 0 \rightarrow y(0) = 0$
$\dot{x}(0) = 0 \rightarrow \dot{y}(0) = 0$

$0 = B_1 + 1.94.10^{-2} \rightarrow B_1 = -1.94.10^{-2}$

$\dot{y}_1(t) = 34\, A_1 \cos 34t - 34\, B_1 \sin 34t$ and $\dot{y}_1(0) = 0$ then $34A_1 = 0 \rightarrow A_1 = 0$

$y_1(t) = 1.94.10^{-2}\,(1 - \cos 34t)$

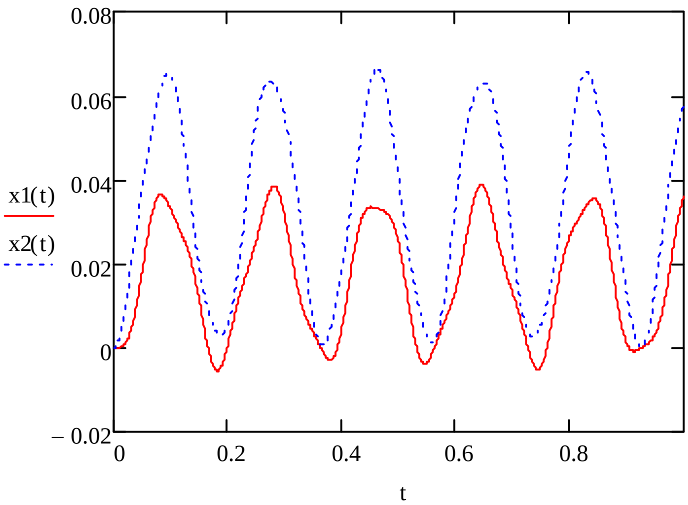
Similarly,

$0 = B_2 - 2.84.10^{-3} \rightarrow B_2 = 2.84.10^{-3}$

$\dot{y}_2(t) = 88\, A_2 \cos 88t - 88\, B_2 \sin 88t$ and $\dot{y}_1(0) = 0$ then $88\, A_2 = 0 \rightarrow A_2 = 0$

$y_2(t) = -2.84.10^{-3}\,(1 - \cos 88t)$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rightarrow \begin{array}{l} x_1 = y_1 + y_2 \\ x_2 = 1.62y_1 - 0.62y_2 \end{array}$$



Further Properties of eigenvalues and eigenvectors:

Theorem : if $\underline{A}\,\underline{x} = \lambda\, \underline{x} \Rightarrow (\lambda_i, \underline{x}^{(i)})$

$\underline{A}^T \underline{y} = \rho\, \underline{y} \Rightarrow (\rho_i, \underline{y}^{(i)})$

i) $\lambda_i = \rho_i \quad \forall_i$

ii) $\underline{x}^{(i)T}\,\underline{y}^{(j)} = \alpha\,\delta_{ij} \qquad \alpha = \underline{x}^{(i)T}\,\underline{y}^{(i)}$

Theorem : Eigenvalues of $\underline{A}^{-1}$ are reciprocals of those of $\underline{A}$ and eigenvectors are equal to those of $\underline{A}$.

Theorem : Eigenvalues of an orthogonal matrix are absolutely equal to unity.

Theorem : If $\underline{A}\,\underline{x} = \lambda\,\underline{x}$ then $\underline{A}^n\,\underline{x} = \lambda^n\,\underline{x}$ where n is a positive integer.

HE. Try to prove them.

Expansion of a vector in terms of eigenvectors of a matrix.

$$\underline{A}\,\underline{x} = \lambda\,\underline{x} \rightarrow \left(\lambda_i, \underline{x}^{(i)}\right)$$

A known vector $\underset{\sim}{z}$ can be expressed in terms of eigenvectors in matrix form.

$$\underline{z} = C_1\,\underline{x}^{(1)} + C_2\,\underline{x}^{(2)} + \dots\dots + C_n\,\underline{x}^{(n)} \qquad \underline{z} = \sum_{i=1}^{n} C_i\,\underline{x}^{(i)}$$

$$\underline{A}^T\,\underline{y} = \lambda\,\underline{y} \rightarrow \left(\lambda_j, \underline{y}^{(j)}\right)$$

Premultiply eqn $\underline{z} = \sum_{i=1}^{n} C_i\,\underline{x}^{(i)}$ by $\underline{y}^{(j)T}$ as follows

$$\underline{y}^{(j)T}\,\underline{z} = \sum_{i=1}^{n} \underline{y}^{(j)T}\,\underline{C}_i\,\underline{x}^{(i)} = \sum_{i=1}^{n} C_i \underbrace{\underline{y}^{(j)T}\,\underline{x}^{(i)}}_{\alpha\delta_{ij}}$$

$$= \sum_{i=1}^{n} \alpha\,C_i\,\delta_{ij} = \alpha \sum_{i=1}^{n} C_i\,\delta_{ij} = \alpha\,C_j$$

$$C_j = \frac{\underline{y}^{(j)T}\,\underline{z}}{\alpha} = \frac{\underline{y}^{(j)T}\,\underline{z}}{\underline{y}^{(j)T}\,\underline{x}^{(j)}} \quad \text{or} \quad C_i = \frac{\underline{y}^{(i)T}\,\underline{z}}{\underline{y}^{(i)T}\,\underline{x}^{(i)}}$$

if $x^{(i)}$ and $y^{(j)}$ are normalized in such a way that $\underline{y}^{(i)T} \cdot \underline{x}^{(j)} = \delta_{ij}$

$\underline{y}^{(i)T}\,\underline{x}^{(i)} = 1$ and $C_i = \underline{y}^{(i)T}\,\underline{z}$; therefore

$$\underline{z} = C_1\,\underline{x}^{(1)} + C_2\,\underline{x}^{(2)} + \dots\dots\dots + C_n\,\underline{x}^{(n)} \quad \text{where} \quad C_i = \underline{y}^{(i)T}\,\underline{z}$$

Solution of a linear system using eigenvector expansion:

$\underline{A}\,\underline{w} = \underline{b} \quad (1)$ w: unknown vector

for a 3x3 system:

$$a_{11}\,w_1 + a_{12}\,w_2 + a_{13}\,w_3 = b_1$$

$$a_{21} w_1 + a_{22} w_2 + a_{23} w_3 = b_2$$
$$a_{31} w_1 + a_{32} w_2 + a_{33} w_3 = b_3$$

$$\underline{w} = \sum_{i=1}^{n} C_i \underline{x}^{(i)} \quad (2) \quad \rightarrow \underline{w} = C_1 \underline{x}^{(1)} + C_2 \underline{x}^{(2)} + \dots\dots\dots + C_n \underline{x}^{(n)} = \sum_{i=1}^{n} C_i \underline{x}^{(i)}$$

$$\underline{A}\, \underline{x}^{(i)} = \lambda_i \underline{x}^{(i)} \quad (3)$$

From Eqs (1) and (2)

$$\underline{A} \sum_{i=1}^{n} C_i \underline{x}^{(i)} = \underline{b} \rightarrow \sum_{i=1}^{n} C_i \underbrace{\underline{A}\, \underline{x}^{(i)}}_{=\lambda_i \underline{x}^{(i)}} = \underline{b}$$

$$\rightarrow \sum_{i=1}^{n} C_i \lambda_i \underline{x}^{(i)} = \underline{b} \qquad (4)$$

$$\underline{A}^T \underline{y}^{(j)} = \lambda_j \underline{y}^{(j)} \qquad (5)$$

Premultiply (4) by $\underline{y}^{(j)T}$ to eliminate $\underline{x}^{(i)}$:

$$\underline{y}^{(j)T} \sum C_i \lambda_i \underline{x}^{(i)} = \underline{y}^{(j)T} \underline{b} \rightarrow \sum_{i=1}^{n} C_i \lambda_i \underbrace{\underline{y}^{(j)T} \underline{x}^{(i)}}_{\alpha \delta_{ij}} = \underline{y}^{(j)T} \underline{b}$$

$$\rightarrow \sum_{i=1}^{n} C_i \lambda_i \alpha \delta_{ij} = \underline{y}^{(j)T} \underline{b}$$

$$C_j \lambda_j \alpha = \underline{y}^{(j)T} \underline{b} \rightarrow C_j = \frac{\underline{y}^{(j)T} \underline{b}}{\alpha \lambda_j}$$

$$C_j = \frac{\underline{y}^{(j)T} \underline{b}}{\underline{y}^{(j)T} \underline{x}^{(j)} \lambda_j}$$

<u>Example</u>

$$\begin{bmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \underline{Q} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}$$

$$\underline{w} = \sum_{i=1}^{3} C_i \underline{x}^{(i)} \qquad \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 4 \\ \lambda_3 = 16 \end{matrix}. \text{ Then, } C_j = \frac{\underline{y}^{(j)T} \underline{b}}{\alpha \lambda_j} \quad \text{or} \quad C_j = \frac{\underline{y}^{(j)T} \underline{b}}{\underline{y}^{(j)T} \underline{x}^{(j)} \lambda_j}$$

$\underline{A}$ is symmetric then $\underline{A} = \underline{A}^T$.

Therefore $\underline{A}$ & $\underline{A}^T$ have same eigenvalues.

$$\Rightarrow C_j = \frac{\underline{x}^{(j)T}\underline{b}}{\partial_j} \quad \underline{x}^{(i)T}\underline{x}^{(j)} = 1$$

$$C_1 = \frac{\underline{x}^{(1)T}\underline{b}}{\lambda_1} = \frac{\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}}{1} = 1 \qquad C_2 = \frac{\underline{x}^{(2)T}\underline{b}}{\lambda_2} = \frac{\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}}{4} = \frac{2}{4\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

$$C_3 = \frac{\underline{x}^{(3)T}\underline{b}}{\lambda_3} = \frac{\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2} & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}}{16} = \frac{0}{16} = 0$$

Then $\begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} + \frac{1}{2\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} + 0 = \begin{bmatrix}1/4\\1/4\\1\end{bmatrix}$

Theorem : if $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ are the eigenvalues of a matrix $\underline{A}$, then,

i. $tr\ \underline{A} = \lambda_1 + \lambda_2 + \ldots\ldots + \lambda_n$ and ii. $det\ \underline{A} = \lambda_1\lambda_2 \ldots\ldots\ldots \lambda_n$

Pr oof of $|\underline{D}| = |\underline{A}|$, $\underline{D} = \underline{Q}^{-1}\underline{A}\underline{Q}$, then $\det(\underline{Q}^{-1}\underline{A}\underline{Q}) = \det \underline{Q}^{-1} \det \underline{A} \det \underline{Q} = \frac{1}{\det \underline{Q}} \det \underline{A} \det \underline{Q} = \det \underline{A}$

$\det \underline{D} = \det \underline{A} = \lambda_1\lambda_2 \ldots\ldots \lambda_n$

Execise: Show (i)

A few more properties of matrices:

- if $\underline{A}$ and $\underline{B}$ are symmetric matrices,
  then $\underline{A} + \underline{B}$ is also symmetric
- if $\underline{A}$ and $\underline{B}$ are symmetric $\rightarrow \underline{A}\underline{B}$ is also symmetric
- $\underline{A}^0 = \underline{I}$
- $\underline{A}^{-n} = \left(\underline{A}^{-1}\right)^n = \underbrace{\underline{A}^{-1}, \underline{A}^{-1}, \underline{A}^{-1} \ldots\ldots\ldots \underline{A}^{-1}}_{\text{n times}}$ n: is a positive integer